

# involve

a journal of mathematics

The financial value of knowing the distribution  
of stock prices in discrete market models

Ayelet Amiran / Fabrice Baudoin / Skylyn Brock / Berend Coster /  
Ryan Craver / Ugonna Ezeaka / Phanael Mariano / Mary Wishart





# The financial value of knowing the distribution of stock prices in discrete market models

Ayelet Amiran, Fabrice Baudoin, Skylyn Brock, Berend Coster,  
Ryan Craver, Ugonna Ezeaka, Phaniel Mariano and Mary Wishart

(Communicated by Jonathon Peterson)

An explicit formula is derived for the value of weak information in a discrete-time model that works for a wide range of utility functions, including the logarithmic utility and power utility. We assume a complete market with a finite number of assets and a finite number of possible outcomes. Explicit calculations are performed for a binomial model with two assets.

## 1. Introduction

Suppose an investor knows the distribution of the prices of the stocks in the market at a future time and this investor wants to optimize her or his expected utility from wealth at that future time. Our basic question is: *What is the financial value of this information?*

Much of the research into utility optimization and the financial value of weak information has been looked at previously in a continuous time setting [Baudoin 2003; Baudoin and Nguyen-Ngoc 2004]. The purpose of this paper is to investigate how to optimize a stock portfolio given weak information in a discrete-time setting. It should be stressed that the results we obtain are new and cannot be obtained as a consequence of the results in [Baudoin 2003; Baudoin and Nguyen-Ngoc 2004].

We will assume that the market is complete. We will also assume that there are no transactions costs. For a definition of complete markets, see [Björk 2009]. The main tool we use in finding the optimal expected utility given the weak information on future stock prices is the martingale method; see [Shreve 2004]. The reader might recognize that the problem treated here is related to robust utility maximization problems, as discussed in [Gilboa and Schmeidler 1989] and later works in mathematical finance by H. Föllmer, A. Gundel and S. Weber.

---

MSC2010: 91G10.

*Keywords:* anticipation, mathematical finance, financial value of weak information, portfolio optimization, discrete market models, insider trading.

As with classical results in this field, we will be looking at the expected utility as opposed to the expected wealth. This is an important difference to note since utility functions allow us to include an individual's attitude towards risk.

## 2. Utility functions

There are many different utility functions used in mathematics and economics to measure an individual's happiness or satisfaction. We denote our utility functions by  $U$ . We require that a utility function is strictly concave, strictly increasing, and continuously differentiable. We assume as in [Baudoin 2003] that

$$\lim_{x \rightarrow 0} U'(x) = +\infty \quad \text{and} \quad \lim_{x \rightarrow \infty} U'(x) = 0. \quad (1)$$

These conditions are sufficient for a utility function to exhibit risk aversion, to satisfy the law of diminishing marginal utility, and to guarantee that an increase in wealth results in an increase in utility. Further, when discussing the risk aversion of our utility functions, we use the absolute and relative risk aversion functions; see [Meyer and Meyer 2005]. We will be looking specifically at three different types of utility functions:

(i) Log utility:  $U(x) = \ln(x)$ ,  $x > 0$ . The log utility function has a constant relative risk aversion of 1. This implies the individual will always take on a constant proportion of risk with respect to their wealth.

(ii) Power utility:  $U(x) = x^\gamma / \gamma$  for  $-\infty < \gamma < 0$  and  $0 < \gamma < 1$  and  $x > 0$ . The power utility function also has a constant relative risk aversion, but the constant value is  $1 - \gamma$ . Thus, the power utility function is less risk-averse compared to the log utility function for  $0 < \gamma < 1$ . In this case, the constant  $\gamma$  reflects the relative risk aversion with the individual becoming more risk-averse as  $\gamma$  approaches 0. If  $-\infty < \gamma < 0$ , the individual is more risk-averse than an individual whose preferences can be described by the logarithmic utility function. As  $\gamma$  approaches  $-\infty$ , the individual becomes more and more risk-averse.

(iii) Exponential utility:  $U(x) = -e^{-\alpha x}$  for  $\alpha > 0$  and  $x \in \mathbb{R}$ . The exponential utility function has a constant absolute risk aversion of 1. Thus, the individual with an exponential utility function will assume a constant amount of risk rather than a constant proportion of risk with respect to their wealth. Notice that the exponential utility function does not satisfy the condition (1), but it is still an interesting function to note, and our results still hold true for this function.

## 3. Modeling the financial value of weak information on discrete-time complete markets with a discrete state space

**Setup.** Suppose we have a market with  $d$  financial assets, and a sample space  $\Omega_1 = \{\omega_1, \dots, \omega_M\}$  of possible outcomes of all the asset prices after one time

period. For all probability measures  $\mathbb{P}$ , we always assume  $\mathbb{P}(\omega_j) > 0$  for all  $j \in \{1, \dots, M\}$ . This is not a restriction since if  $\mathbb{P}(\omega_j) = 0$ , then we exclude  $\omega_j$  from  $\Omega_1$ . Let  $N$  be our final time period, and let  $\vec{S}_n \in \mathbb{R}^d$  denote the asset prices at time  $n$  where  $n \in \{0, 1, \dots, N\}$ . Further, let the random variable  $V_n$  denote the value of the portfolio at time  $n$ . Denote the initial wealth of the investor  $V_0$  by  $v$ . Without loss of generality we can assume one of the assets is a risk-free asset. We define  $r$  to be the rate of return of the risk-free asset. We will denote by  $\mathcal{M}$  the set of equivalent<sup>1</sup> probability measures under which discounted stock prices are martingales. Furthermore, we will assume our market is free from arbitrage. Thus, we can assume that the set  $\mathcal{M}$  is nonempty. For a complete market,  $\mathcal{M}$  is a singleton, say  $\mathcal{M} = \{\tilde{\mathbb{P}}\}$ , where  $\tilde{\mathbb{P}}$  is the unique probability measure under which discounted stock prices are martingales; see [Björk 2009] for more details about arbitrage, completeness, and equivalent martingale measures. We denote by  $\Psi^v$  the set of self-financing portfolios given initial wealth  $v$ . The probability measure  $\tilde{\mathbb{P}}$  basically represents the “knowledge” of the uninformed investor. Notice that by Jensen’s inequality this is the same as having no information at all, since it is optimal to invest in the risk-free asset only.

**3.1. Weak anticipation.** Now suppose we have some weak anticipation (weak information) regarding the prices of assets at our final time period. That is to say, we know the distribution of  $\vec{S}_N$ . We will denote this distribution by  $\nu$ . Let  $\Omega$  denote the path space of the ( $M$ -dimensional) stock price process  $\{\vec{S}_n\}_{1 \leq n \leq N}$ . Further, let  $\mathcal{A}$  be the (finite) set of possible asset prices at time  $N$ . Note  $|\mathcal{A}| \leq M^N$ .

**Definition.** The probability measure  $\mathbb{P}^\nu$  defined by

$$\mathbb{P}^\nu(\omega) := \sum_{\vec{x} \in \mathcal{A}} \tilde{\mathbb{P}}(\omega \mid \vec{S}_N = \vec{x}) \nu(\vec{S}_N = \vec{x})$$

is called the minimal probability measure associated with the weak information  $\nu$ , where  $\tilde{\mathbb{P}} \in \mathcal{M}$  is an (remember  $\mathcal{M}$  is a singleton in a complete market) equivalent martingale measure.

In the sense of the following proposition,  $\mathbb{P}^\nu$  is minimal in the set of probability measures  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  such that  $\mathbb{Q}(\vec{S}_N = \vec{x}) = \nu(\vec{S}_N = \vec{x})$  for all  $\vec{x} \in \mathcal{A}$ . We denote this set by  $\mathcal{E}^\nu$ .

**Proposition 3.1.** *Let  $\phi$  be a convex function. Then*

$$\min_{\mathbb{Q} \in \mathcal{E}^\nu} \tilde{\mathbb{E}} \left[ \phi \left( \frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}} \right) \right] = \tilde{\mathbb{E}} \left[ \phi \left( \frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}} \right) \right],$$

where  $d\mathbb{Q}/d\tilde{\mathbb{P}}$  denotes the Radon–Nikodym derivative of  $\mathbb{Q}$  with respect to  $\tilde{\mathbb{P}}$ .

<sup>1</sup>In our finite discrete sample space, by equivalent we simply mean, for all  $i \in \{1, 2, \dots, M\}$ ,  $\mathbb{Q}(\omega_i) > 0$ .

*Proof.* Let  $\vec{x} \in \mathcal{A}$  and  $\mathbb{Q} \in \mathcal{E}^\nu$  be given. Then,

$$\tilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}} \mid \vec{S}_N = \vec{x}\right] = \frac{v(\vec{S}_N = \vec{x})}{\tilde{\mathbb{P}}(\vec{S}_N = \vec{x})}.$$

Let  $\phi$  be a convex function. Then from the conditional version of Jensen's inequality

$$\phi\left(\frac{v(\vec{S}_N = \vec{x})}{\tilde{\mathbb{P}}(\vec{S}_N = \vec{x})}\right) = \phi\left(\tilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}} \mid \vec{S}_N = \vec{x}\right]\right) \leq \tilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}\right) \mid \vec{S}_N = \vec{x}\right].$$

Taking the expected value on both sides, we get

$$\tilde{\mathbb{E}}\left[\phi\left(\frac{v(S_N)}{\tilde{\mathbb{P}}(S_N)}\right)\right] = \tilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}}\right)\right] \leq \tilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}\right)\right],$$

and the result is proved.  $\square$

**3.2. Value of weak information.** Since an insider's anticipation has a different final time distribution than an uninformed investor's, it is natural to find a way to characterize the value of this information. Since we focused on maximizing our utility of wealth rather than the monetary value of wealth, we will define our value accordingly.

**Definition.** The *financial value of weak information* is the lowest expected utility that can be gained from anticipation. We write

$$u(v, \nu) = \min_{\mathbb{Q} \in \mathcal{E}^\nu} \max_{\psi \in \Psi^\nu} \mathbb{E}^\mathbb{Q}[U(V_N)].$$

Our main theorem is the following:

**Theorem 3.2.** *The financial value of weak information in a complete market is*

$$u(v, \nu) = \max_{\psi \in \Psi^\nu} \mathbb{E}^\nu[U(V_N)] = \mathbb{E}^\nu\left[U\left(I\left(\frac{\lambda(v)}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right],$$

where  $\lambda(v)$  is determined by

$$\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda(v)}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] = v,$$

where  $\tilde{\mathbb{P}} \in \mathcal{M}$  is the unique probability measure under which the prices are martingales. Moreover, the optimal wealth at time  $n$ ,  $\hat{V}_n$ , is given by

$$\hat{V}_n = \frac{1}{(1+r)^{N-n}} \sum_{\omega \in \Omega} I\left(\frac{\lambda(v)}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}(\omega)\right) \tilde{\mathbb{P}}(\omega \mid \vec{S}_n) \quad \text{for } n \in \{0, 1, \dots, N\}.$$

At time  $n$ , the optimal amount to purchase of the  $i$ -th linearly independent asset is

$$\delta_n^i = \sum_{j=1}^M (D_{n+1}^{-1})_{i,j} \hat{V}_{n+1}(\omega_j) \quad \text{for } n \in \{0, 1, \dots, N-1\},$$

where

$$D_{n+1} = \begin{bmatrix} S_{n+1}^1(\omega_1) & S_{n+1}^2(\omega_1) & \cdots & S_{n+1}^M(\omega_1) \\ S_{n+1}^1(\omega_2) & S_{n+1}^2(\omega_2) & \cdots & S_{n+1}^M(\omega_2) \\ \vdots & \vdots & & \vdots \\ S_{n+1}^1(\omega_M) & S_{n+1}^2(\omega_M) & \cdots & S_{n+1}^M(\omega_M) \end{bmatrix}$$

is the matrix of  $M$  linearly independent asset prices at time  $n+1$ ,  $(D_{n+1}^{-1})_{i,j}$  represents the element  $(i, j)$  of the matrix  $D_{n+1}^{-1}$ , and  $\widehat{V}_{n+1}$  comes from the above equation.

*Proof.* We will proceed by rewriting  $\max_{\psi \in \Psi^v} \mathbb{E}^{\mathbb{Q}}[U(V_N)]$ . In order to do this, we need the convex conjugate  $\widetilde{U}(y) := \max_{x>0} [U(x) - xy]$ ; see [Karatzas et al. 1991]. We form the Lagrangian for solving  $\max_{\psi \in \Psi^v} \mathbb{E}^{\mathbb{Q}}[U(V_N)]$  by

$$\mathcal{L}(\lambda) = \mathbb{E}^{\mathbb{Q}}[U(V_N)] + \lambda \left[ v - \mathbb{E}^{\mathbb{Q}} \left[ \frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}} \frac{V_N}{(1+r)^N} \right] \right].$$

Now using  $\widetilde{U}$ , substituting in for  $V_N$  from the martingale method (see the Appendix), and doing algebra, we can rewrite our Lagrangian as

$$\mathcal{L}(\lambda) = \lambda v + \widetilde{\mathbb{E}} \left[ \frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}} \widetilde{U} \left( \frac{\lambda}{(1+r)^N} \frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}} \right) \right].$$

Thus, we deduce

$$\begin{aligned} u(v, v) &= \min_{\mathbb{Q} \in \mathcal{E}^v} \min_{\lambda > 0} \left[ \lambda v + \widetilde{\mathbb{E}} \left[ \frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}} \widetilde{U} \left( \frac{\lambda}{(1+r)^N} \frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}} \right) \right] \right] \\ &= \min_{\lambda > 0} \left[ \lambda v + \min_{\mathbb{Q} \in \mathcal{E}^v} \widetilde{\mathbb{E}} \left[ \frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}} \widetilde{U} \left( \frac{\lambda}{(1+r)^N} \frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}} \right) \right] \right]. \end{aligned}$$

Since the convexity of  $\widetilde{U}$  implies the function mapping  $z \mapsto z\widetilde{U}(\lambda/((1+r)^N z))$  is convex, we can use Proposition 3.1 to get

$$u(v, v) = \min_{\lambda > 0} \left[ \lambda v + \widetilde{\mathbb{E}} \left[ \frac{d\mathbb{P}^v}{d\widetilde{\mathbb{P}}} \widetilde{U} \left( \frac{\lambda}{(1+r)^N} \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^v} \right) \right] \right].$$

Taking the derivative now with respect to  $\lambda$  and setting it equal to 0, we find

$$v = \widetilde{\mathbb{E}} \left[ \frac{1}{(1+r)^N} I \left( \frac{\lambda^*(v)}{(1+r)^N} \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^v} \right) \right],$$

where  $\lambda^*(v)$  is the minimizer. Now,

$$u(v, v) = \lambda^*(v)v + \widetilde{\mathbb{E}} \left[ \frac{d\mathbb{P}^v}{d\widetilde{\mathbb{P}}} \widetilde{U} \left( \frac{\lambda^*(v)}{(1+r)^N} \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^v} \right) \right] = \mathbb{E}^v \left[ U \left( I \left( \frac{\lambda^*(v)}{(1+r)^N} \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^v} \right) \right) \right].$$

Thus, we have shown the first part of the theorem. Now note that the discounted optimal wealth process  $\{\widehat{V}_n/(1+r)^n\}_{0 \leq n \leq N}$  is a martingale under  $\widetilde{\mathbb{P}}$  (see the Appendix). As a result,

$$\widehat{V}_n = \frac{1}{(1+r)^{N-n}} \widetilde{\mathbb{E}}[\widehat{V}_N | \vec{S}_n] = \frac{1}{(1+r)^{N-n}} \sum_{\omega \in \Omega} I\left(\frac{\lambda(v)}{(1+r)^N} \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^v}(\omega)\right) \widetilde{\mathbb{P}}(\omega | \vec{S}_n)$$

for all  $n \in \{0, 1, \dots, N\}$ . Further, note that wealth is determined by your portfolio from the previous time period and the current prices. Thus,

$$\widehat{V}_{n+1} = D_{n+1} \vec{\delta}_n,$$

so we have

$$D_{n+1}^{-1} \widehat{V}_{n+1} = \vec{\delta}_n. \quad \square$$

**Remark.** We know from [Björk 2009] that the matrix of all asset prices in the complete market has rank  $M$ . Therefore, we can choose  $M$  linearly independent assets to invest in. Further, note that the optimal amount to purchase for each asset is only unique when  $M = d$ .

**Definition.** We define the *additional value of weak information* as the extra utility gained from investing with anticipation instead of just putting all of your wealth in the risk-free asset, which we define by

$$F(v, v) = u(v, v) - U(v(1+r)^N).$$

**Definition.** We also define the *ratio of added value to the total value* by

$$\pi(v, v) = \frac{F(v, v)}{u(v, v)} = 1 - \frac{U(v(1+r)^N)}{u(v, v)}.$$

As a consequence of Theorem 3.2 we obtain the following interpretation of the additional value of weak information for the log utility function.

**Corollary 3.3.** *The additional value of weak information for the log utility function is given by the relative entropy of  $v$  with respect to  $\widetilde{\mathbb{P}}_{\vec{S}_N}$ :*

$$F(v, v) = \mathbb{E}^v \left[ \ln \left( \frac{dv}{d\widetilde{\mathbb{P}}_{\vec{S}_N}} \right) \right].$$

*Proof.* We first solve for  $\lambda$ :

$$v = \widetilde{\mathbb{E}} \left[ \frac{1}{(1+r)^N} I \left( \frac{\lambda}{(1+r)^N} \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^v} \right) \right] = \widetilde{\mathbb{E}} \left[ \frac{1}{(1+r)^N} \frac{(1+r)^N}{\lambda} \frac{d\mathbb{P}^v}{d\widetilde{\mathbb{P}}} \right] \implies \lambda = \frac{1}{v}.$$



Substituting for  $\lambda$  in our value of weak information equation, we thus have

$$\begin{aligned} u(v, v) &= \mathbb{E}^v \left[ U \left( I \left( \frac{\lambda}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^v} \right) \right) \right] \\ &= \mathbb{E}^v \left[ \ln \left( \frac{(1+r)^N}{1/v} \frac{d\mathbb{P}^v}{d\tilde{\mathbb{P}}} \right) \right] = \ln(v(1+r)^N) + \mathbb{E}^v \left[ \ln \left( \frac{d\mathbb{P}^v}{d\tilde{\mathbb{P}}} \right) \right]. \end{aligned}$$

This implies the additional value of weak information for the log utility is

$$F(v, v) = \mathbb{E}^v \left[ \ln \left( \frac{d\mathbb{P}^v}{d\tilde{\mathbb{P}}} \right) \right] = \mathbb{E}^v \left[ \ln \left( \frac{dv}{d\tilde{\mathbb{P}}_{\tilde{S}_N}} \right) \right],$$

where the last equality follows from the definition of  $\mathbb{P}^v$ .  $\square$

Just like the log utility, we can also find the financial value of weak information for the power utility.

**Corollary 3.4.** *The value of weak information for the power utility function is given by*

$$u(v, v) = \frac{v^\gamma (1+r)^{N\gamma}}{\gamma (\tilde{\mathbb{E}}[(d\tilde{\mathbb{P}}/d\mathbb{P}^v)^{1/(\gamma-1)}])^{\gamma-1}} \mathbb{E}^v \left[ \left( \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^v} \right)^{\gamma/(\gamma-1)} \right].$$

*Proof.* We now will solve for the value of  $\lambda$ :

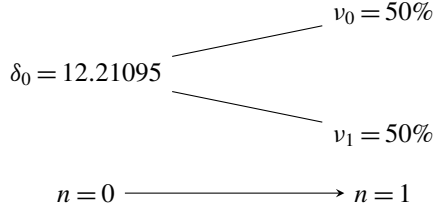
$$\tilde{\mathbb{E}} \left[ \frac{1}{(1+r)^N} \left( \frac{\lambda}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^v} \right)^{1/(\gamma-1)} \right] = v \quad \Rightarrow \quad \lambda = \left( \frac{v(1+r)^{N\gamma/(\gamma-1)}}{\tilde{\mathbb{E}}[(d\tilde{\mathbb{P}}/d\mathbb{P}^v)^{1/(\gamma-1)}]} \right)^{\gamma-1}.$$

Substituting in for  $\lambda$ , we get

$$\begin{aligned} u(v, v) &= \mathbb{E}^v \left[ U \left( I \left( \frac{\lambda}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^v} \right) \right) \right] \\ &= \mathbb{E}^v \left[ \frac{1}{\gamma} \left( \left( \frac{v(1+r)^{N\gamma/(\gamma-1)}}{\tilde{\mathbb{E}}[(d\tilde{\mathbb{P}}/d\mathbb{P}^v)^{1/(\gamma-1)}]} \right)^{\gamma-1} \frac{1}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^v} \right)^{\gamma/(\gamma-1)} \right] \\ &= \frac{v^\gamma (1+r)^{N\gamma}}{\gamma (\tilde{\mathbb{E}}[(d\tilde{\mathbb{P}}/d\mathbb{P}^v)^{1/(\gamma-1)}])^{\gamma-1}} \mathbb{E}^v \left[ \left( \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^v} \right)^{\gamma/(\gamma-1)} \right]. \quad \square \end{aligned}$$

#### 4. Complete markets: the binomial model

**Single-period binomial model.** We first will focus on a single-period binomial model with two assets: one risk-free asset with payoff  $1+r$ , and one risky asset with payoffs  $S_0(1+h)$  if the stock goes up, and  $S_0(1-k)$  if the stock goes down, where we assume  $S_0 > 0$  and  $k < 1$ . In order to have an arbitrage-free market, we



**Figure 1.** An example of a single-period binomial model using the log utility, where the parameter values are  $r = .032$ ,  $h = .09$ ,  $k = .019$ ,  $v = 200.0$ , and  $s = 20.0$ .

require  $h > r > -k$ . Since there is only one risky asset, we will denote the amount of units owned of the risky asset at time  $n$  by  $\delta_n$ .

Figure 1 shows a basic single-period binomial using the log utility. It represents the amount of stock you should buy initially,  $\delta_0$ . From here there are only two outcomes for our final time; the stock price will either go up or down.

**Example 1** (log utility). When looking at the log utility function, we begin by maximizing  $\mathbb{E}[U(V_N)]$  with respect to  $\delta$ . We then are able to obtain our equation for the optimal number of shares with respect to wealth,  $\hat{\delta}$ , in a single-period model:

$$\hat{\delta}_0 = \frac{v(1+r)(v_0(h-r) + v_1(-k-r))}{-s(h-r)(-k-r)}.$$

**Example 2** (power utility). As in the log utility case, we solve for our optimal number of shares with respect to wealth,  $\hat{\delta}_0$ , in a single-period model:

$$\hat{\delta}_0 = \frac{((v_0(h-r))^{1/(\gamma-1)} - (v_1(-k-r))^{1/(\gamma-1)})(1+r)v}{(v_1(-k-r))^{1/(\gamma-1)}s(-k-r) - (v_0(h-r))^{1/(\gamma-1)}s(h-r)}.$$

**Example 3** (exponential utility). Similarly to the previously examined utilities, we solve for the optimal number of shares with respect to wealth,  $\hat{\delta}$ , in a single-period model for the exponential utility:

$$\hat{\delta}_0 = \frac{\ln(v_0(h-r)) - \ln(-v_1(-k-r))}{s(h+k)}.$$

***N-period binomial model.*** In binomial models, everything can be explicitly computed. For instance, the following proposition gives the formula for the transition probabilities of the minimal probability  $\mathbb{P}^v$ . It is easy to establish by using the formula for conditional probabilities and straightforward combinatorial arguments. We note that  $\{S_n\}_{1 \leq n \leq N}$  is a Markov chain under the probability  $\mathbb{P}^v$ .

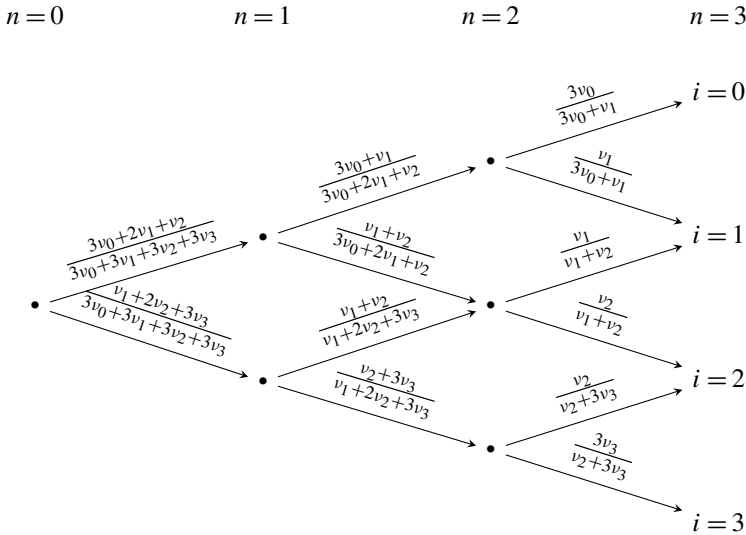
**Proposition 4.1.** Let  $l \in \{1, \dots, N-1\}$  and  $i \in \{0, \dots, N-l\}$ . Then

$$\begin{aligned} \mathbb{P}^v(S_{N-l+1} = (1+h)S_{N-l} \mid S_{N-l} = (1+h^{N-l-i})(1-k)^i S_0) \\ = \frac{\sum_{j=0}^{l-1} \binom{l-1}{j} (N-i-j) \cdots (N-i-(l-1))(i+1)(i+2) \cdots (i+j) v_{i+j}}{\sum_{j=0}^l \binom{l}{j} (N-i-j) \cdots (N-i-(l-1))(i+1)(i+2) \cdots (i+j) v_{i+j}} \end{aligned}$$

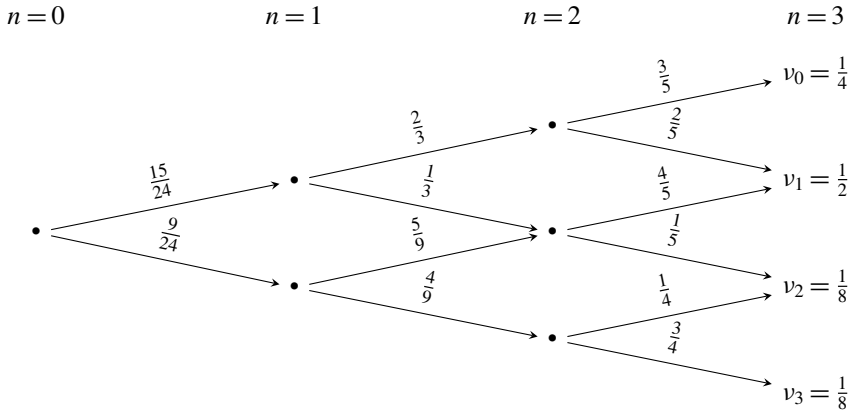
and

$$\begin{aligned} \mathbb{P}^v(S_{N-l+1} = (1-k)S_{N-l} \mid S_{N-l} = (1+h^{N-l-i})(1-k)^i S_0) \\ = \frac{\sum_{j=0}^{l-1} \binom{l-1}{j} (N-i-j-1) \cdots (N-i-(l-1))(i+1) \cdots (i+j+1) v_{i+j+1}}{\sum_{j=0}^l \binom{l}{j} (N-i-j) \cdots (N-i-(l-1))(i+1)(i+2) \cdots (i+j) v_{i+j}}. \end{aligned}$$

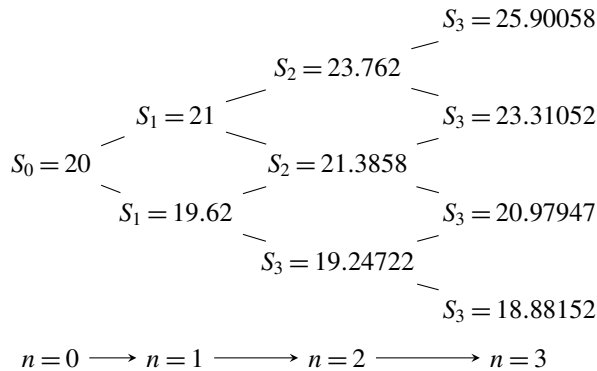
**Example 4** (log utility). Figure 5 shows an example of two different 3-period binomial trees with set values. The first tree shows the values of  $\delta$  at time  $n$  when the anticipation has a uniform distribution. The second tree, however, shows an optimistic anticipation example. One can see how the amount of stocks in which one should invest changes depending on the distribution of the anticipation. For example, one would want to buy more stocks in an optimistic model because there is a better chance of the stock increasing in price as time goes on than in the model where all of the probabilities are the same. Negative values of  $\delta$  correspond to short-selling the asset.



**Figure 2.**  $\mathbb{P}^v$  for a 3-period binomial model.



**Figure 3.**  $\mathbb{P}^\nu$  for a 3-period binomial model for a specific choice of  $\nu$ .



**Figure 4.** A 3-period binomial tree showing the values of  $S_n$ , where the parameters are  $r = .032$ ,  $h = .09$ ,  $k = .019$ ,  $v = 200.0$ , and  $s = 20.0$ .

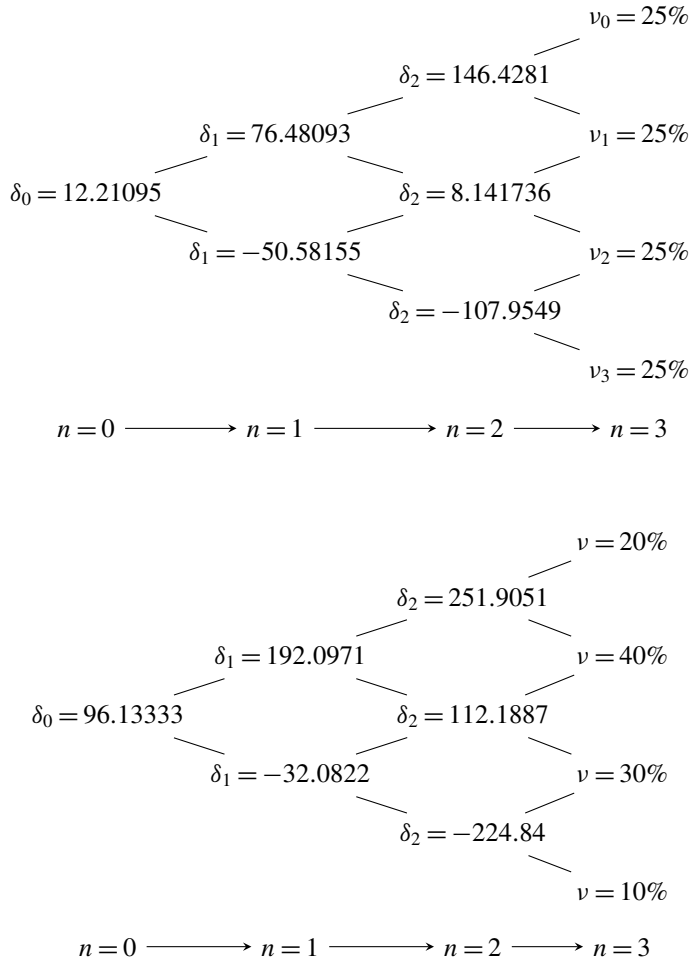
Recall from Corollary 3.3 the additional value of weak information for the log utility is

$$F(v, \nu) = \mathbb{E}^\nu \left[ \ln \left( \frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}} \right) \right],$$

and the proportion is

$$\pi(v, \nu) = \frac{\mathbb{E}^\nu [\ln(d\mathbb{P}^\nu / d\tilde{\mathbb{P}})]}{\ln(v(1+r)^N) + \mathbb{E}^\nu [\ln(d\mathbb{P}^\nu / d\tilde{\mathbb{P}})]}.$$

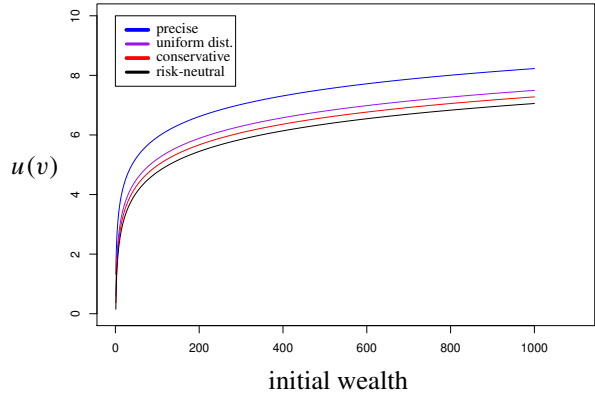
Note that  $F(v, \nu)$  is only a function of  $\nu$ , so for any fixed  $\nu$ , we have that  $F(v, \nu)$  is constant. Furthermore,  $\pi(v, \nu)$  is a decreasing function of  $v$  for any fixed  $\nu$ . As



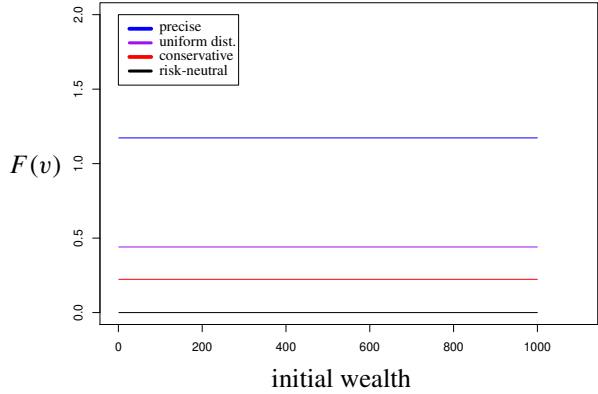
**Figure 5.** 3-period binomial trees showing the values of  $\delta$  for various anticipations of  $v$  using the log utility, where the parameters are  $r = .032$ ,  $h = .09$ ,  $k = .019$ ,  $v = 200.0$ , and  $s = 20.0$ .

a result, the wealthier you are, the less proportion of utility you are gaining as a result of anticipation. In a 5-period binomial model, with the four anticipations below, we can look at the above functions as functions of  $v$ :

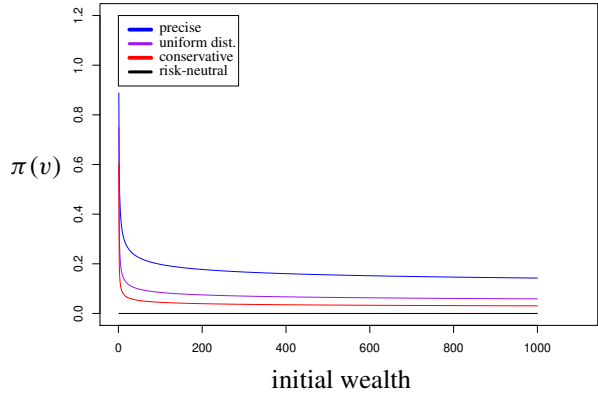
- Precise:  $\{0.01, 0.01, 0.01, 0.95, 0.01, 0.01\}$ .
- Uniform distribution:  $\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$ .
- Conservative:  $\{0.1, 0.2, 0.2, 0.2, 0.2, 0.1\}$ .
- Risk-neutral:  $v = \tilde{\mathbb{P}}$ .



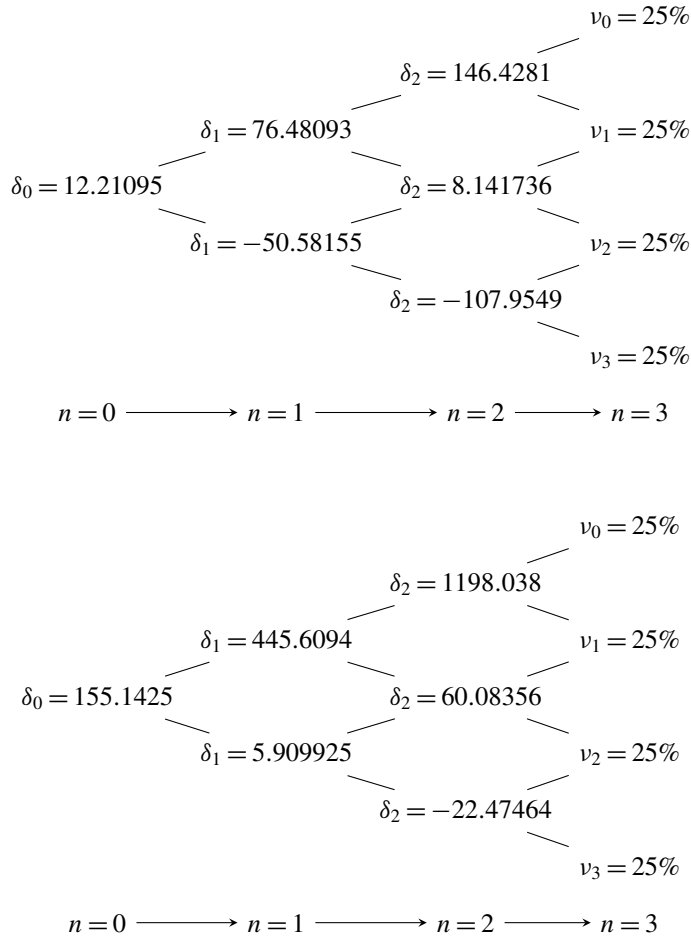
**Figure 6.** Value of weak information, given  $r = 3\%$ ,  $h = 8\%$ ,  $k = 4\%$ , using the log utility. The legend labels the curves in order, top to bottom.



**Figure 7.** Additional value of weak information, given  $r = 3\%$ ,  $h = 8\%$ ,  $k = 4\%$ , using the log utility. Legend labels curves in order.



**Figure 8.** Proportion of value added, given  $r = 3\%$ ,  $h = 8\%$ ,  $k = 4\%$ , using the log utility. Legend labels curves in order.

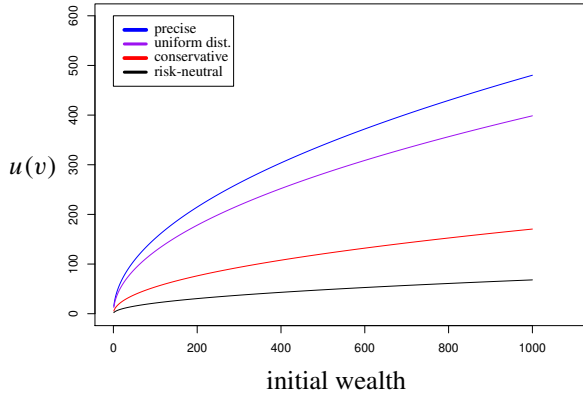


**Figure 9.** Two different 3-period binomial trees showing the values of  $\delta$  for equal anticipations of  $v$  using the log utility (top) and the power utility (bottom), where the constants are the same as Figure 5. In the power utility model,  $\gamma = .5$ .

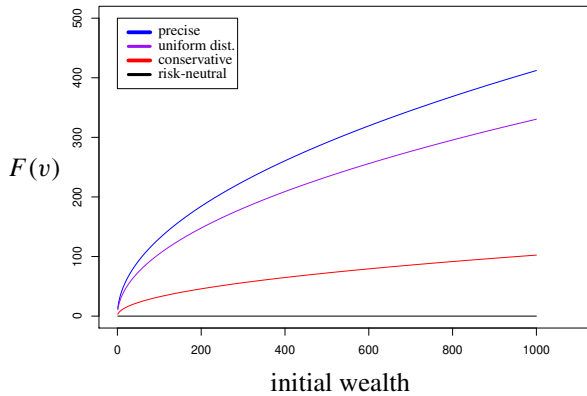
**Example 5** (power utility). Figure 9 shows the difference between the log and power utilities. As the log utility is a more relatively risk-averse utility function (for  $\gamma = 0.5$ ), the absolute value of  $\delta$  tends to be smaller when compared to the power utility function.

From Corollary 3.4 we have that the additional value for the power utility is

$$F(v, v) = \frac{v^\gamma (1+r)^{N\gamma}}{\gamma (\tilde{\mathbb{E}}[(d\tilde{\mathbb{P}}/d\mathbb{P}^v)^{1/(\gamma-1)})]^{\gamma-1}} \mathbb{E}^v \left[ \left( \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^v} \right)^{\gamma/(\gamma-1)} \right] - \frac{v^\gamma (1+r)^{N\gamma}}{\gamma},$$



**Figure 10.** Value of weak information, given  $r = 3\%$ ,  $h = 8\%$ ,  $k = 4\%$ , using the power utility. Legend labels curves in order.



**Figure 11.** Additional value of weak information, given  $r = 3\%$ ,  $h = 8\%$ ,  $k = 4\%$ , using the power utility. Legend labels curves in order.

and the proportion is

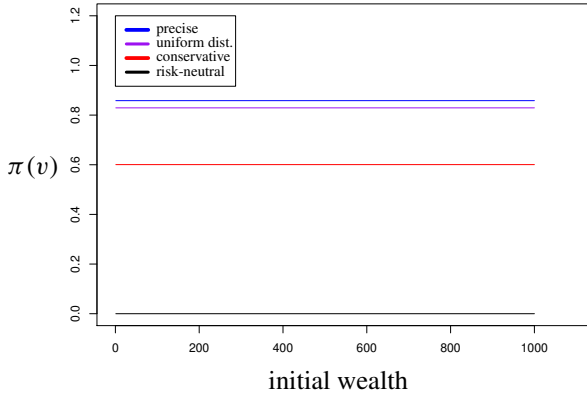
$$\pi(v, v) = 1 - \frac{1}{\mathbb{E}^v[(d\tilde{\mathbb{P}}/d\mathbb{P}^v)^{\gamma/(\gamma-1)}] \cdot (\tilde{\mathbb{E}}[(d\tilde{\mathbb{P}}/d\mathbb{P}^v)^{1/(\gamma-1)}])^{1-\gamma}}.$$

For the power utility, we have the opposite relationship for a fixed  $v$  with the proportion remaining constant and the added value being an increasing function of initial wealth.

**Example 6** (exponential utility). We can also find the financial value of weak information for exponential utility.

$$\mathbb{E}^v[-e^{-a\hat{V}_N}] = e^{-v\alpha(1+r)^N - \sum_{i=0}^N \binom{N}{i} \tilde{p}^{N-i} \tilde{q}^i \ln((\binom{N}{i} \cdot \tilde{p}^{N-i} \tilde{q}^i / v_i))}.$$





**Figure 12.** Proportion of value added, given  $r = 3\%$ ,  $h = 8\%$ ,  $k = 4\%$ , using the power utility. Legend labels curves in order.

We begin by solving for  $\lambda$ .

$$\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^v}\right)\right] = v.$$

We use this equation and then plug in for  $I$ :

$$\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} \frac{-1}{\alpha} \ln\left(\frac{\lambda}{\alpha(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^v}\right)\right] = v.$$

We then solve for  $\lambda$ :

$$\lambda = \alpha(1+r)^N e^{-v\alpha(1+r)^N - \mathbb{E}^v[\tilde{d\mathbb{P}}/d\mathbb{P}^v \ln(\tilde{d\mathbb{P}}/d\mathbb{P}^v)]}.$$

Finally we can plug our  $I$  and our  $\lambda$  into our equation for the financial value of weak information to solve for the value as it specifically relates to exponential utility:

$$\begin{aligned} u(v, v) &= \mathbb{E}^v\left[U\left(I\left(\frac{\lambda}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^v}\right)\right)\right] \\ &= \mathbb{E}^v[-e^{-a(-1/\alpha) \ln(\lambda/(\alpha(1+r)^N) \cdot (\tilde{d\mathbb{P}}/d\mathbb{P}^v))}] \\ &= e^{-v\alpha(1+r)^N - \sum_{i=0}^N \binom{N}{i} \tilde{p}^{N-i} \tilde{q}^i \ln(\binom{N}{i} \cdot \tilde{p}^{N-i} \tilde{q}^i / v_i)}. \end{aligned}$$

## Appendix

The following is with respect to the general discrete case in a complete market. As in Section 3, we denote by  $\Psi^v$  the set of self-financing portfolios given initial wealth  $v$ .

**Theorem A.1.** *The discounted wealth process is a martingale under the martingale measure  $\mathbb{Q}$ .*

*Proof.* See [Runggaldier 2005]. □

**Theorem A.2.** *Maximizing  $\mathbb{E}[U(V_N)]$  over the set of self-financing portfolios  $\Psi^v$  is equivalent to maximizing  $\mathbb{E}[U(V_N)]$  subject to  $\tilde{\mathbb{E}}[U(V_N)] = v$ , with  $\tilde{\mathbb{P}}$  being the unique equivalent martingale measure.*

*Proof.* See [Rásonyi and Stettner 2005, Lemma 4.9]. □

**Theorem A.3.**

$$\widehat{V}_N = I\left(\frac{\lambda}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\right).$$

*More specifically, optimal terminal wealth  $\widehat{V}_N$  is attained when  $\lambda$  satisfies*

$$v = \tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\right)\right].$$

*Proof.* See [Runggaldier 2005, p. 16]. □

### Acknowledgments

This research was funded by the NSF grant DMS 1659643. The authors would like to thank Oleksii Mostovyi for several instructive discussions and comments on this work.

### References

- [Baudoin 2003] F. Baudoin, “Modeling anticipations on financial markets”, pp. 43–94 in *Paris-Princeton Lectures on Mathematical Finance, 2002*, edited by R. A. Carmona et al., Lecture Notes in Math. **1814**, Springer, 2003. MR Zbl
- [Baudoin and Nguyen-Ngoc 2004] F. Baudoin and L. Nguyen-Ngoc, “The financial value of a weak information on a financial market”, *Finance Stoch.* **8**:3 (2004), 415–435. MR Zbl
- [Björk 2009] T. Björk, *Arbitrage theory in continuous time*, 3rd ed., Oxford University Press, 2009.
- [Gilboa and Schmeidler 1989] I. Gilboa and D. Schmeidler, “Maxmin expected utility with nonunique prior”, *J. Math. Econom.* **18**:2 (1989), 141–153. MR Zbl
- [Karatzas et al. 1991] I. Karatzas, J. P. Lehoczky, S. E. Shreve, and G.-L. Xu, “Martingale and duality methods for utility maximization in an incomplete market”, *SIAM J. Control Optim.* **29**:3 (1991), 702–730. MR Zbl
- [Meyer and Meyer 2005] D. J. Meyer and J. Meyer, “Relative risk aversion: what do we know?”, *J. Risk Uncertainty* **31**:3 (2005), 243–262. Zbl
- [Rásonyi and Stettner 2005] M. Rásonyi and L. Stettner, “On utility maximization in discrete-time financial market models”, *Ann. Appl. Probab.* **15**:2 (2005), 1367–1395. MR Zbl
- [Runggaldier 2005] W. Runggaldier, “Portfolio optimization in discrete time”, preprint, 2005, available at [https://www.math.unipd.it/runggaldier/MPS\\_ru.pdf](https://www.math.unipd.it/runggaldier/MPS_ru.pdf).
- [Shreve 2004] S. E. Shreve, *Stochastic calculus for finance, I: The binomial asset pricing model*, Springer, 2004. MR Zbl

Received: 2018-11-08 Accepted: 2019-01-26

aamiran@umass.edu	<i>University of Massachusetts, Amherst, MA, United States</i>
fabrice.baudoin@uconn.edu	<i>University of Connecticut, Storrs, CT, United States</i>
snbrock@mavs.coloradomesa.edu	<i>Colorado Mesa University, Grand Junction, CO, United States</i>
berend.coster@uconn.edu	<i>University of Connecticut, Storrs, CT, United States</i>
rcraver@terpmail.umd.edu	<i>University of Maryland, College Park, MD, United States</i>
uezeaka@umass.edu	<i>University of Massachusetts, Amherst, MA, United States</i>
phanu9000@sbcglobal.net	<i>Purdue University, West Lafayette, IN, United States</i>
wishartmar@my.easternct.edu	<i>Eastern Connecticut State University, Willimantic, CT, United States</i>



# involve

msp.org/involve

## INVOLVE YOUR STUDENTS IN RESEARCH

*Involve* showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

## MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

## BOARD OF EDITORS

Colin Adams	Williams College, USA	Chi-Kwong Li	College of William and Mary, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Robert B. Lund	Clemson University, USA
Martin Bohner	Missouri U of Science and Technology, USA	Gaven J. Martin	Massey University, New Zealand
Nigel Boston	University of Wisconsin, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of N Carolina, Chapel Hill, USA	Frank Morgan	Williams College, USA
Pietro Cerone	La Trobe University, Australia	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Scott Chapman	Sam Houston State University, USA	Zuhair Nashed	University of Central Florida, USA
Joshua N. Cooper	University of South Carolina, USA	Ken Ono	Emory University, USA
Jem N. Corcoran	University of Colorado, USA	Yuval Peres	Microsoft Research, USA
Toka Diagana	Howard University, USA	Y.-F. S. Pétermann	Université de Genève, Switzerland
Michael Dorff	Brigham Young University, USA	Jonathon Peterson	Purdue University, USA
Sever S. Dragomir	Victoria University, Australia	Robert J. Plemmons	Wake Forest University, USA
Joel Foisy	SUNY Potsdam, USA	Carl B. Pomerance	Dartmouth College, USA
Errin W. Fulp	Wake Forest University, USA	Vadim Ponomarenko	San Diego State University, USA
Joseph Gallian	University of Minnesota Duluth, USA	Bjorn Poonen	UC Berkeley, USA
Stephan R. Garcia	Pomona College, USA	József H. Przytycki	George Washington University, USA
Anant Godbole	East Tennessee State University, USA	Richard Rebarber	University of Nebraska, USA
Ron Gould	Emory University, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Javier Rojo	Oregon State University, USA
Jim Haglund	University of Pennsylvania, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Johnny Henderson	Baylor University, USA	Hari Mohan Srivastava	University of Victoria, Canada
Glenn H. Hurlbert	Arizona State University, USA	Andrew J. Sterge	Honorary Editor
Charles R. Johnson	College of William and Mary, USA	Ann Trenk	Wellesley College, USA
K. B. Kulasekera	Clemson University, USA	Ravi Vakil	Stanford University, USA
Gerry Ladas	University of Rhode Island, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
David Larson	Texas A&M University, USA	John C. Wierman	Johns Hopkins University, USA
Suzanne Lenhart	University of Tennessee, USA	Michael E. Zieve	University of Michigan, USA

## PRODUCTION

Silvio Levy, Scientific Editor


Cover: Alex Scorpan

See inside back cover or [msp.org/involve](http://msp.org/involve) for submission instructions. The subscription price for 2019 is US \$195/year for the electronic version, and \$260/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1444-4184 electronic, 1444-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2019 Mathematical Sciences Publishers

# involve

2019      vol. 12      no. 5

Orbigraphs: a graph-theoretic analog to Riemannian orbifolds	721
KATHLEEN DALY, COLIN GAVIN, GABRIEL MONTES DE OCA, DIANA OCHOA, ELIZABETH STANHOPE AND SAM STEWART	
Sparse neural codes and convexity	737
R. AMZI JEFFS, MOHAMED OMAR, NATCHANON SUAYSOM, ALEINA WACHTEL AND NORA YOUNGS	
The number of rational points of hyperelliptic curves over subsets of finite fields	755
KRISTINA NELSON, JÓZSEF SOLYMOSI, FOSTER TOM AND CHING WONG	
Space-efficient knot mosaics for prime knots with mosaic number 6	767
AARON HEAP AND DOUGLAS KNOWLES	
Shabat polynomials and monodromy groups of trees uniquely determined by ramification type	791
NAIOMI CAMERON, MARY KEMP, SUSAN MASLAK, GABRIELLE MELAMED, RICHARD A. MOY, JONATHAN PHAM AND AUSTIN WEI	
On some edge Folkman numbers, small and large	813
JENNY M. KAUFMANN, HENRY J. WICKUS AND STANISŁAW P. RADZISZOWSKI	
Weighted persistent homology	823
GREGORY BELL, AUSTIN LAWSON, JOSHUA MARTIN, JAMES RUDZINSKI AND CLIFFORD SMYTH	
Leibniz algebras with low-dimensional maximal Lie quotients	839
WILLIAM J. COOK, JOHN HALL, VICKY W. KLIMA AND CARTER MURRAY	
Spectra of Kohn Laplacians on spheres	855
JOHN AHN, MOHIT BANSIL, GARRETT BROWN, EMILEE CARDIN AND YUNUS E. ZEYTUNCU	
Pairwise compatibility graphs: complete characterization for wheels	871
MATTHEW BEAUDOUIN-LAFON, SERENA CHEN, NATHANIEL KARST, DENISE SAKAI TROXELL AND XUDONG ZHENG	
The financial value of knowing the distribution of stock prices in discrete market models	883
AYELET AMIRAN, FABRICE BAUDOIN, SKYLYN BROCK, BEREND COSTER, RYAN CRAVER, UGONNA EZEAKA, PHANUEL MARIANO AND MARY WISHART	



1944-4176(2019)12:5;1-0