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Occurrence graphs of patterns in permutations

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We define the occurrence graph $G_p(\pi)$ of a pattern p in a permutation π as the graph whose vertices are the occurrences of p in π, with edges between the vertices if the occurrences differ by exactly one element. We then study properties of these graphs. The main theorem in this paper is that every hereditary property of graphs gives rise to a permutation class.

1. Introduction

The research area of permutation patterns can be traced back to [MacMahon 1915, Section III, Chapter V] where it is shown that permutations without an increasing subsequence of length 3 (avoiding 123 in the language introduced below) are counted by the Catalan numbers. Another famous result is the Erdős–Szekeres theorem [1935] which says that a permutation of length $(n-1)(m-1)+1$ has an increasing subsequence of length n (the pattern $12\cdots n$) or a decreasing subsequence of length m (the pattern $m\cdots 21$). The field came into its own when Knuth [1968] showed that "stack-sortable" permutations are the 231-avoiding permutations and are enumerated by the Catalan numbers. Since then dozens of papers have been written about enumerations of permutations avoiding patterns, their structure, and connections to other objects in mathematics. See [Kitaev 2011] for an overview. The goal of this paper is to connect the study of permutation patterns with properties of graphs.

We define the *occurrence graph* $G_p(\pi)$ of a pattern p in a permutation π as the graph where each vertex represents an occurrence of p in π. Vertices share an edge if the occurrences they represent differ by exactly one element. We study properties of these graphs and show that every *hereditary property* of graphs gives rise to a *permutation class*, which we define below.

The motivation for defining these graphs comes from the algorithm discussed in the proof of the simultaneous shading lemma by Claesson, Tenner and Ulfarsson

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Keywords: graph, permutation, subgraph, pattern.

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[Claesson et al. 2015, Lemma 7.6]. The steps in that algorithm can be thought of as constructing a path in an occurrence graph, terminating at a desirable occurrence of a pattern.

2. Basic definitions

We begin by reviewing some standard definitions.

Definition 2.1. A *graph* is an ordered pair $G = (V, E)$, where V is a set of *vertices* and E is a set of two-element subsets of V. The elements $\{u, v\} \in E$ are called *edges* and connect the vertices. Two vertices u and v are *neighbors* if $\{u, v\} \in E$. The *degree* of a vertex v is the number of neighbors it has. A graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq \{\{u, v\} \in E : u, v \in V'\}$.

The reader might have noticed that our definition of a graph excludes those with loops and multiple edges between vertices. We often write uv as shorthand for $\{u, v\}$ and in case of ambiguity we use $V(G)$ and $E(G)$ instead of V and E.

Definition 2.2. Two graphs G and H are *isomorphic* if there exists a bijection from $V(G)$ to $V(H)$ such that two vertices in G are neighbors if and only if the corresponding vertices (according to the bijection) in H are neighbors. We denote this by $G \cong H$.

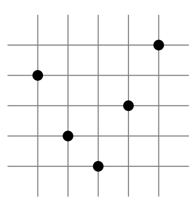
We let $[\![1, n]\!]$ denote the integer interval $\{1, \dots, n\}$.

Definition 2.3. A *permutation of length n* is a bijective function $\sigma : [\![1, n]\!] \to [\![1, n]\!]$. We denote the permutation by $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$. The permutation $\mathrm{id}_n = 12\cdots n$ is the *identity permutation* of length n.

The set of permutations of length n is denoted by $\mathfrak{S}_n$. The set of all permutations is $\mathfrak{S} = \bigcup_{n=0}^{+\infty} \mathfrak{S}_n$. Note that $\mathfrak{S}_0 = \{\mathscr{E}\}$, where $\mathscr{E}$ is the empty permutation, and $\mathfrak{S}_1 = \{1\}$. There are $n!$ permutations of length n.

Definition 2.4. A *grid plot* or *grid representation* of a permutation $\pi \in \mathfrak{S}_n$ is the subset $\mathrm{Grid}(\pi) = \{(i, \pi(i)) : i \in [\![1, n]\!]\}$ of the Cartesian product $[\![1, n]\!]^2 = [\![1, n]\!] \times [\![1, n]\!]$.

Example 2.5. Let $\pi = 42135$. The grid representation of π is

Grid(42135) =  .

The central definition in the theory of permutation patterns is how permutations lie inside other (larger) permutations. Before we define that precisely we need a preliminary definition:

Definition 2.6. Let $a_1, \dots, a_k$ be distinct integers. The *standardization* of the string $a_1 \cdots a_k$ is the permutation $\sigma \in \mathfrak{S}_k$ such that $a_1 \cdots a_k$ is order isomorphic to $\sigma(1) \cdots \sigma(k)$. In other words, for every $i \neq j$ we have $a_i < a_j$ if and only if $\sigma(i) < \sigma(j)$. We denote this by $\mathrm{st}(a_1 \cdots a_k) = \sigma$.

For example st(253) = 132 and st(132) = 132.

Definition 2.7. Let p be a permutation of length k. We say that the permutation $\pi \in \mathfrak{S}_n$ *contains* p if there exist indices $1 \leq i_1 < \cdots < i_k \leq n$ such that $\mathrm{st}(\pi(i_1) \cdots \pi(i_k)) = p$. The subsequence $\pi(i_1) \cdots \pi(i_k)$ is an *occurrence* of p in π with the *index set* $\{i_1, \dots, i_k\}$. The increasing sequence $i_1 \cdots i_k$ will be used to denote the order-preserving injection $i : [\![1, k]\!] \to [\![1, n]\!]$, $j \mapsto i_j$, which we call the *index injection* of p into π for this particular occurrence.

The set of all index sets of p in π is the *occurrence set* of p in π, denoted by $V_p(\pi)$. If π does not contain p, then π *avoids* p. In this context the permutation p is called a (*classical permutation*) *pattern*.

Unless otherwise stated, we write the index set $\{i_1, \dots, i_n\}$ in ordered form, i.e., such that $i_1 < \cdots < i_n$, in accordance with how we write the index injection.

The set of all permutations that avoid p is $\mathrm{Av}(p)$. More generally for a set of patterns M we define

$$\mathrm{Av}(M) = \bigcap_{p \in M} \mathrm{Av}(p).$$

Example 2.8. The permutation 42135 contains five occurrences of the pattern 213, namely 425, 415, 435, 213 and 215. The occurrence set is

$$V_{213}(42135) = \{\{1, 2, 5\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\}.$$

The permutation 42135 avoids the pattern 132.

Sets of permutations that can be defined by the avoidance of patterns are given a special name:

Definition 2.9. A set of permutations $\mathscr{C}$ that is *closed downwards*, i.e., if $\pi \in \mathscr{C}$ then $p \in \mathscr{C}$ for every pattern p in π, is called a *permutation class*. A permutation class can be written as $\mathrm{Av}(M)$, where M is a set of classical permutation patterns. If M is minimal, then it is called the *basis* of the class.

3. Occurrence graphs

We now formally define occurrence graphs.

Definition 3.1. For a pattern p of length k and a permutation π we define the *occurrence graph* $G_p(\pi)$ of p in π as follows:

- The set of vertices is $V_p(\pi)$, the occurrence set of p in π.

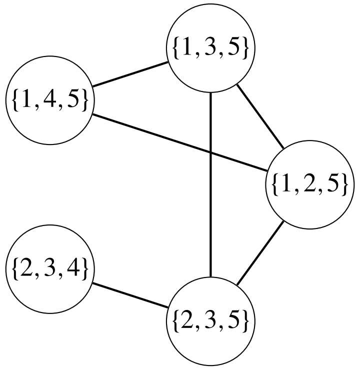


Figure 1. The occurrence graph $G_{213}(42135)$.

- uv is an edge in $G_p(\pi)$ if the vertices $u = \{u_1, \ldots, u_k\}$ and $v = \{v_1, \ldots, v_k\}$ in $V_p(\pi)$ differ by exactly one element, i.e., if

$$|u \setminus v| = 1 = |v \setminus u|.$$

Example 3.2. In Example 2.8 we derived the occurrence set $V_{213}(42135)$. We compute the edges of $G_{213}(42135)$ by comparing the vertices two at a time to see if the sets differ by exactly one element. The graph is shown in Figure 1.

Remark 3.3. For a permutation π of length n the graph $G_{\mathscr{E}}(\pi)$ is a graph with one vertex and no edges and $G_1(\pi)$ is a clique on n vertices.

Following the definition of these graphs there are several natural questions that arise. For example, for a fixed pattern p, which occurrence graphs $G_p(\pi)$ satisfy a given graph property, such as being connected or being a tree? Before we answer questions of this sort we consider a simpler question: what can be said about the graph $G_{12}(\mathrm{id}_n)$?

4. The pattern $p = 12$ and the identity permutation

In this section we only consider the pattern $p = 12$ and let $n \geq 2$. For this choice of p and a fixed n the identity permutation $\pi = 1 \cdots n$ contains the most occurrences of p. Indeed, every set $\{i, j\}$ with $i \neq j$ is an index set of p in π. We can choose this pair in

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

different ways. Therefore, this is the size of the vertex set of $G = G_p(\pi)$.

Every vertex $u = \{i, j\}$ in G is connected to $n-2$ vertices $v = \{i, j'\}$, $j' \neq j$, and $n-2$ vertices $w = \{i', j\}$, $i' \neq i$. Thus, the degree of every vertex in G is $2(n-2)$. By

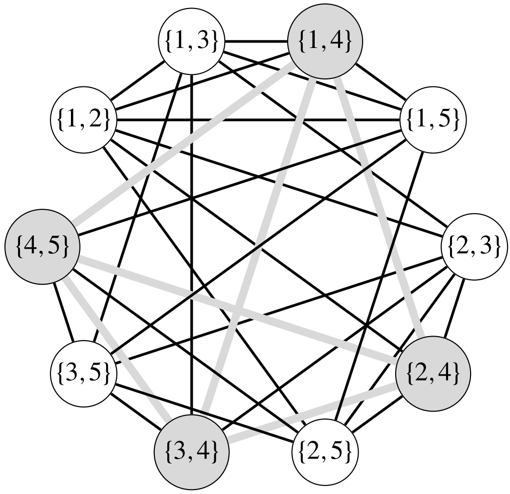


Figure 2. The graph $G_{12}(12345)$.

summing over the set of vertices and dividing by 2 we get the number of edges in G:

$$|E(G)| = \frac{n(n-1)(n-2)}{2} = 3\binom{n}{3}.$$

A triangle in G consists of three vertices u, v, w with edges uv, vw, wu. If $u = \{i, j\}$ (not necessarily in ordered form) then we can assume v is $\{j, k\}$. For this triplet to be a triangle w must connect to both u and v, and therefore w must either be the index set $\{i, k\}$ or $\{j, j'\}$, where $j' \neq i, k$. In the first case, we just need to choose three indices i, j, k. In the second case we start by choosing the common index k and then we choose the remaining indices. Thus the number of triangles in G is

$$\binom{n}{3} + n\binom{n-1}{3} = (n-2)\binom{n}{3}.$$

Example 4.1. The graph $G_{12}(12345)$ is pictured in Figure 2. It has 10 vertices, 30 edges, and 30 triangles. It also has 5 subgraphs isomorphic to K_4, one of them highlighted with thicker gray edges and gray vertices.

The following proposition generalizes the observations above to larger cliques.

Proposition 4.2. *For $n > 0$, the number of cliques of size $k > 3$ in $G_{12}(\mathrm{id}_n)$ is*

$$(k+1)\binom{n}{k+1} = n\binom{n-1}{k}.$$

Proof. The vertices $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_k, b_k\}$ in a clique of size $k > 3$ must have a common index, say $\ell = a_1 = a_2 = \cdots = a_k$, without loss of generality. The remaining indices $b_1, b_2, \ldots, b_k$ can be chosen as any subset of the other $n-1$ indices. This explains the right-hand side of the equation in the proposition. □

5. Hereditary properties of graphs

Intuitively one might think that if a pattern p is contained inside a larger pattern q, then one of the occurrence graphs $G_p(\pi)$ and $G_q(\pi)$ (for any permutation π) would be contained inside the other. But this is not the case as the following examples demonstrate.

Example 5.1. (1) Let $p = 12$, $q = 231$ and $\pi = 3421$. The occurrence sets are $V_p(\pi) = \{\{1, 2\}\}$ and $V_q(\pi) = \{\{1, 2, 3\}, \{1, 2, 4\}\}$. The cardinality of the set $V_p(\pi)$ is smaller than the cardinality of $V_q(\pi)$.

(2) If on the other hand $p = 12$, $q = 123$ and $\pi = 123$ then the occurrence sets are $V_p(\pi) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and $V_q(\pi) = \{\{1, 2, 3\}\}$. Thus, in this case, the cardinality of $V_p(\pi)$ is larger than the cardinality of $V_q(\pi)$.

However, for a fixed pattern p, we obtain an inclusion of one occurrence graph in another, in Proposition 5.4. First we need a lemma.

Lemma 5.2. *Let p be a pattern and π, σ be two permutations. For an occurrence of π in σ the index injection induces an injection $\Phi_p : V_p(\pi) \to V_p(\sigma)$.*

Proof. Let p, π, σ be permutations of lengths l, m, n respectively. Every $v = \{i_1, \dots, i_l\} \in V_p(\pi)$ is an index set of p in π with index injection i. Let j be an index injection for an index set $\{j_1, \dots, j_m\}$ of π in σ. It is easy to see that $u = \{j_{i_1}, \dots, j_{i_l}\}$ is an index set of p in σ because $j \circ i$ is an index injection of p into σ. Define $\Phi_p(v) = u$. □

Example 5.3. Let $p = 12$, $\pi = 132$ and $\sigma = 24153$. There are three occurrences of π in σ: 243, 253 and 153 with respective index injections 125, 145 and 345.

For a given index injection, say $i = 345$, we obtain the injection Φ_p by mapping every $\{v_1, v_2\} \in V_p(\pi)$ to $\{i_{v_1}, i_{v_2}\} \in V_p(\sigma)$. We calculate that Φ_p maps $\{1, 2\}$ to $\{i_1, i_2\} = \{3, 4\}$ and $\{1, 3\}$ to $\{i_1, i_3\} = \{3, 5\}$; see Figure 3.

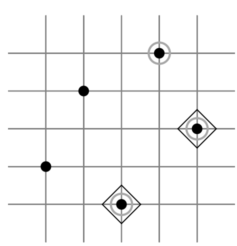


Figure 3. The occurrence of π in σ that is defined by the index injection $i = 345$ is highlighted with gray circles. The occurrence set $\{1, 3\}$ of p in π is mapped with the injection Φ_p, induced by i, to the index set $\{3, 5\}$ of p in σ, highlighted with black diamonds.

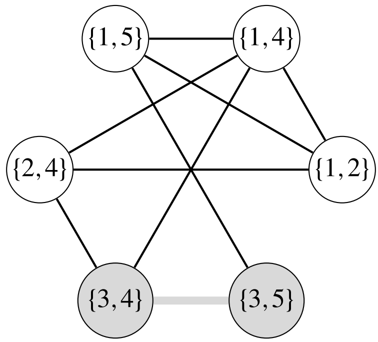


Figure 4. The graph $G_{12}(24153)$ with a highlighted subgraph isomorphic to $G_{12}(132)$.

Proposition 5.4. *Let p be a pattern and π, σ be two permutations. For an occurrence of π in σ the index injection induces an isomorphism of the occurrence graph $G_p(\pi)$ with a subgraph of $G_p(\sigma)$.*

Proof. From Lemma 5.2 we have the injection $\Phi_p : V_p(\pi) \to V_p(\sigma)$. We need to show for every $uv \in E(G_p(\pi))$ that $\Phi_p(u)\Phi_p(v) \in E(G_p(\sigma))$. Let uv be an edge in $G_p(\pi)$, where $u = \{u_1, \ldots, u_l\}$ and $v = \{v_1, \ldots, v_l\}$. For every index injection j of π into σ, the vertices u, v map to $\Phi_p(u) = \{j(u_1), \ldots, j(u_l)\}$, $\Phi_p(v) = \{j(v_1), \ldots, j(v_l)\}$ respectively. Since j is an injection, there exists an edge between these two vertices in $G_p(\sigma)$. □

Example 5.5. We will continue with Example 5.3 and show how the index injection $i = 345$ defines a subgraph of $G_p(\sigma)$ which is isomorphic to $G_p(\pi)$. The occurrence graph of p in π is a graph on two vertices $\{1, 2\}$ and $\{1, 3\}$ with an edge between them. The occurrence graph $G_p(\sigma)$ with the highlighted subgraph induced by i is shown in Figure 4.

The next example shows that different occurrences of π in σ do not necessarily lead to different subgraphs of $G_p(\sigma)$.

Example 5.6. If $p = 12$, $\pi = 312$ and $\sigma = 3412$ there are two occurrences of π in σ. The index injections are $i = 134$ and $i' = 234$. However, as $(i_2, i_3) = (i'_2, i'_3)$ and $\{2, 3\}$ is the only index set of p in π, we obtain the same injection Φ_p and therefore the same subgraph of $G_p(\sigma)$ for both index injections.

We call a property of a graph G *hereditary* if it is invariant under isomorphisms and for every subgraph of G the property also holds. For example the properties of being a forest, bipartite, planar or k-colorable are hereditary properties, while being a tree is not hereditary. A set of graphs defined by a hereditary property is a *hereditary class*.

p	basis	numerical sequence	OEIS
12	123, 1432, 3214	1, 2, 5, 12, 26, 58, 131, 295	A116716
123	1234, 12543, 14325, 32145	1, 2, 6, 23, 100, 462, 2207, 10758	
132	1432, 12354, 13254, 13452, 15234, 21354, 23154, 31254, 32154	1, 2, 6, 23, 95, 394, 1679, 7358	

Table 1. Experimental results for bipartite occurrence graphs, computed with permutations up to length 8.

Given c, a property of graphs, we define a set of permutations:

$$\mathscr{G}_{p,c} = \{\pi \in \mathfrak{S} : G_p(\pi) \text{ has property } c\}.$$

We can now state the main theorem of the paper.

Theorem 5.7. *Let c be a hereditary property of graphs. For any pattern p the set $\mathscr{G}_{p,c}$ is a permutation class*; *i.e.*, *there exists a set of classical patterns M such that*

$$\mathscr{G}_{p,c} = \operatorname{Av}(M).$$

Proof. Let σ be a permutation such that $G_p(\sigma)$ satisfies the hereditary property c and let π be a pattern in σ. By Proposition 5.4 the graph $G_p(\pi)$ is isomorphic to a subgraph of $G_p(\sigma)$ and thus inherits the property c. □

In the remainder of this section we consider two hereditary classes of graphs: bipartite graphs and forests. Recall that a nonempty simple graph on n vertices ($n > 0$) is a *tree* if and only if it is connected and has $n-1$ edges. An equivalent condition is that the graph has at least one vertex and no simple cycles (a sequence of unique vertices $v_1, \ldots, v_k$ with edges $v_1v_2, \ldots, v_{k-1}v_k, v_kv_1$). A *forest* is a disjoint union of trees. The empty graph is a forest but not a tree. *Bipartite* graphs are graphs that can be colored with two colors in such a way that no edge joins two vertices with the same color. We note that every forest is a bipartite graph.

Table 1 shows experimental results, obtained using software developed by Magnusson and the second author [Magnusson and Ulfarsson 2012], on which occurrence graphs with respect to the patterns $p = 12$, $p = 123$, $p = 132$ are bipartite. Permutations and patterns have the eight symmetries of the square, as can be seen from their grid representation. We only consider one representative from each symmetry class.

In the following theorem we verify the statements in line 1 of Table 1. We leave the remainder of the table as conjectures.

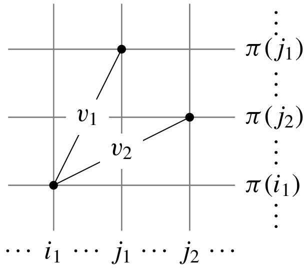


Figure 5. The vertices v_1 and v_2 (shown as line segments inside the permutation π) share the index i_1.

Theorem 5.8. *Let c be the property of being bipartite. Then*

$$\mathscr{G}_{12,c} = \text{Av}(123, 1432, 3214).$$

The OEIS sequence A116716 *enumerates a symmetry of this permutation class.*

The proof of this theorem relies on a proposition characterizing the cycles in the graphs under consideration.

Proposition 5.9. *If the graph* $G_{12}(\pi)$ *has a cycle of length* $k > 4$ *then it also has a cycle of length* 3.

Proof. Let π be a permutation such that $G_{12}(\pi)$ contains a cycle of length $k > 4$. Label the vertices in the cycle $v_1, \dots, v_k$ with $v_l = \{i_l, j_l\}$, $i_l < j_l$, for $l = 1, \dots, k$.

The vertices v_1 and v_2 in the cycle have exactly one index in common. If $i_2 = j_1$ then the vertices $v_1, v_2, \{i_1, j_2\}$ form a triangle. So we can assume $i_1 = i_2$. If $j_1 < j_2$ and $\pi(j_1) < \pi(j_2)$ (or $j_1 > j_2$ and $\pi(j_1) > \pi(j_2)$) then $u = \{j_1, j_2\}$ is an occurrence of 12 in π, forming a triangle v_1, v_2, u. So either $j_1 > j_2$ and $\pi(j_1) < \pi(j_2)$ holds, or, without loss of generality (see Figure 5), $j_1 < j_2$ and $\pi(j_1) > \pi(j_2)$.

Next we look at the edge v_2v_3 in the cycle. If the vertices have the index i_1 in common then v_1, v_2, v_3 form a triangle in $G_{12}(\pi)$. So assume that v_2 and v_3 have the index j_2 in common with the conditions $i_3 > i_1$ and $\pi(i_3) < \pi(i_1)$ (because else there are more vertices and edges forming a cycle of length 3 in $G_{12}(\pi)$). Continuing down this road we know that v_3v_4 is an edge with shared index i_3 and conditions $j_3 > i_3$ and $\pi(j_3) < \pi(j_1)$; see Figure 6, where we consider the case $i_3 > j_1$, and $\pi(j_3) < \pi(i_1)$.

Graphically, it is quite obvious that we cannot extend the path in Figure 6 with more southwest-northeast line segments (a sequence of vertices $v_5, \dots, v_k$) such that the extension closes the path into a cycle without adding more edges (line segments) between vertices that are not adjacent in the cycle and thus forming a cycle of length 3 in the occurrence graph. More precisely, for an edge between v_k and v_1 to exist we must have $v_k = \{i_k, j_k\}$ with a nonempty intersection with v_1. Analyzing each of these cases completes the proof. □

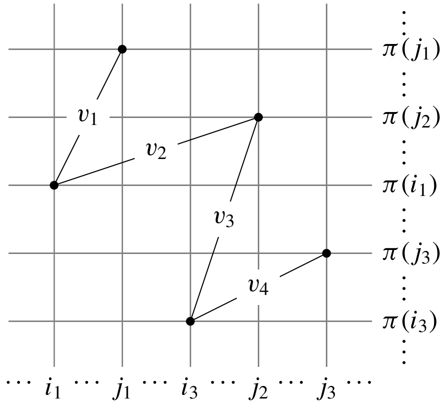


Figure 6. The vertices v_1, v_2, v_3, v_4.

Proof of Theorem 5.8. If π contains any of the patterns 123, 1432, 3214 then $G_p(\pi)$ contains a subgraph that is isomorphic to a triangle. So if $\pi \notin \mathrm{Av}(123, 1432, 3214)$ then $G_{12}(\pi)$ contains an odd cycle and is therefore not bipartite.

On the other hand, let π be a permutation such that $G_{12}(\pi)$ is not bipartite. Then the occurrence graph contains an odd cycle which by Proposition 5.9 implies the graph has a cycle of length 3. The indices corresponding to this cycle form a pattern of length 3 or 4 in π with occurrence graph that is a cycle of length 3. It is easy to see that the only permutations of this length with occurrence graph a cycle of length 3 are 123, 1432 and 3214. Therefore π must contain at least one of the patterns. □

Table 2 considers occurrence graphs that are forests.

p	basis	numerical sequence	OEIS
12	123, 1432, 2143, 3214	1, 2, 5, 11, 24, 53, 117, 258	A052980
123	1234, 12543, 13254, 14325, 21354, 21435 32145	1, 2, 6, 23, 97, 429, 1947, 8959	
132	1432, 12354, 12453, 12534, 13254, 13452, 14523, 15234, 21354, 21453, 21534, 23154, 31254, 32154	1, 2, 6, 23, 90, 359, 1481, 6260	

Table 2. Experimental results for occurrence graphs that are forests, computed with permutations up to length 8.

In the following theorem we verify the statements in line 1 of Table 2. We leave the remainder of the table as conjectures.

Theorem 5.10. *Let c be the property of being a forest. Then*

$$\mathscr{G}_{12,c} = \mathrm{Av}(123, 1432, 2143, 3214).$$

Proof. If π contains the pattern 2143 then $G_{12}(\pi)$ contains a subgraph that is isomorphic to a cycle of length 4, according to Proposition 5.4, because $G_{12}(2143)$ is a cycle of length 4. If π contains any of the patterns 123, 1432, 3214 then its occurrence graph is not bipartite by Theorem 5.8, and in particular is not a forest.

On the other hand, let π be a permutation such that $G_{12}(\pi)$ is not a forest. Then the occurrence graph contains a cycle. Proposition 5.9 implies that the cycle must have length either 3 or 4. But it is easy to see that the only permutations with occurrence graphs that are cycles of length 3 or 4 are 123, 1432, 2143, 3214. Therefore π must contain at least one of the patterns. □

6. Nonhereditary properties of graphs

This section is devoted to graph properties that are not hereditary. Thus Theorem 5.7 does not guarantee the permutations whose occurrence graphs satisfy the property form a pattern class. Experimental results in Table 3 seem to suggest that some properties still give rise to permutation classes.

To describe permutations π such that $G_{12}(\pi)$ is connected, we need the language of mesh patterns, which we briefly review here. A *mesh pattern* is a pair (p, s) where p is a classical pattern and the *mesh* s is a subset of $[\![0, |p|]\!] \times [\![0, |p|]\!]$. The

property	basis	numerical sequence	OEIS
connected	see Figure 7	1, 2, 6, 23, 111, 660, 4656, 37745	
chordal	1234, 1243, 1324, 2134, 2143	1, 2, 6, 19, 61, 196, 630, 2025	
clique	1234, 1243, 1324, 1342, 1423, 2134, 2143, 2314, 2413, 3124, 3142, 3412	1, 2, 6, 12, 20, 30, 42, 56	A002378 from n=2
tree	very large nonclassical basis	0, 1, 4, 9, 16, 25, 36, 49	A000290

Table 3. Experimental results for the pattern $p = 12$, computed with permutations up to length 8.

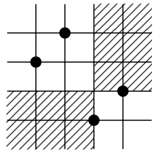


Figure 7. The mesh pattern $m = (p, s)$, where $p = 3412$ and s consists of the boxes (0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4).

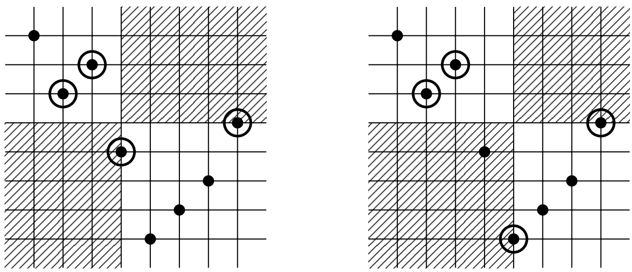


Figure 8. Two occurrences of 3412 in 86741235. Only the left one is an occurrence of the mesh pattern in Figure 7.

elements of s are often called (*shaded*) *boxes*, and informally they denote areas which permutation points are not allowed to occupy in a valid occurrence of (p, s).

For the formal definition of these generalized patterns see [Brändén and Claesson 2011]. We give an example here to illustrate the role the mesh plays.

Example 6.1. There are seven occurrences of the classical pattern $p = 3412$ in the permutation $\pi = 86741235$. In Figure 8 we have highlighted two of them: the subsequences 6745 and 6715. Only the first one is an occurrence of the mesh pattern $m = (p, s)$ in Figure 7, since the mesh can be "stretched" over the grid of the permutation without containing any points. In the second occurrence the point 4 in the permutation occupies a "forbidden" region defined by the mesh.

Theorem 6.2. *Let c be the property of being connected. Then*

$$\mathscr{G}_{12,c} = \mathrm{Av}(m),$$

where m is the mesh pattern in Figure 7. The generating function for the enumeration of these permutations is

$$\frac{F(x) - x}{(1 - x)^2} + \frac{1}{1 - x},$$

where $F(x) = 1 - 1/\sum k!\, x^k$ is the generating function for the skew-indecomposable permutations (see, e.g., [Comtet 1974, p. 261]).

Before we prove the theorem we recall that the *skew-sum* of two permutations π and σ is the permutation $\pi \ominus \sigma$ obtained by adding the length of σ to every letter of π and then appending σ to it. For example $132 \ominus 2413 = 5762413$. We say that a permutation is *skew-decomposable* if it can be written as a skew-sum of two nonempty permutations. Otherwise it is *skew-indecomposable*. Every permutation can be written as a skew-sum of skew-indecomposable permutations, and we call that the *skew-decomposition* of the permutation.

Proof. If the graph $G_{12}(\pi)$ is disconnected then π has two occurrences of 12 in distinct skew-components, A and B, which we can take to be consecutive in the skew-decomposition of $\pi = \cdots \ominus A \ominus B \ominus \cdots$. Let ab be any occurrence of 12 in A. Let u be the highest point in B and v be the leftmost point in B. Then $abuv$ is an occurrence of the mesh pattern. It is clear that an occurrence $abuv$ of the mesh pattern will correspond to two vertices ab, uv in the occurrence graph, and the shadings ensure that there is no path between them.

The enumeration follows from the fact that these permutations must have no, or exactly one, skew-component of size greater than 1. The first case is counted by $1/(1-x)$, while the second case is counted by $(F(x)-x)/(1-x)^2$. □

Note that our software suggests a very large nonclassical basis for the permutations with a tree as an occurrence graph. We omit displaying this basis here. However, since a graph is a tree if and only if it is a nonempty connected forest we obtain:

Corollary 6.3. *Let c be the property of being a tree. Then*

$$\mathcal{G}_{12,c} = \text{Av}(123, 1432, 2143, 3214, m) \setminus \text{Av}(12),$$

where m is the mesh pattern in Figure 7.

Proof. This follows from Theorems 5.10 and 6.2. We must remove the decreasing permutations since they have empty occurrence graphs. □

We end with proving the enumeration for the permutations in the corollary above. The proof is a rather tedious, but simple, induction proof.

Theorem 6.4. *The number of permutations of length n in $\mathcal{G}_{12,\text{tree}}$ is $(n-1)^2$.*

7. Future work

We expect the conjectures in lines 2 and 3 in Tables 1 and 2 to follow from an analysis of the cycle structure of occurrence graphs with respect to the patterns 123 and 132, similar to what we did in Proposition 5.9 for the pattern 12.

Other natural hereditary graph properties to consider would be k-colorable graphs, for $k > 2$, as these are supersets of bipartite graphs. Also planar graphs, which lie between forests and 4-colorable graphs.

It might also be interesting to consider the intersection $\bigcap_{p \in M} \mathcal{G}_{p,c}$ where M is some set of patterns, perhaps all.

We would like to note that Smith (personal communication, 2016) independently defined occurrence graphs and used them to prove a result on the shellability of a large class of intervals of permutations.

Appendix: Proof of Theorem 6.4

We start by introducing a new notation.

Definition A.1. Let $\pi \in \mathfrak{S}_n$ and k be an integer such that $1 \leq k \leq n+1$. The *k-prefix of π* is the permutation $\pi' \in \mathfrak{S}_{n+1}$ defined by $\pi'(1) = k$ and

$$\pi'(i+1) = \begin{cases} \pi(i) & \text{if } \pi(i) < k, \\ \pi(i)+1 & \text{if } \pi(i) \geq k \end{cases}$$

for $i = 1, \ldots, n$. We denote π' by $k \succ \pi$. In a similar way we define the *k-postfix of π* as the permutation $\pi \prec k$ in $\mathfrak{S}_{n+1}$.

Example A.2. Let $\pi = 42135$ and $k = 2$. Visually, if we draw the grid representation of π, we put the new number k to the left on the x-axis and raise all the numbers $\geq k$ on the y-axis by 1. Thus, $2 \succ 42135 = 253146$, as in Figure 9.

We note that for every permutation $\pi' \in \mathfrak{S}_{n+1}$ there is one and only one pair (k, π) such that $\pi' = k \succ \pi$. We let $k = \pi'(1)$ and $\pi = \mathrm{st}(\pi'(2) \cdots \pi'(n+1))$.

Proof of Theorem 6.4. We start by considering three base cases.

For $n = 1$ the occurrence graph is the empty graph. For $n = 2$ we get two occurrence graphs: $G_{12}(12)$ is a graph with a single vertex and $G_{12}(21)$ is the empty graph. For $n = 3$ we have $3! = 6$ different permutations π. Of those we calculate that 132, 213, 231 and 312 result in connected occurrence graphs on one or two vertices but $G_{12}(123)$ is a triangle and $G_{12}(321)$ is the empty graph.

We have thus shown that the claimed enumeration is true for $n = 1, 2, 3$.

For the inductive step we assume $n \geq 4$ and let π be a permutation of length n. We look at four different cases of k to construct $\pi' = k \succ \pi$. We let x, y and z be the indices of $n-1$, n and $n+1$ in π' respectively.

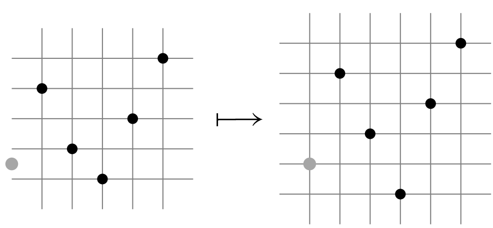


Figure 9. The 2-prefix of 42135 is 253146.

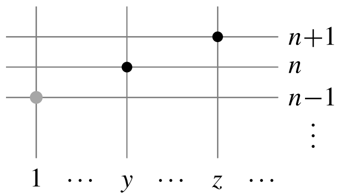


Figure 10. $k = n - 1$ and $y < z$.

(I) $\underline{k \le n-2}$: The index sets $\{1, x\}$, $\{1, y\}$ and $\{1, z\}$ of 12 in π' all share exactly one common element and thus form a triangle in $G_{12}(\pi')$. Therefore there are no permutations π such that the occurrence graph $G_{12}(\pi')$ is a tree.

(II) $\underline{k = n-1}$: Let $T(n+1)$ denote the number of permutations π' of length $n+1$ with $\pi'(1) = n-1$ such that $G_{12}(\pi')$ is a tree. Note that $T(1) = T(2) = 0$, $T(3) = 1$ and $T(4) = 2$. In order to obtain a formula for T we need to look at a few subcases:

(i) If $y < z$ then $\{1, y\}$, $\{1, z\}$ and $\{y, z\}$ form a triangle in $G_{12}(\pi')$; see Figure 10. Independent of the permutation π, the graph $G_{12}(\pi')$ is not a tree.

(ii) Assume $y > z$ and $z \ne 2$, as in Figure 11. Then $\pi'(2) < n-1$ and $\{1, z\}$, $\{2, z\}$, $\{2, y\}$ and $\{1, y\}$ form a cycle of length 4 in $G_{12}(\pi')$, resulting in it not being a tree.

(iii) Assume $y > z$ and $z = 2$, as in Figure 12. If $y \ge 5$ then the vertices $\{1, y\}$, $\{3, y\}$ and $\{4, y\}$ form a cycle in $G_{12}(\pi')$. If $y = 3$ then $\{1, 2\}$ and $\{1, 3\}$ will be an isolated path component in $G_{12}(\pi')$, making $\pi' = (n-1)(n+1)n(n-2)\cdots 1$ the only permutation such that the occurrence graph $G_{12}(\pi')$ is a tree. If $y = 4$, we need to consider further subcases for the value of $\pi'(3)$.

(a) If $\pi'(3) \le n-4$ then $\pi'(3)n$, $\pi'(3)(n-2)$ and $\pi'(3)(n-3)$ are all occurrences of 12 in π', with the respective index sets forming a triangle in $G_{12}(\pi')$.

(b) If $\pi'(3) = n-2$ then $\pi' = (n-1)(n+1)(n-2)n(n-3)\cdots 1$ is the only permutation resulting in $G_{12}(\pi')$ being a tree.

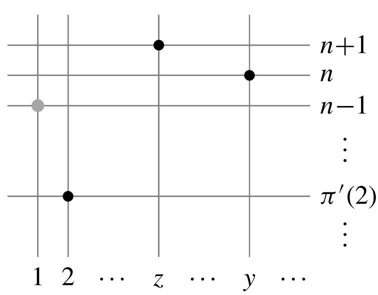


Figure 11. $k = n - 1$, $y > z$ and $z \ne 2$.

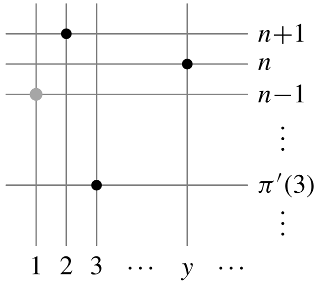


Figure 12. $k = n - 1$, $y > z$ and $z = 2$.

(c) If $\pi'(3) = n-3$ then we look at Figure 13. The permutation $\sigma = \text{st}(\pi'(3) \cdots \pi'(n+1))$ is just like π' in the case $k = n-1$ and $z = 2$, only the length of σ is $n-1$. Because $\{1, 2\}$ is a vertex in $G_{12}(\sigma)$, the occurrence graph of 12 in σ is not the empty graph. Thus it is easy to see that $G_{12}(\pi')$ is a tree if and only if $G_{12}(\sigma)$ is a tree, and according to the aforementioned case there are $T(n-1)$ such permutations σ.

Summing up these possibilities we get a total of $1+1+T(n-1)$ permutations π' making the occurrence graph a tree, i.e., $T(n+1) = 2 + T(n-1)$. Because $T(4) = 2$ and $T(3) = 1$, we deduce that $T(n+1) = n-1$.

The whole case $k = n-1$ gives us that there are $n-1$ permutations π' such that $G_{12}(\pi')$ is a tree.

(III) $\underline{k = n}$: We need to examine three subcases:

(i) If $z \geq 4$ then $\{1, z\}$, $\{2, z\}$, $\{3, z\}$ are all index sets of 12 in π', forming a triangle in $G_{12}(\pi')$.

(ii) If $z = 3$, then $\{1, 3\}$ is an index set of 12 in π making the occurrence graph $G_{12}(\pi)$ nonempty; see Figure 14.

If $\pi'(2) \leq n-2$ then $\pi'(2)(n+1)$, $\pi'(2)(n-1)$ and $\pi'(2)(n-2)$ are all occurrences of 12 in π', resulting in $G_{12}(\pi')$ having a triangle. If $\pi'(2) = n-1$ then $\{1, 3\}$ and $\{2, 3\}$ is an isolated path component in $G_{12}(\pi')$ and $\pi' =$

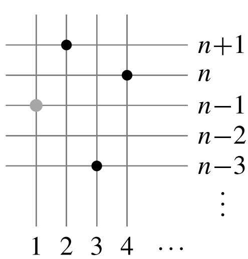


Figure 13. $k = n - 1$, $y = 4$ and $z = 2$.

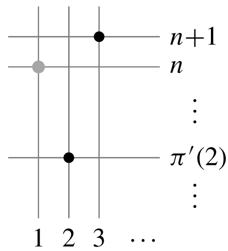


Figure 14. $k = n$ and $z = 3$.

$n(n-1)(n+1)(n-2)\cdots 1$ is the only permutation such that the occurrence graph is a tree. We therefore assume $\pi'(2) = n-2$; see Figure 15.

Let $\sigma = \mathrm{st}(\pi'(2)\cdots\pi'(n+1))$. Note that the occurrence graphs $G_{12}(\pi')$ and $G_{12}(\sigma)$ are the same except the former has the extra vertex $\{1, 2\}$ and an edge connecting it to a graph corresponding to $G_{12}(\sigma)$. Therefore, $G_{12}(\pi')$ is a tree if and only if $G_{12}(\sigma)$ is a tree.

Note that $\sigma(1) = n-2$ and $\sigma(2) = n$ and therefore σ is like π' in the case $k = n-1$ and $z = 2$ as in Figure 12, only of length n instead of $n+1$. By the same reasoning as in that case, the number of permutations σ (and therefore π') such that $G_{12}(\pi')$ is a tree is $T(n) = n-2$.

(iii) If $z = 2$, then $\{1, 2\}$ is an isolated vertex in $G_{12}(\pi')$; see Figure 16. The occurrence graph of 12 in π' is a tree if and only if $G_{12}(\pi)$ is the empty graph, which is true if and only if π is the decreasing permutation. Therefore there is only one permutation $\pi' = n(n+1)(n-1)\cdots 1$ such that $G_{12}(\pi')$ is a tree.

To sum up the case $k = n$ there are $1 + (n-2) + 1 = n$ permutations π' such that $G_{12}(\pi')$ is a tree.

(IV) $\underline{k = n+1}$: Every occurrence $\pi(i)\pi(j)$ of 12 in π is also an occurrence of 12 in π', but with index set $\{i+1, j+1\}$ instead of $\{i, j\}$. There are no more occurrences of 12 in π' because $\pi'(1) = n+1 > \pi'(j')$ for every $j' > 1$ so $\pi'(1)\pi'(j')$ is not an occurrence of 12 for any $j' > 1$.

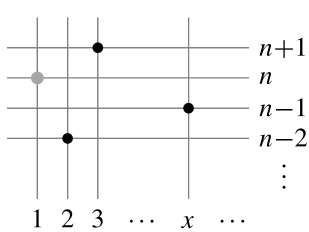


Figure 15. $k = n$, $z = 3$ and $\pi'(2) = n-2$.

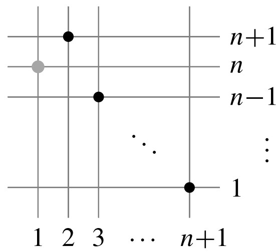


Figure 16. $k = n$ and $z = 2$.

This means that $G_{12}(\pi') \cong G_{12}(\pi)$, so by the induction hypothesis we obtain that there are $(n-1)^2$ permutations π' such that the occurrence graph is a tree for this value of k.

To sum up the four instances there is a total of $0 + (n-1) + n + (n-1)^2 = n^2$ permutations π' such that $G_{12}(\pi')$ is a tree. □

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bjk17@hi.is Department of Mathematics, University of Iceland, Reykjavik, Iceland

henningu@ru.is School of Computer Science, Reykjavik University, Reykjavik, Iceland

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