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Pairwise compatibility graphs: complete characterization for wheels

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A simple graph G is a pairwise compatibility graph (PCG) if there exists an edge-weighted tree T with positive weights and nonnegative numbers $d_{\min}$ and $d_{\max}$ such that the leaves of T are exactly the vertices of G, and uv is an edge in G if and only if the sum of weights of edges on the unique path between u and v in T is at least $d_{\min}$ and at most $d_{\max}$. We show that a wheel on n vertices is a PCG if and only if $n \leq 8$, settling an open problem proposed by Calamoneri and Sinaimeri (*SIAM Review* **58**:3 (2016), 445–460). Our approach is based on unavoidable binary classifications of the edges in the complement of wheels that are PCGs. (Note: during the review process of our work, we learned that the same result has been obtained independently with an alternative proof.)

1. Introduction

Edge-weighted rooted trees are common graph models used in phylogenetics, a branch of biology that studies the evolutionary history and relationships of sets of taxa, i.e., organisms sharing similar characteristics (e.g., species, populations). In such a phylogenetic tree, a leaf represents a taxon, an internal vertex represents a possible common ancestor of its descendant leaves, and the weight of an edge may be interpreted as the length of the evolutionary history separating the species or populations represented by its two incident vertices. One of the first illustrations of a phylogenetic tree appeared in Charles Darwin's groundbreaking work [1859].

In computational biology, the problem of reconstructing an optimal phylogenetic tree from a given set of taxa is complex [Calamoneri and Sinaimeri 2016], and so researchers have focused on constrained instances of this problem. For example, since very large and very small distances between pairs of taxa in the evolutionary history may have a negative impact on the performance of reconstruction algorithms, bounding these distances is a natural constraint [Kearney et al. 2003]. In graph-theoretical

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terms, let G be a graph where each vertex represents a taxon and uv be an edge in G if the evolutionary distance between vertices u and v is within an acceptable range. One is interested in finding an edge-weighted tree T with positive weights and nonnegative numbers $d_{\min}$ and $d_{\max}$ such that the set of leaves of T is exactly the set of vertices of G, and uv is an edge in G if and only if the sum of weights of edges on the unique path between u and v in T is at least $d_{\min}$ and at most $d_{\max}$. If such T, $d_{\min}$ and $d_{\max}$ exist, then we say that G is a *pairwise compatibility graph* (PCG) with *witness tree T bounded* by $d_{\min}$ and $d_{\max}$, or simply $G = \mathrm{PCG}(T, d_{\min}, d_{\max})$. For any two vertices u and v in G (not necessarily adjacent), $\mathrm{d}(u, v)$ will denote the sum of weights of the edges on the unique path in T between the leaves u and v (for simplicity, we omitted the subscript in $\mathrm{d}_T(u, v)$ which is traditionally used to denote the weighted distance between any pair of vertices u and v in T).

The literature suggests that the PCG recognition problem is difficult, and it has been conjectured to be NP-hard [Durocher et al. 2015]. Since no complete characterization of PCGs is currently known, a large portion of the existing research has focused on determining whether particular graphs are PCGs or not. The following are some examples of the known classes of PCGs: graphs with at most seven vertices [Calamoneri et al. 2013a; Phillips 2002]; bipartite graphs with at most eight vertices [Mehnaz and Rahman 2013]; cycles, single-chord cycles, cacti, tree power graphs, Steiner and phylogenetic k-power graphs [Mehnaz and Rahman 2013; Yanhaona et al. 2009]; trees, ladders, triangle-free outerplanar 3-graphs [Salma et al. 2013]; Dilworth 2 graphs [Calamoneri and Petreschi 2014]; split matrogenic graphs and certain superclasses [Calamoneri et al. 2013b]. Some particular graphs that are not PCGs have also been identified: a nonbipartite circular arc graph on 8 vertices, a bipartite graph on 15 vertices, and a planar graph on 20 vertices [Yanhaona et al. 2009; 2010]. Recently, two results involving the complement G^c of a graph G provided additional tools in the study of PCGs [Hossain et al. 2017]: if G^c is acyclic then G is a PCG; if G^c contains two vertex-disjoint chordless cycles without an edge simultaneously incident to both cycles, then G is not a PCG. One instance relevant to our work is the class of *k-leaf power graphs* which are PCGs, where $d_{\min} = 0$ and $d_{\max} = k$. It is well known that these graphs are strongly chordal, i.e., chordal and sun-free [Farber 1983]; however, the converse is not true [Bibelnieks and Dearing 1993]. In fact, no complete characterization of k-leaf power graphs is known except when $k \leq 4$ [Brandstädt and Le 2006; Brandstädt et al. 2008; Dom et al. 2004; 2005; Rautenbach 2006].

From the references above and from our recent experience, we have learned that many of the existing results concerning the PCG recognition problem required determination and clever, nontrivial approaches to generate witness trees or to show that none exist. Nevertheless, the efforts behind these approaches may not be readily apparent since they often describe witness trees without providing a clear discussion of what drives their particular structures. Perhaps for these reasons there are still many open problems in the area, as mentioned in the comprehensive survey [Calamoneri and Sinaimeri 2016], including the following:

Open Problem 1. Find other graph classes that do not belong to the PCG class.

Open Problem 2. It is not known whether or not wheels on at least eight nodes are PCGs.

We add one more class for Open Problem 1 while settling Open Problem 2 in our main result:

Theorem 1.1. *Wheels on n vertices are PCGs if and only if $n \leq 8$.*

We will be using the following notation throughout this work. The *wheel* W_n with order $n \geq 4$ has vertices $w_1, w_2, \dots, w_n$, edges $w_i w_n$ for $i = 1, 2, \dots, n-1$, edges $w_i w_{i+1}$ for $i = 1, 2, \dots, n-2$, and edge $w_1 w_{n-1}$. The cycle induced by the vertices $w_1, w_2, \dots, w_{n-1}$ is called the *rim* of the wheel.

Figure 1 shows the wheel W_8 and a witness tree T bounded by $d_{\min} = 5.5$ and $d_{\max} = 7.5$, that is, $W_8 = \text{PCG}(T, 5.5, 7.5)$. This claim can be easily verified using the information in Table 1, where for each entry (i, j), the corresponding column header is $\text{d}(w_i, w_j)$ for T in Figure 1 (pairs in bold correspond to the edges in W_8).

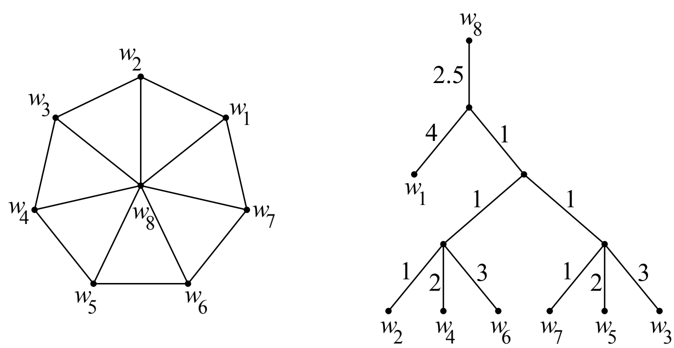

Figure 1. Wheel W_8 on the left and a witness tree T on the right with $W_8 = \text{PCG}(T, 5.5, 7.5)$.

3	4	5	5.5	6	6.5	7	7.5	8	9
(2,4)	(2,6)	(2,5)	**(2,8)**	**(2,3)**	**(1,8)**	**(1,2)**	**(3,8)**	(1,4)	(1,3)
(5,7)	(2,7)	(3,5)	**(7,8)**	**(4,5)**	**(4,8)**	**(1,7)**	**(6,8)**	(1,5)	(1,6)
	(3,7)	(4,6)		**(6,7)**	**(5,8)**	**(3,4)**		(3,6)	
		(4,7)			**(5,6)**				

Table 1. For each entry (i, j), the corresponding column header is $\text{d}(w_i, w_j)$ for T in Figure 1 (pairs in bold correspond to the edges in W_8).

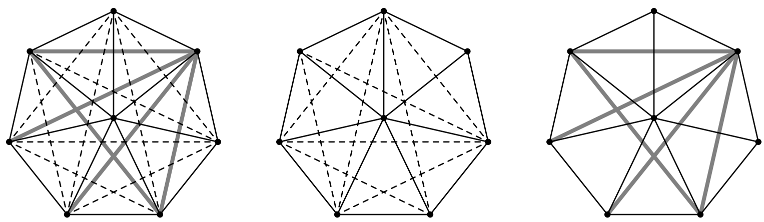

Figure 2. Edges in W_8 are solid black, light edges in $(W_8)^c$ are dashed and heavy edges in $(W_8)^c$ are thick gray.

Generating this witness tree for W_8 was far from trivial. A brute-force computation approach was infeasible due to the large number of trees with eight leaves and the infinite number of choices for their edge-weights and bounds. We relied on potential binary classifications of the edges in the complement $(W_8)^c$ of W_8, more specifically, which edge uv in $(W_8)^c$ could be *light*, i.e., $\mathrm{d}(u, v) < d_{\min}$, and which could be *heavy*, i.e., $\mathrm{d}(u, v) > d_{\max}$. Using general results that do not require the knowledge of an exact witness tree and bounds, we generated the configuration of light and heavy edges given on the left-most graph in Figure 2, where edges in W_8 are solid black, light edges in $(W_8)^c$ are dashed, and heavy edges in $(W_8)^c$ are thick gray. The center and right-most graphs in this figure are provided for clarity and show W_8 together with only light and with only heavy edges, respectively. The exact steps to obtain this configuration are omitted, as they are similar to the steps presented in the proof of Theorem 2.6 in Section 2. From this configuration, we were able to obtain the witness tree T and bounds in Figure 1 by inspection.

Recall that all graphs with at most seven vertices are PCGs. Theorem 1.1 will follow, given that we have shown here that W_8 is also a PCG and will show in Section 2 that no W_n for $n \geq 9$ is a PCG.

During the review process of our work, we learned that Theorem 1.1 has been verified independently in the arXiv manuscript [Baiocchi et al. 2017] which was later presented as the conference extended abstract [Baiocchi et al. 2018]. In [Baiocchi et al. 2018], the edges of a PCG are colored black, and edges in the complement are colored red if they are light and white if they are heavy. Several forbidden tricolored structures are identified. The general approach assumes that W_n for $n \geq 9$ is a PCG and these forbidden structures are used in an exhaustive case discussion to reach a contradiction. Our approach is similar in the sense that it focuses on certain unavoidable binary configurations of edges and, indeed, one of the forbidden structures identified in [Baiocchi et al. 2018] (namely $\mathbf{f\text{-}c}(2K_2)a$, coincides with the configuration H_5 described in our Lemma 2.4). Nevertheless, we believe our proof streamlines the case discussion by generating a sequence of unavoidable light edges until the forbidden configuration H_5 is achieved.

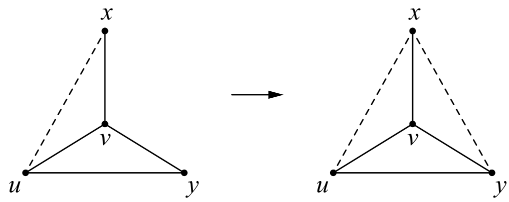

Figure 3. Configuration H_1 (left), where xy is an edge in G^c and $\mathrm{d}(u, v) \geq \mathrm{d}(v, x)$, implies H_2 (right), as shown in Lemma 2.2.

2. Wheels with more than eight vertices are not PCGs

Key to our discussion is the following useful result that allows for distance comparisons between certain pairs of leaves in general edge-weighted trees.

Result 2.1 [Yanhaona et al. 2010]. Let T be an edge-weighted tree and let u, v, x be three leaves in T such that $\mathrm{d}(u, v) = \max\{\mathrm{d}(u, v), \mathrm{d}(v, x), \mathrm{d}(x, u)\}$. If y is a leaf other than u, v, x, then $\mathrm{d}(x, y) \leq \mathrm{d}(u, y)$ or $\mathrm{d}(x, y) \leq \mathrm{d}(v, y)$. □

We will apply Result 2.1 to the witness trees of certain PCGs in Lemmas 2.2, 2.3 and 2.4. These lemmas will be vital tools used to show that W_n is not a PCG when $n \geq 9$ in Theorem 2.6. We first extend the definitions of light and heavy edges mentioned in Section 1 to general PCGs; that is, given $G = \mathrm{PCG}(T, d_{\min}, d_{\max})$, we say that an edge uv in G^c is *light* if $\mathrm{d}(u, v) < d_{\min}$ and is *heavy* if $\mathrm{d}(u, v) > d_{\max}$. Any future figures will continue using the conventions given in Figure 2: edges in G are solid black, light edges in G^c are dashed, and heavy edges in G^c are thick gray.

Lemma 2.2. *Let $G = \mathrm{PCG}(T, d_{\min}, d_{\max})$. If G and G^c contain the edges in the configuration H_1 in Figure 3 (left), where xy is an edge in G^c and $\mathrm{d}(u, v) \geq \mathrm{d}(v, x)$, then xy must be light as indicated in the configuration H_2 in Figure 3 (right).*

Proof. Since $\mathrm{d}(u, v) \geq \mathrm{d}(v, x)$ and xu is light, we have $\mathrm{d}(u, v) = \max\{\mathrm{d}(u, v), \mathrm{d}(v, x), \mathrm{d}(x, u)\}$. By Result 2.1, $\mathrm{d}(x, y) \leq \mathrm{d}(u, y)$ or $\mathrm{d}(x, y) \leq \mathrm{d}(v, y)$. But $\mathrm{d}(u, y) \leq d_{\max}$ and $\mathrm{d}(v, y) \leq d_{\max}$ because uy and vy are edges in G, therefore $\mathrm{d}(x, y) \leq d_{\max}$. This latter inequality combined with the fact that xy is an edge in G^c implies $\mathrm{d}(x, y) < d_{\min}$ and therefore xy is light. □

Lemma 2.3. *Let $G = \mathrm{PCG}(T, d_{\min}, d_{\max})$. If G and G^c contain the edges in the configuration H_3 in Figure 4 (left), where uv is an edge in G^c, then uv must be light as indicated in the configuration H_4 in Figure 4 (right).*

Proof. Suppose by contradiction that uv is heavy. Since xu and vx are edges in G, we must have $\mathrm{d}(x, u) \leq d_{\max}$ and $\mathrm{d}(v, x) \leq d_{\max}$; hence $\mathrm{d}(u, v) = \max\{\mathrm{d}(u, v), \mathrm{d}(v, x), \mathrm{d}(x, u)\}$ and by Result 2.1, $\mathrm{d}(x, y) \leq \mathrm{d}(u, y)$ or $\mathrm{d}(x, y) \leq \mathrm{d}(v, y)$. But uy

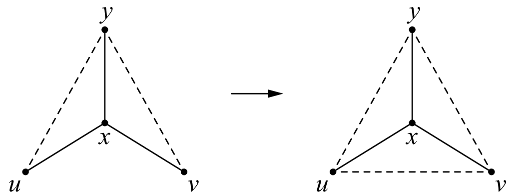

Figure 4. Configuration H_3 (left), where uv is an edge in G^c, implies H_4 (right), as shown in Lemma 2.2.

and vy are light, that is, $d(u, y) < d_{\min}$ and $d(v, y) < d_{\min}$, which would imply $d(x, y) < d_{\min}$, contradicting the fact that $d(x, y) \geq d_{\min}$ as xy is an edge in G. □

For later discussions, it is important to note the differences between the vertex labels in Figures 3 and 4 (e.g., v and x are the only vertices of degree 3 in each respective figure). These labels were chosen so that Result 2.1 could be readily applied in the proofs of Lemmas 2.2 and 2.3, respectively.

Lemma 2.4. *Let $G = \mathrm{PCG}(T, d_{\min}, d_{\max})$. G and G^c cannot contain the configuration H_5 of Figure 5.*

Proof. Suppose by contradiction that G and G^c contain the configuration H_5. Since xu and vx are light, we must have $d(x, u) < d_{\min}$ and $d(v, x) < d_{\min}$. But uv is an edge in G, so we have $d(u, v) \geq d_{\min}$. Hence $d(u, v) = \max\{d(u, v), d(v, x), d(x, u)\}$ and by Result 2.1, $d(x, y) \leq d(u, y)$ or $d(x, y) \leq d(v, y)$. But uy and vy are light, that is, $d(u, y) < d_{\min}$ and $d(v, y) < d_{\min}$, which would imply $d(x, y) < d_{\min}$, contradicting the fact that $d(x, y) \geq d_{\min}$ as xy is an edge in G. □

In the proof of Theorem 2.6, we will assume by contradiction that W_n is a PCG for some $n \geq 9$ and apply Lemmas 2.2 and 2.3 repeatedly until a contradiction to Lemma 2.4 is reached. To be able to set this argument in motion, we need to verify the existence of a particular light edge. For each $p = 2, 3, \ldots, n-3$, we define a *p-light edge* in $(W_n)^c$ to be a light edge with ends connected by a path on the rim of W_n with exactly p edges (note that a p-light edge is also an $(n-p-1)$-light edge).

Lemma 2.5. *If $n \geq 5$ and $W_n = \mathrm{PCG}(T, d_{\min}, d_{\max})$, then there exists a p-light edge for each $p = 2, 3, \ldots, n-3$.*

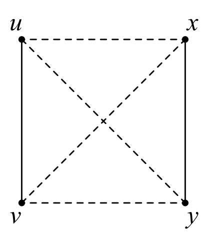

Figure 5. Forbidden configuration H_5 in Lemma 2.4.

Proof. Let $W_n = \mathrm{PCG}(T, d_{\min}, d_{\max})$ with $n \geq 5$. Since p-light edges are $(n-p-1)$-light edges, it is enough to verify the lemma for $p = 2, 3, \ldots, \lfloor (n-1)/2 \rfloor$. We will proceed by induction on p.

The rim of W_n is a chordless cycle; hence W_n is not chordal and consequently not strongly chordal. Recall from Section 1 that k-leaf power graphs are strongly chordal so W_n is not a k-leaf power graph; that is, $d_{\min} > 0$ and there exists at least one light edge (if there are no light edges, then uv would be an edge in G if and only if $0 \leq \mathrm{d}(u, v) \leq d_{\max}$, and hence W_n would be a k-leaf power graph). Choose a light edge with ends that minimize the distance on the rim of W_n (i.e., the number of edges on the shortest path between these ends using only edges on the rim) over all light edges, and let m be this smallest distance. We may assume without loss of generality that $w_1 w_{m+1}$ is this selected light edge and $\mathrm{d}(w_{m+1}, w_n) \geq \mathrm{d}(w_1, w_n)$ (if not, rotate and/or reverse the labels on the rim). Clearly, $2 \leq m \leq \lfloor (n-1)/2 \rfloor$. If $m > 2$, since $w_1 w_m$ is an edge in $(W_n)^c$ and $\mathrm{d}(w_{m+1}, w_n) \geq \mathrm{d}(w_1, w_n)$, then applying Lemma 2.2 with $u = w_{m+1}, v = w_n, x = w_1, y = w_m$ would imply $xy = w_1 w_m$ is light with ends connected by a path on the rim with $m-1$ edges, which contradicts the minimality of m. Hence $m = 2$; that is, $w_1 w_3$ is light with ends connected by the path $w_1 w_2 w_3$ on the rim. Thus, there is a 2-light edge in W_n, and the basis of the induction has been established.

Assume for $2 \leq p < \lfloor (n-1)/2 \rfloor$ that there exists a p-light edge and we will show that there exists a $(p+1)$-light edge, concluding our inductive argument. Rotate and/or reverse the labels on the rim so that $w_1 w_{p+1}$ is this p-light edge and $\mathrm{d}(w_{p+1}, w_n) \geq \mathrm{d}(w_1, w_n)$. Note that since $n \geq 5$ and $p < \lfloor (n-1)/2 \rfloor$, we have $p+2 < \lfloor (n-1)/2 \rfloor + 2 \leq n-1$ so $w_1 w_{p+2}$ is an edge in $(W_n)^c$. Applying Lemma 2.2 with $u = w_{p+1}, v = w_n, x = w_1, y = w_{p+2}$ we conclude that $xy = w_1 w_{p+2}$ is a $(p+1)$-light edge, and so our induction is complete. □

We can confirm that this lemma holds in the instance of W_8 presented in Figure 2; for example, $w_1 w_3$ is a 2- and 5-light edge, and $w_1 w_4$ is a 3- and 4-light edge.

Applications of Lemma 2.2 similar to the two discussed in the proof of Lemma 2.5 will occur multiple times in the proof of Theorem 2.6, and so we will use the abbreviated notation $(i, j, k) \xrightarrow{2.2} (j, k)$ to indicate that $w_j w_k$ is an edge in $(W_n)^c$, $\mathrm{d}(w_i, w_n) \geq \mathrm{d}(w_j, w_n)$, and setting $u = w_i, v = w_n, x = w_j, y = w_k$ we have the configuration H_1 in Figure 3 (left); therefore applying Lemma 2.2 implies $xy = w_j w_k$ is light. With this notation, the two applications of Lemma 2.2 in the proof of Lemma 2.5 would simply read $(m+1, 1, m) \xrightarrow{2.2} (1, m)$ and $(p+1, 1, p+2) \xrightarrow{2.2} (1, p+2)$, respectively. In the same spirit, we also define the abbreviated notation $(i, j, k) \xrightarrow{2.3} (i, j)$ to indicate that $w_i w_j$ is an edge in $(W_n)^c$ and setting $u = w_i$, $v = w_j, x = w_n, y = w_k$ we have the configuration H_3 in Figure 4 (left); therefore applying Lemma 2.3 implies $uv = w_i w_j$ is light.

Theorem 2.6. *If $n \geq 9$, then W_n is not a PCG.*

Proof. Let $n \geq 9$ and suppose by contradiction that $W_n = \text{PCG}(T, d_{\min}, d_{\max})$. From Lemma 2.5, there exists a 4-light edge. We may assume without loss of generality that $w_2 w_6$ is a 4-light edge and that $\text{d}(w_6, w_n) \geq \text{d}(w_2, w_n)$ (if not, rotate and/or reverse the labels on the rim). The proof proceeds by adding light edges forced by Lemmas 2.2 and 2.3 until we reach the configuration H_5 featured in Figure 5, which would contradict Lemma 2.4. We begin by observing that $(6, 2, 5) \xrightarrow{2.2} (2, 5)$ and $(6, 2, 7) \xrightarrow{2.2} (2, 7)$. The three current light edges are shown in the configuration G_1 of Figure 6. We split the discussion into two cases:

Case 1: Suppose $\text{d}(w_5, w_n) \geq \text{d}(w_2, w_n)$. Hence $(5, 2, 4) \xrightarrow{2.2} (2, 4)$, with current light edges shown in the configuration G_2 of Figure 6. In addition, $(4, 7, 2) \xrightarrow{2.3} (4, 7)$ and $(5, 7, 2) \xrightarrow{2.3} (5, 7)$, with current light edges shown in the configuration G_3 of Figure 6. Let us first examine the subcase where $\text{d}(w_7, w_n) < \text{d}(w_2, w_n)$. Since $n \geq 9$, we have that $w_1 w_7$ is an edge in $(W_n)^c$ and is in fact a light edge, since $(2, 7, 1) \xrightarrow{2.2} (7, 1)$. We then have $(1, 4, 7) \xrightarrow{2.3} (1, 4)$ and $(1, 5, 7) \xrightarrow{2.3} (1, 5)$. The current light edges are shown in the configuration G_4 of Figure 6 and therefore we reached the configuration H_5 with $u = w_1,\ v = w_2,\ x = w_4,\ y = w_5$ (boxed vertices), a contradiction. We now focus on the remaining subcase where $\text{d}(w_7, w_n) \geq \text{d}(w_2, w_n)$ and reset our current light edges to those shown in configuration G_3 of Figure 6. First observe that $(7, 2, 8) \xrightarrow{2.2} (2, 8)$. We then have $(4, 8, 2) \xrightarrow{2.3} (4, 8)$ and $(5, 8, 2) \xrightarrow{2.3} (5, 8)$. The current light edges are shown in the configuration G_5 of Figure 6 and therefore we reached the configuration H_5 with $u = w_4,\ v = w_5,\ x = w_7,\ y = w_8$ (boxed vertices), a contradiction.

Case 2: Suppose $\text{d}(w_5, w_n) < \text{d}(w_2, w_n)$ and reset our current light edges to those shown in configuration G_1 of Figure 6. Hence $(2, 5, 1) \xrightarrow{2.2} (5, 1)$ and $(2, 5, 3) \xrightarrow{2.2} (5, 3)$ with current light edges shown in the configuration G_6 of Figure 6. Let us first examine the subcase where $\text{d}(w_5, w_n) \geq \text{d}(w_1, w_n)$, thus $(5, 1, 6) \xrightarrow{2.2} (1, 6)$. The current light edges are shown in the configuration G_7 of Figure 6 and therefore we reached the configuration H_5 with $u = w_1,\ v = w_2,\ x = w_5,\ y = w_6$ (boxed vertices), a contradiction. We now focus on the remaining subcase where $\text{d}(w_5, w_n) < \text{d}(w_1, w_n)$ and reset our current light edges to those shown in configuration G_6 of Figure 6. First observe that $(1, 5, n-1) \xrightarrow{2.2} (5, n-1)$. We then have $(2, n-1, 5) \xrightarrow{2.3} (2, n-1)$ and $(3, n-1, 5) \xrightarrow{2.3} (3, n-1)$. Now we have $(6, n-1, 2) \xrightarrow{2.3} (6, n-1)$ (note that $w_6 w_{n-1}$ is an edge in $(W_n)^c$ since $n \geq 9$) and can finally conclude that $(3, 6, n-1) \xrightarrow{2.3} (3, 6)$. The current light edges are shown in the configuration G_8 of Figure 6 and therefore we reached the configuration H_5 with $u = w_2,\ v = w_3,\ x = w_5,\ y = w_6$ (boxed vertices), a contradiction.

Since contradictions were reached in all possible cases, the theorem holds. □

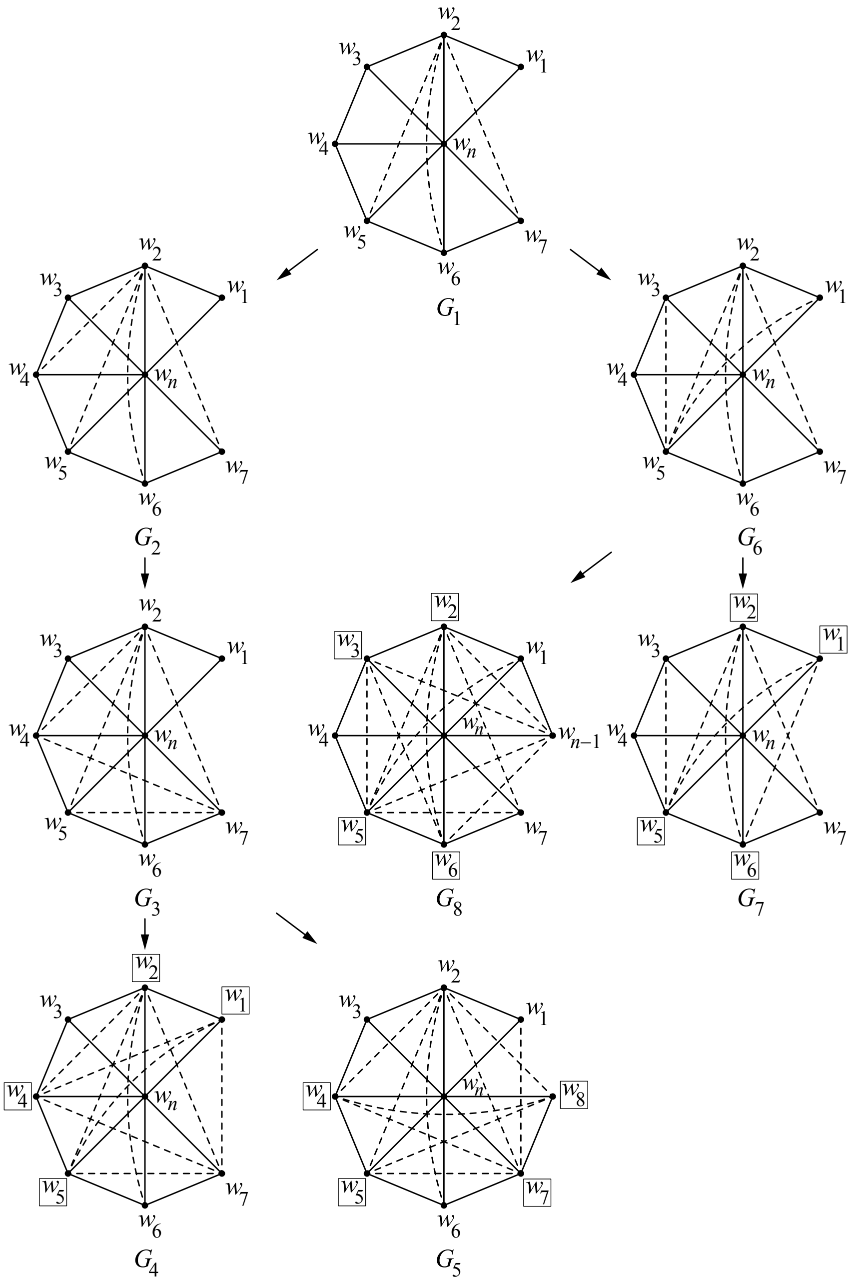

Figure 6. Configurations from the proof of Theorem 2.6.

A series of steps based on Lemmas 2.2, 2.3, and 2.4, similar to the ones described in the proof of Theorem 2.6, could be applied to W_8 to construct complete configurations of light and heavy edges that do not contain H_5 of Figure 5. After

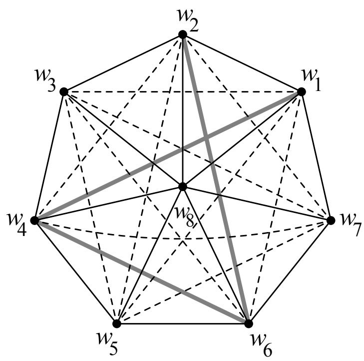

Figure 7. Invalid configuration for W_8 and $(W_8)^c$ in Lemma 2.7.

exhaustive case discussions (omitted for the sake of brevity), we found only two of these configurations, namely the configurations in Figures 2 and 7. The former allowed us to prove that W_8 is a PCG as shown in Section 1. Interestingly, the latter is not a valid configuration for W_8 and $(W_8)^c$ as verified in Lemma 2.7.

Lemma 2.7. *If* $W_8 = \mathrm{PCG}(T, d_{\min}, d_{\max})$, *then its corresponding light and heavy edges cannot be described by the configuration in Figure 7* (*up to rotating and/or reversing the vertex labels on the rim*).

Proof. Suppose the lemma does not hold. We examine three cases:

Case 1: $\mathrm{d}(w_1, w_2) = \max\{\mathrm{d}(w_1, w_2), \mathrm{d}(w_2, w_8), \mathrm{d}(w_8, w_1)\}$. Apply Result 2.1 with $u = w_1$, $v = w_2$, $x = w_8$, $y = w_5$ to conclude $\mathrm{d}(w_8, w_5) \leq \mathrm{d}(w_1, w_5)$ or $\mathrm{d}(w_5, w_8) \leq \mathrm{d}(w_2, w_5)$. But w_1w_5 and w_2w_5 are light which would imply $\mathrm{d}(w_8, w_5) < d_{\min}$, contradicting the fact that w_8w_5 is an edge in W_8.

Case 2: $\mathrm{d}(w_2, w_8) = \max\{\mathrm{d}(w_1, w_2), \mathrm{d}(w_2, w_8), \mathrm{d}(w_8, w_1)\}$. Apply Result 2.1 with $u = w_2$, $v = w_8$, $x = w_1$, $y = w_4$ to conclude $\mathrm{d}(w_1, w_4) \leq \mathrm{d}(w_2, w_4)$ or $\mathrm{d}(w_1, w_4) \leq \mathrm{d}(w_8, w_4)$. If $\mathrm{d}(w_1, w_4) \leq \mathrm{d}(w_2, w_4)$, then $\mathrm{d}(w_1, w_4) < d_{\min}$ since w_2w_4 is light; if $\mathrm{d}(w_1, w_4) \leq \mathrm{d}(w_8, w_4)$, then $\mathrm{d}(w_1, w_4) \leq d_{\max}$ since w_8w_4 is an edge in W_8; both options contradict the fact that w_1w_4 is heavy.

Case 3: $\mathrm{d}(w_8, w_1) = \max\{\mathrm{d}(w_1, w_2), \mathrm{d}(w_2, w_8), \mathrm{d}(w_8, w_1)\}$. Given the symmetry of the configuration in Figure 7, this case can be verified as in Case 2 if we rotate the vertex labels around the rim one unit counterclockwise and then reverse their order clockwise.

Since contradictions were reached in all possible cases, the lemma holds. □

3. Closing remarks

We proved that W_8 is a PCG, but W_n for $n \geq 9$ are not PCGs, settling an open problem proposed in [Calamoneri and Sinaimeri 2016]. The difficulty in showing

W_8 is a PCG stemmed from the many degrees of freedom one has in constructing potential witness trees — as both the tree's structure and its edge weights must be specified, the collection of candidate witness trees is both very large and highly varied. A natural direction for future work would be to ask whether some subfamilies of trees could be conclusively ruled out as witness trees. Our results followed from a series of lemmas concerning light and heavy edges. While considerably distanced from the properties of any underlying witness tree, this layer of abstraction is nonetheless extremely useful. We have presented here a collection of general tools concerning configurations of heavy and/or light edges, but this set is by no means exhaustive — indeed, Lemma 2.7 hints at other families of forbidden subgraphs. We hope to see expanded results, both in terms of composition and complexity of such configurations, in the months and years to come.

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