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Algorithms for classifying points in a 2-adic Mandelbrot set

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In her Ph.D. thesis, Jacqueline Anderson identified a nonarchimedean set similar in spirit to the Mandelbrot set which appears to exhibit a fractal-like boundary. We continue this research by presenting algorithms for determining when rational points lie in this set. We then prove that certain infinite families of points lie in (or out) of this set, giving greater resolution to the self-similarity present in this set.

1. Introduction

The Mandelbrot set and its higher-dimensional analogues are well-known sources of continuing research. These sets, which are defined via an archimedean metric, exhibit fascinating fractal-like boundaries. In this paper, we continue the study of a nonarchimedean (2-adic) set which appears to also have an interesting fractal-like boundary [Anderson 2013; Silverman 2013]. The definition of this set, which will be given shortly, is similar to that of the more familiar Mandelbrot set and its higher-dimensional variants. However, since this set is nonarchimedean, determining which elements are in the set is more difficult. To this end, we present two algorithms (Algorithms 4.7 and 5.3) which often determine when points lie in this set. The results of these algorithms reveal a variety of patterns. At various points we take note of patterns which appear to persist indefinitely, and when possible, prove this is indeed the case (Theorems 5.5 and 5.7). We note that Theorem 5.5 in particular expands upon results in [Anderson 2013].

In Section 2 we review some basic facts concerning fields with nonarchimedean absolute values, including a brief introduction to the field of p-adic numbers $\mathbb{Q}_p$, its ring of integers $\mathbb{Z}_p$, and the field $\mathbb{C}_p$, the topological completion of the algebraic closure of $\mathbb{Q}_p$. We review the definition of the Mandelbrot set in Section 3 and discuss generalizations of the Mandelbrot set to $\mathbb{C}_p$ along with stating an important critical radius bound given in [Anderson 2013].

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Sections 4 and 5 explore the 2-adic Mandelbrot set $\mathcal{M}_{3,2}$ in more detail, which can be thought of as the set of

$$f(x) = f_{\alpha,\beta}(x) = x^3 - \tfrac{3}{2}(\alpha+\beta)x^2 + 3\alpha\beta x, \tag{1-1}$$

with $\alpha, \beta \in \mathbb{C}_p$, for which both $\{f^n(\alpha)\}$ and $\{f^n(\beta)\}$ are bounded. Considering general $\alpha, \beta \in \mathbb{C}_p$ is beyond this scope of this paper; we restrict our attention to determining when $f_{\alpha,\beta}$ is in $\mathcal{M}_{3,2}$ for $\alpha, \beta \in \mathbb{Q}_p$. Section 4 begins with a number of elementary results, the proofs of which rely on little more than basic properties of p-adic numbers. Although fundamentally simple, these results, along with the critical radius bound given in [Anderson 2013], enable us to determine when $f_{\alpha,\beta}$ is in $\mathcal{M}_{3,2}$ for most $\alpha, \beta \in \mathbb{Q}_2$. There are instances where membership of $f_{\alpha,\beta}$ in $\mathcal{M}_{3,2}$ cannot be determined from these results. We break down such instances into two cases, one of which (the case where $\alpha, \beta \in \mathbb{Z}_p$ with $\alpha+\beta$ odd) is the primary focus of the remainder of this paper. For this case, Algorithm 4.7 often determines when $f_{\alpha,\beta}$ is in $\mathcal{M}_{3,2}$. Results of this algorithm are displayed in Figures 1 and 2. We close Section 4 by noting the difficulty of extending this algorithm beyond the case at hand.

Section 5 is focused entirely on understanding the structure of the intersection $\mathcal{M}_{3,2} \cap \{f_{\alpha,0} : \alpha \in \mathbb{Q}_2\}$. Prior work on this case was presented in [Anderson 2013, §6], resulting in the observation that $\mathcal{M}_{3,2}$ appears to have a fractal-like boundary at $\alpha = 1$. We continue this analysis by studying the sequence $\{x_n\}$ (defined in Lemma 5.1) as a proxy for $\{f^n(\alpha)\}$. We adapt the algorithm presented in Section 4 to the analysis of $\{x_n\}$ and present the results of this algorithm in Figures 3 and 4. By working with $\{x_n\}$, certain patterns in $\mathcal{M}_{3,2} \cap \{f_{\alpha,0} : \alpha \in \mathbb{Q}_2\}$ become more readily apparent. Theorems 5.5 and 5.7 classify certain classes of $f_{\alpha,0}$. Section 5 concludes with a discussion on how further improvements in the classification of $f_{\alpha,0}$ might be obtained.

2. Fields with nonarchimedean absolute values

We begin by reviewing some basic facts about nonarchimedean absolute values. We refer the reader to [Gouvêa 1993; Koblitz 1977] for more thorough introductions. Recall that an *absolute value* $|\cdot|$ on a field $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that for all $x, y \in \mathbb{K}$,

(a) $|x| = 0$ if and only if $x = 0$,

(b) $|xy| = |x|\,|y|$,

(c) $|x+y| \le |x| + |y|$.

If in addition we have that for all $x, y \in \mathbb{K}$

(d) $|x+y| \le \max\{|x|, |y|\}$,

then $|\cdot|$ is said to be *nonarchimedean*; otherwise $|\cdot|$ is *archimedean*. Note that (d) (the ultrametric inequality) implies (c) (the triangle inequality). For nonarchimedean absolute values, one can show that for all $x, y \in \mathbb{K}$

$$|x| < |y| \quad \Longrightarrow \quad |x+y| = |y|. \tag{2-1}$$

Furthermore, (d) and (2-1) can be extended to any number of elements. For instance,

$$|x+y+z| \le \max\{|x|, |y|, |z|\}, \tag{2-2}$$

$$|x|, |y| < |z| \quad \Longrightarrow \quad |x+y+z| = |z| \tag{2-3}$$

for all $x, y, z \in \mathbb{K}$. A field $\mathbb{K}$ equipped with a nonarchimedean absolute value has associated with it a topology induced by the metric $d(x, y) = |x - y|$. Such topological spaces are called *ultrametric spaces*. Absolute values are *equivalent* if they induce identical topologies on $\mathbb{K}$.

To define a nonarchimedean absolute value on $\mathbb{Q}$, we fix a prime number p. Let $v_p(n) = \max\{k \in \mathbb{Z}_{\ge 0} : p^k \mid n\}$, where $n \in \mathbb{Z}$. We then extend v_p to $\mathbb{Q}$ by defining $v_p(a/b) = v_p(a) - v_p(b)$ for $a, b \in \mathbb{Z}$, $b \ne 0$. The function v_p is called the *p-adic valuation* on $\mathbb{Q}$. The *p-adic absolute value* on $\mathbb{Q}$ is $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ such that $|x|_p = p^{-v_p(x)}$, where $x \in \mathbb{Q}$. One can show that $|\cdot|_p$ is a nonarchimedean absolute value on $\mathbb{Q}$ and that each nonarchimedean absolute value on $\mathbb{Q}$ is equivalent to a p-adic absolute value (Ostrowski's theorem). The set of *p-adic numbers*, denoted by $\mathbb{Q}_p$, is the completion of $\mathbb{Q}$ under the p-adic absolute value. This completion is obtained by taking the quotient of the ring of Cauchy sequences in $\mathbb{Q}$ (with respect to the topology induced by the p-adic absolute value) over the ideal of sequences in $\mathbb{Q}$ that converge to 0. The real numbers can be constructed similarly, but with the topology on $\mathbb{Q}$ induced from the usual archimedean absolute value. The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, allowing the p-adic absolute value on $\mathbb{Q}$ to be extended to $\mathbb{Q}_p$, which we also denote by $|\cdot|_p$. One can show that the range of $|\cdot|_p$ on $\mathbb{Q}_p$ is $\{p^k : k \in \mathbb{Z}\}$.

Each $x \in \mathbb{Q}_p$ has a unique representation in the form of a finite-tailed Laurent series in p:

$$x = \sum_{n=n_0}^{\infty} a_n p^n = a_{n_0} p^{n_0} + a_{n_0+1} p^{n_0+1} + \cdots, \tag{2-4}$$

where $n_0 \in \mathbb{Z}$, $a_n \in \{0, 1, \ldots, p-1\}$, and $a_{n_0} \ne 0$. Let $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \le 1\}$, which is called the set of *p-adic integers*. If $x \in \mathbb{Z}_p$ then the Laurent series representation of x has $n_0 \ge 0$. For $x, y \in \mathbb{Z}_p$ and $m \in \mathbb{Z}_{>0}$, we say that $x \equiv y \pmod{p^m}$ if there exists $c \in \mathbb{Z}_p$ such that $x - y = c \cdot p^m$. By (2-4), we see that for given $x \in \mathbb{Z}_p$ and $m \in \mathbb{Z}_{>0}$, there exists a unique $y \in \mathbb{Z}$ such that $x \equiv y \pmod{p^m}$ and $0 \le y < p^m$.

Like $\mathbb{R}$, the set $\mathbb{Q}_p$ is not algebraically closed. Let $\overline{\mathbb{Q}}_p$ denote the algebraic closure of $\mathbb{Q}_p$. We extend $|\cdot|_p$ to $\overline{\mathbb{Q}}_p$ by defining $|\alpha|_p = |N_{\mathbb{Q}_p(\alpha)/\mathbb{Q}_p}(\alpha)|_p^{1/m}$, where

$\alpha \in \overline{\mathbb{Q}}_p$ and $m = [\mathbb{Q}_p(\alpha) : \mathbb{Q}_p]$.[1] Unlike $\mathbb{C}$, the algebraic closure of $\mathbb{R}$, $\overline{\mathbb{Q}}_p$ is not topologically complete. To remedy this, let $\mathbb{C}_p$ denote the (topological) completion of $\overline{\mathbb{Q}}_p$, with $|\cdot|_p$ extending in the natural way. One can show that $\mathbb{C}_p$ is not only topologically complete but also algebraically complete. As one might suspect, $|\cdot|_p$ on $\mathbb{C}_p$ (hence also on $\overline{\mathbb{Q}}_p$ and $\mathbb{Q}_p$) is a nonarchimedean absolute value.

3. Mandelbrot sets over *p*-adic numbers

We begin by recalling the definition of the Mandelbrot set; further details can be found in [Beardon 1991; Devaney 1989]. Consider $f(z) \in \mathbb{C}[z]$ with $\deg(f) = 2$. For $n \in \mathbb{Z}_{>0}$, let $f^n(z)$ denote the n-th iterate of $f(z)$. For $\alpha \in \mathbb{C}$, let $\{f^n(\alpha)\}$ denote the sequence of n-th iterates of $f(z)$ evaluated at $z = \alpha$. We call $\{f^n(\alpha)\}$ the *orbit* of α under f. We say that $f(z)$ is *critically bounded* if $\{f^n(\alpha)\}$ is a bounded sequence for the critical point $\alpha \in \mathbb{C}$ of $f(z)$.[2] Let $g(z) = h \circ f \circ h^{-1}(z)$, where $h(z) = az + b \in \mathbb{C}[z]$ with $a \neq 0$. We say that $g(z)$ is a *linear conjugate* of $f(z)$. By choosing a and b appropriately, $g(z) = z^2 + c$ for some unique $c \in \mathbb{C}$. One can show that $f(z)$ is critically bounded if and only if $g(z)$ is critically bounded. Notice $g(z)$ has $\alpha = 0$ as its only critical point. The *Mandelbrot set* is

$$\begin{aligned} \mathcal{M} &= \{c \in \mathbb{C} : f_c(z) = z^2 + c \in \mathbb{C}[z] \text{ is critically bounded}\} \\ &= \{c \in \mathbb{C} : \{f_c^n(0)\} \text{ is bounded}\}. \end{aligned} \tag{3-1}$$

Since critical boundedness is well-defined up to linear conjugation, we can also think of $\mathcal{M}$ as the set of classes of linearly conjugate quadratic polynomials in $\mathbb{C}[z]$ that are critically bounded. The Mandelbrot set has a fractal-like boundary, the study of which is an active area of research [Dudko 2017; Lomonaco and Petersen 2017].

The definitions and analysis above translate without issue to $\mathbb{C}_p$. Thus it is natural to wonder whether a similarly defined set in $\mathbb{C}_p$ might also have a fractal-like boundary. Unfortunately, the natural candidate for a p-adic analogue to $\mathcal{M}$,

$$\{c \in \mathbb{C}_p : f_c(z) = z^2 + c \in \mathbb{C}_p[z] \text{ is critically bounded}\}, \tag{3-2}$$

is simply the unit disk $\{c \in \mathbb{C}_p : |c|_p \leq 1\}$ [Anderson 2013, Theorem 4.1]. However, it appears that more interesting sets can be obtained by considering polynomials of higher degree.

Consider $f(z) \in \mathbb{C}_p[z]$ with $d = \deg(f)$. We say that $f(z) \in \mathbb{C}_p[z]$ is *critically bounded* if $\{f^n(\alpha)\}$ is a bounded sequence for all critical points $\alpha \in \mathbb{C}_p$ of $f(z)$. As with quadratics, critical boundedness is well-defined up to linear conjugation,

[1] Here, $N_{\mathbb{Q}_p(\alpha)/\mathbb{Q}_p} : \mathbb{Q}_p(\alpha) \to \mathbb{Q}_p$ denotes the *norm* defined in field theory.

[2] A fundamental result in complex dynamics states that the Julia set of $f(z)$ is connected if and only if $f(z)$ is critically bounded. This result helps to explain why the classification of critically bounded $f(z)$ is of interest in its own right.

and in light of this, we can restrict our attention to monic $f(z)$ such that $f(0) = 0$ (every class of linearly conjugate degree-d polynomials has a representative of this form); that is to say, we consider

$$\mathcal{P}_{d,p} = \{x^d + a_{d-1}x^{d-1} + \cdots + a_1 x : (a_{d-1}, \ldots, a_1) \in \mathbb{C}_p^{d-1}\}. \tag{3-3}$$

Let

$$\mathcal{M}_{d,p} = \{f \in \mathcal{P}_{d,p} : f \text{ is critically bounded}\}. \tag{3-4}$$

If $p \geq d$ then $\mathcal{M}_{d,p} = \{f \in \mathcal{P}_{d,p} : |\alpha|_p \leq 1 \text{ for all critical points } \alpha \in \mathbb{C}_p \text{ of } f\}$ [Anderson 2013, Theorem 4.1, Proposition 4.2], in which case $\mathcal{M}_{d,p}$ lacks a fractal-like boundary. However, if $p < d$ then $\mathcal{M}_{d,p}$ may have a more intricate structure [Anderson 2013, §6]. Such sets are called *p-adic Mandelbrot sets*.

Let

$$r(d, p) = \sup_{f \in \mathcal{M}_{d,p}} \max_{\substack{\alpha \in \mathbb{C}_p \\ f'(\alpha)=0}} \{-v_p(\alpha)\}, \tag{3-5}$$

which is called the *critical radius* of $\mathcal{M}_{d,p}$. As can be easily checked, if α is a critical point of $f \in \mathcal{M}_{d,p}$ then $|\alpha|_p \leq p^{r(d,p)}$. In [Anderson 2013, Theorem 1.2], it was shown that for $d/2 < p < d$, we have $r(d, p) = p/(d-1)$. Therefore, if $d/2 < p < d$ then the critical points of $f \in \mathcal{M}_{d,p}$ are contained in a disk of radius $p^{p/(d-1)}$.

4. Determining when $f_{\alpha,\beta} \in \mathcal{M}_{3,2}$ for $\alpha, \beta \in \mathbb{Q}_2$

We focus our attention on $\mathcal{M}_{3,2}$, which, as was shown in [Anderson 2013, §6], appears to have a fractal-like boundary. Let $f \in \mathcal{P}_{3,2}$. Then

$$\begin{aligned} f(x) = f_{\alpha,\beta}(x) &= x^3 - \tfrac{3}{2}(\alpha+\beta)x^2 + 3\alpha\beta x \\ &= x\bigl(x^2 - \tfrac{3}{2}(\alpha+\beta)x + 3\alpha\beta\bigr), \end{aligned} \tag{4-1}$$

where $\alpha, \beta \in \mathbb{C}_2$ are the critical points of $f(x)$. Since $d/2 < p < d$ for $d = 3$ and $p = 2$, we have $r(3, 2) = 2/(3-1) = 1$. Therefore all critical points of $f \in \mathcal{M}_{3,2}$ are contained in a disk of radius $p^{p/(d-1)} = 2$. Consequently, we only consider $f \in \mathcal{P}_{3,2}$ with $|\alpha| \leq 2$ and $|\beta| \leq 2$, where we write $|\cdot|$ for $|\cdot|_2$. Because of the complexity involved in dealing with elements of $\mathbb{C}_2$, we restrict our attention to $\alpha, \beta \in \mathbb{Q}_2$.

Lemma 4.1. *Let $f = f_{\alpha,\beta}$ with $\alpha, \beta \in \mathbb{Q}_2$ such that $|\alpha|, |\beta| \leq 2$. If $|f^m(\alpha)| > 4$ for some $m \in \mathbb{Z}_{>0}$ then* $\lim_{n\to\infty} |f^n(\alpha)| = \infty$, *and so $f \notin \mathcal{M}_{3,2}$.*

Proof. Since $|f^m(\alpha)| > 4$, we know $|f^m(\alpha)| = 2^k$ for some $k > 2$. Observe

$$|f^m(\alpha)^2| = 2^{2k}, \quad \bigl|-\tfrac{3}{2}(\alpha+\beta)f^m(\alpha)\bigr| \leq 2^{k+2}, \quad |3\alpha\beta| \leq 4.$$

Since $|f^m(\alpha)^2|$ is the largest of the quantities above, by (2-3)

$$|f^{m+1}(\alpha)| = |f^m(\alpha)|\left|f^m(\alpha)^2 - \tfrac{3}{2}(\alpha+\beta)f^m(\alpha) + 3\alpha\beta\right| = 2^k 2^{2k} = 2^{3k}.$$

By induction, $|f^{m+n}(\alpha)| = 2^{3^n k}$ for all $n \geq 1$. Thus $\lim_{n\to\infty} |f^n(\alpha)| = \infty$. □

Since we are considering $|\alpha|, |\beta| \leq 2$, we have $\alpha = a/2$ and $\beta = b/2$ for some $a, b \in \mathbb{Z}_2$. We say that $c \in \mathbb{Z}_2$ is *odd* if $|c| = 1$ and *even* if $|c| < 1$.

Proposition 4.2. *Let $f = f_{\alpha,\beta}$ with $\alpha = a/2$, $\beta = b/2$, $a, b \in \mathbb{Z}_2$:*

(a) *If $a + b$ is odd then $f \notin \mathcal{M}_{3,2}$.*

(b) *If a and b are odd and $a + b \equiv 2 \pmod 4$ then $f \notin \mathcal{M}_{3,2}$.*

(c) *If a and b are even and $a + b \equiv 0 \pmod 4$ (i.e., $\alpha, \beta \in \mathbb{Z}_2$ and $\alpha + \beta$ even) then $f \in \mathcal{M}_{3,2}$.*

Proof. For part (a), assume without loss of generality that a is odd and b is even. Thus $|\alpha| = 2$ and $|\beta| \leq 1$. Observe

$$|\alpha^2| = 4, \quad \left|-\tfrac{3}{2}(\alpha+\beta)\alpha\right| = 8, \quad |3\alpha\beta| \leq 2.$$

Since $\left|-\tfrac{3}{2}(\alpha+\beta)\alpha\right|$ is the largest of the quantities above, by (2-3)

$$|f(\alpha)| = |\alpha|\left|\alpha^2 - \tfrac{3}{2}(\alpha+\beta)\alpha + 3\alpha\beta\right| = 16.$$

Therefore by Lemma 4.1, $f \notin \mathcal{M}_{3,2}$.

For part (b), since $a + b \equiv 2 \pmod 4$, we know $a + b = 2k$ for some odd $k \in \mathbb{Z}_2$. Observe

$$|f(\alpha)| = \left|\frac{a}{2}\right|\left|\frac{a^2}{4} - \frac{3(a+b)a}{8} + \frac{3ab}{4}\right| = 2\left|\frac{a^2 - 3ka + 3ab}{4}\right|.$$

Since $a^2 - 3ka + 3ab$ is odd, $|f(\alpha)| = 8$. Thus by Lemma 4.1, $f \notin \mathcal{M}_{3,2}$.

For part (c), since $\alpha + \beta$ is even, $|\alpha + \beta| \leq \frac{1}{2}$. So if $|x| \leq 1$ then by (2-2)

$$\begin{aligned}|f(x)| &= |x|\left|x^2 - \tfrac{3}{2}(\alpha+\beta)x + 3\alpha\beta\right| \\ &\leq \max\left\{|x^2|, \left|-\tfrac{3}{2}(\alpha+\beta)\right||x|, |3\alpha\beta|\right\} \leq 1.\end{aligned}$$

Since $|\alpha|, |\beta| \leq 1$, we have $|f(\alpha)|, |f(\beta)| \leq 1$. By induction, $|f^n(\alpha)|, |f^n(\beta)| \leq 1$ for all $n \geq 1$. Thus $\{f^n(\alpha)\}$ and $\{f^n(\beta)\}$ are bounded, and hence $f \in \mathcal{M}_{3,2}$. □

The following lemma gives an improvement on Lemma 4.1 when $a + b$ is even. Although this improvement is slight, having it will make the classification of $f_{\alpha,\beta}$, given by Algorithm 4.7 simpler than it would be otherwise as well as yield simpler proofs for other results given in the remainder of this paper.

Lemma 4.3. *Let* $f = f_{\alpha,\beta}$, *with* $\alpha = a/2,\ \beta = b/2,\ a, b \in \mathbb{Z}_2$ *such that* $a+b$ *is even. If* $|f^m(\alpha)| \geq 4$ *for some* $m \in \mathbb{Z}_{>0}$ *then* $f \notin \mathcal{M}_{3,2}$.

Proof. If $|f^m(\alpha)| > 4$ then, by Lemma 4.1, $f \notin \mathcal{M}_{3,2}$. Thus it remains to consider the case when $|f^m(\alpha)| = 4$. Observe

$$|f^m(\alpha)^2| = 16, \qquad \left|-\frac{3}{2}\left(\frac{a+b}{2}\right) f^m(\alpha)\right| \leq 8, \qquad |3\alpha\beta| \leq 4.$$

Since $|f^m(\alpha)^2|$ is the largest of the quantities above, by (2-3)

$$|f^{m+1}(\alpha)| = |f^m(\alpha)| \left| f^m(\alpha)^2 - \frac{3}{2}\left(\frac{a+b}{2}\right) f^m(\alpha) + 3\alpha\beta \right| = 64.$$

Therefore by Lemma 4.1, $f \notin \mathcal{M}_{3,2}$. □

Notice that Proposition 4.2 considers all possibilities for $a, b \in \mathbb{Z}_2$ except for

(i) a and b even and $a+b \equiv 2 \pmod 4$ (i.e., $\alpha, \beta \in \mathbb{Z}_2$ with $\alpha+\beta$ odd),

(ii) a and b odd and $a+b \equiv 0 \pmod 4$.

We consider each of these cases separately, focusing primarily on (i) and briefly addressing (ii) at the end of this section.

Lemma 4.4. *Let* $f = f_{\alpha,\beta}$ *with* $\alpha, \beta \in \mathbb{Z}_2$ *and* $\alpha+\beta$ *odd. If* $|f^m(\alpha)| \leq \frac{1}{2}$ *for some* $m \in \mathbb{Z}_{>0}$ *then* $\{f^n(\alpha)\}$ *is bounded. Furthermore*:

(a) *If* $|f^m(\alpha)| \leq \frac{1}{4}$ *for some* $m \in \mathbb{Z}_{>0}$, *then* $\lim_{n\to\infty} f^n(\alpha) = 0$ (*i.e.*, α *is in the basin of attraction of zero*).

(b) *If* $|f^m(\alpha)| = \frac{1}{2}$ *for some* $m \in \mathbb{Z}_{>0}$, *then* $|f^{m+n}(\alpha)| = \frac{1}{2}$ *for all* $n \in \mathbb{Z}_{>0}$.

Proof. Suppose $|f^m(\alpha)| = 2^{-k}$ for some $k \in \mathbb{Z}_{\geq 1}$. Since $\alpha+\beta$ is odd, α and β have opposite parity. Thus

$$|f^m(\alpha)^2| = 2^{-2k}, \quad \left|-\tfrac{3}{2}(\alpha+\beta) f^m(\alpha)\right| = 2^{1-k}, \quad |3\alpha\beta| \leq \tfrac{1}{2}. \tag{4-2}$$

If $k \geq 2$ then by (2-2)

$$|f^{m+1}(\alpha)| = |f^m(\alpha)| \left| f^m(\alpha)^2 - \tfrac{3}{2}(\alpha+\beta) f^m(\alpha) + 3\alpha\beta \right| \leq 2^{-k-1}.$$

By induction, $|f^{m+n}(\alpha)| \leq 2^{-k-n}$ for all $n \geq 1$. Thus $\lim_{n\to\infty} f^n(\alpha) = 0$.

If instead $k = 1$ then $\left|-\frac{3}{2}(\alpha+\beta) f^m(\alpha)\right| = 1$ is the largest of the terms in (4-2). Thus by (2-3),

$$|f^{m+1}(\alpha)| = |f^m(\alpha)| \left| f^m(\alpha)^2 - \tfrac{3}{2}(\alpha+\beta) f^m(\alpha) + 3\alpha\beta \right| = \tfrac{1}{2}.$$

By induction, $|f^{m+n}(\alpha)| = \frac{1}{2}$ for all $n \geq 1$. □

Suppose $\alpha, \beta \in \mathbb{Z}_2$ with $\alpha+\beta$ odd. Without loss of generality, suppose throughout that α is odd and β is even. Notice that if $\beta \equiv 2 \pmod 4$ then $|\beta| = \frac{1}{2}$, and if $\beta \equiv 0 \pmod 4$ then $|\beta| \leq \frac{1}{4}$. With this in mind, the following proposition follows from the proof of Lemma 4.4 if we replace $f^m(\alpha)$ with β.

Proposition 4.5. *Let $f = f_{\alpha,\beta}$, with $\alpha, \beta \in \mathbb{Z}_2$, with α odd and β even. Then $\{f^n(\beta)\}$ is bounded. Furthermore*:

(a) *If $\beta \equiv 0 \pmod 4$ then $\lim_{n\to\infty} f^n(\beta) = 0$.*

(b) *If $\beta \equiv 2 \pmod 4$ then $|f^n(\beta)| = \frac{1}{2}$ for all $n \in \mathbb{Z}_{>0}$.*

The following result follows from Lemmas 4.3 and 4.4 and Proposition 4.5.

Proposition 4.6. *Let $f = f_{\alpha,\beta}$, with $\alpha, \beta \in \mathbb{Z}_2$, with α odd and β even*:

(a) *There exists $n \in \mathbb{Z}_{>0}$ such that $|f^n(\alpha)| \geq 4$ if and only if $f \notin \mathcal{M}_{3,2}$.*

(b) *If there exists $n \in \mathbb{Z}_{>0}$ such that $|f^n(\alpha)| \leq \frac{1}{2}$ then $f \in \mathcal{M}_{3,2}$.*

For given $\alpha, \beta \in \mathbb{Z}_2$ with α odd and β even, we can sometimes use Proposition 4.6 to determine whether $f = f_{\alpha,\beta} \in \mathcal{M}_{3,2}$. We do so by selecting an upper bound N and computing $f^n(\alpha)$ for $1 \leq n \leq N$. Proposition 4.6 can then be applied, except when $|f^n(\alpha)| \in \{1, 2\}$ for $1 \leq n \leq N$. Taking larger N may bring resolution in cases such as these, but as a practical matter, computing $f^n(\alpha)$ can slow substantially for large n. Indeed, even if $\alpha, \beta \in \mathbb{Z}$, we often find that $f^n(\alpha)$ consists of rational numbers whose numerators (and sometimes denominators) are rapidly increasing in size with respect to the archimedean absolute value, even if $|f^n(\alpha)| \in \{1, 2\}$ for $1 \leq n \leq N$.

Instead of considering fixed $\alpha, \beta \in \mathbb{Z}_2$ with α odd and β even, a more computationally efficient and general method is to first fix $j, k \in \mathbb{Z}_{>0}$ and fix $a_0, b_0 \in \mathbb{Z}$ such that a_0 is odd, b_0 is even, $0 \leq a_0 < 2^j$, and $0 \leq b_0 < 2^k$. Then consider all $\alpha \in D(a_0, 2^{-j})$ and $\beta \in D(b_0, 2^{-k})$, where

$$D(d, 2^{-m}) = \{x \in \mathbb{Q}_p : |x - d| \leq 2^{-m}\} = \{x \in \mathbb{Q}_p : x \equiv d \pmod{2^m}\} \tag{4-3}$$

for $d \in \mathbb{Q}_p$ and $m \in \mathbb{Z}$. Equivalently, we can also think of $\alpha = a_0 + 2^j p$ and $\beta = b_0 + 2^k q$ for indeterminates $p, q \in \mathbb{Z}_2$. Let $f = f_{\alpha,\beta}$, which we now think of as being dependent upon p and q.

Consider $z = z(p, q) \in \mathbb{Q}_2[p, q]$. We abuse notation by saying that $z \in \mathbb{Z}_2$ if $z(p_0, q_0) \in \mathbb{Z}_2$ for any values $p_0, q_0 \in \mathbb{Z}_2$. We say that $z \notin \mathbb{Z}_2$ otherwise. It is easy to check that under this convention, $4f(\alpha) \in \mathbb{Z}_2$. Thus

$$j_1 = \max\{j' \in \mathbb{Z}_{\geq 0} : 4f(\alpha) \equiv c \pmod{2^{j'}} \text{ for some fixed } c \in \mathbb{Z}\} \tag{4-4}$$

is well-defined with the understanding that the statement $4f(\alpha) \equiv c \pmod{2^{j'}}$ for some fixed $c \in \mathbb{Z}$ means that regardless of whatever values in $\mathbb{Z}_2$ we may assign to p

in $\alpha = a_0 + 2^j p$ or to q in $\beta = b_0 + 2^k q$, we always have that $4f(\alpha) \equiv c \pmod{2^{j'}}$ for the same fixed $c \in \mathbb{Z}$. Alternatively, we can understand (4-4) as asserting that $4f(\alpha) \in D(c, 2^{-j_1})$.

Therefore

$$4f(\alpha) = c_1 + 2^{j_1} p_1$$

for indeterminate $p_1 \in \mathbb{Z}_2$, which itself is dependent upon p and q, and unique $c_1 \in \mathbb{Z}$ such that $0 \le c_1 < 2^{j_1}$. To illustrate this, consider $a_0 = 3$, $b_0 = 4$, $j = 2$, and $k = 3$. Then, with some algebraic simplifications, we find that

$$\begin{aligned} 4f(\alpha) &= 4(a_0+2^j p)\big((a_0+2^j p)^2 - \tfrac{3}{2}(a_0+2^j p+b_0+2^k q)(a_0+2^j p) \\ &\qquad\qquad + 3(a_0+2^j p)(b_0+2^k q)\big) \\ &= 162 + 2^3(3^2\cdot 5p + 2^2\cdot 3p^2 - 2^4 p^3 + 2\cdot 3^3 q + 2^4\cdot 3^2 pq + 2^5\cdot 3p^2 q) \\ &= 2 + 2^3(2^2\cdot 5 + 3^2\cdot 5p + 2^2\cdot 3p^2 - 2^4 p^3 + 2\cdot 3^3 q + 2^4\cdot 3^2 pq + 2^5\cdot 3p^2 q). \end{aligned}$$

This shows that $c_1 = 2$ and $j_1 = 3$, with indeterminate

$$p_1 = 2^2\cdot 5 + 3^2\cdot 5p + 2^2\cdot 3p^2 - 2^4 p^3 + 2\cdot 3^3 q + 2^4\cdot 3^2 pq + 2^5\cdot 3p^2 q$$

(i.e., p_1 is a polynomial in p and q).

Continuing in this manner, we recursively define (possibly finite) sequences $\{c_n\}$ and $\{j_n\}$ as follows. Suppose $4f((c_n + 2^{j_n} p_n)/4) \in \mathbb{Z}_2$; we have established this for $n = 0$ if we take $c_0 = 4a_0$, $j_0 = j + 2$, and $p_0 = p$. Then

$$j_{n+1} = \max\left\{ j' \in \mathbb{Z}_{\ge 0} : 4f\left(\frac{c_n + 2^{j_n} p_n}{4}\right) \equiv c \pmod{2^{j'}} \text{ for some fixed } c \in \mathbb{Z} \right\} \tag{4-5}$$

is well-defined (with an important disclaimer in the following paragraph). Thus

$$4f\left(\frac{c_n + 2^{j_n} p_n}{4}\right) = c_{n+1} + 2^{j_{n+1}} p_{n+1}$$

for indeterminate $p_{n+1} \in \mathbb{Z}_2$ (ultimately expressible in terms of p and q) and unique $c_{n+1} \in \mathbb{Z}$ such that $0 \le c_{n+1} < 2^{j_{n+1}}$. If $4f((c_n + 2^{j_n} p_n)/4) \notin \mathbb{Z}_2$ then we terminate $\{c_n\}$ and $\{j_n\}$ at index n.

We have already noted that $4f(\alpha) = c_1 + 2^{j_1} p_1$, and, as one can easily check, we also have

$$4f^n(\alpha) = 4f\left(\frac{c_{n-1} + 2^{j_{n-1}} p_{n-1}}{4}\right) = c_n + 2^{j_n} p_n \tag{4-6}$$

for $n \in \mathbb{Z}_{>0}$ whenever c_n and j_n are defined, provided that we think of p_n as being expressed in terms of p and q (i.e., as an element of $\mathbb{Z}[p, q]$). Thinking of p_n as a polynomial on two variables, it may be the case that $p_n : \mathbb{Z}_2^2 \to \mathbb{Z}_2$ is not surjective. However, determining the range of p_n is complicated. Rather than keeping track of this in our algorithm, we view p_n as an independent indeterminate when computing j_{n+1}; that is, we think of p_n as being able to take on any value in $\mathbb{Z}_2$. By divorcing the relationship of p_n to p and q, the statement that $4f^n(\alpha) = c_n + 2^{j_n} p_n$ is, strictly speaking, false. Nevertheless, we will continue to write $4f^n(\alpha) = c_n + 2^{j_n} p_n$ with the understanding that for any choice of values in $\mathbb{Z}_2$ substituted in for p and q there exists a corresponding value in $\mathbb{Z}_2$ that when substituted in for p_n produces $4f^n(\alpha) = c_n + 2^{j_n} p_n$.

There is a close connection between $|f^n(\alpha)|$ and c_n. If $j_n > 0$ and $c_n \neq 0$, then there exists $e \in \mathbb{Z}_{\geq 0}$ such that $c_n \equiv 2^e \pmod{2^{e+1}}$ and $e + 1 \leq j_n$. Thus by (4-6),

$$\begin{aligned} c_n \equiv 2^e \pmod{2^{e+1}} &\iff c_n + 2^{j_n} p_n \equiv 2^e \pmod{2^{e+1}} \\ &\iff 4f^n(\alpha) \equiv 2^e \pmod{2^{e+1}} \\ &\iff |4f^n(\alpha)| = 2^{-e} \\ &\iff |f^n(\alpha)| = 2^{2-e}. \end{aligned} \tag{4-7}$$

Therefore by Proposition 4.6, $f \notin \mathcal{M}_{3,2}$ if $e = 0$ and $f \in \mathcal{M}_{3,2}$ if $e \geq 3$. Because of this, we terminate $\{c_n\}$ and $\{j_n\}$ at index n if $e = 0$ or $e \geq 3$.

If instead $c_n = 0$ then by (4-6), $|4f^n(\alpha)|$ is dependent upon p_n, and so we cannot know the exact value of $|f^n(\alpha)|$ from c_n and j_n. However by (4-6),

$$\begin{aligned} c_n \equiv 0 \pmod{2^{j_n}} &\implies 4f^n(\alpha) \equiv 0 \pmod{2^{j_n}} \\ &\implies |4f^n(\alpha)| \leq 2^{-j_n} \\ &\implies |f^n(\alpha)| \leq 2^{2-j_n}. \end{aligned} \tag{4-8}$$

This shows that if $j_n \geq 3$ and $c_n = 0$ then $f \in \mathcal{M}_{3,2}$ by Proposition 4.6(b). If $j_n \leq 2$ and $c_n = 0$ then we cannot determine the behavior of $\{f^n(\alpha)\}$; indeed, one can check that if $j_n = 0$ then $4f((c_n + 2^{j_n} p_n)/4) \notin \mathbb{Z}_2$, if $c_n = 0$ and $j_n = 1$ then $j_{n+1} = 0$, and if $c_n = 0$ and $j_n = 2$ then $c_{n+1} = 0$ and $j_{n+1} = 1$. Given that the behavior of $\{f^n(\alpha)\}$ when $c_n = 0$ is either fully classified (as when $j_n \geq 3$) or impossible to determine (as when $j_n \leq 2$), we terminate $\{c_n\}$ and $\{j_n\}$ at index n whenever $c_n = 0$.

As stated earlier, we terminate $\{c_n\}$ and $\{j_n\}$ at index n if $4f((c_n + 2^{j_n} p_n)/4) \notin \mathbb{Z}_2$. However, the additional termination criteria we just gave (i.e., terminate if $c_n \neq 0$ and $e = 0$ or $e \geq 3$, or if $c_n = 0$) makes it so that we never have to test for this. To see why this is the case, notice that we only compute c_{n+1} and j_{n+1} if $j_n \geq 2$ and $c_n \equiv 2 \pmod 4$ or if $j_n \geq 3$ and $c_n \equiv 4 \pmod 8$ (i.e., when $e = 1, 2$ in (4-7)). One can show that in either of these cases $4f((c_n + 2^{j_n} p_n)/4) \in \mathbb{Z}_2$.

In addition to the previously mentioned criteria for determining when $f \in \mathcal{M}_{3,2}$, we also have

$$\text{if } c_n = c_{n+\ell} \text{ and } j_n = j_{n+\ell} \text{ for some } n, \ell \in \mathbb{Z}_{>0} \text{ then } f \in \mathcal{M}_{3,2}. \tag{4-9}$$

Indeed, if $c_n = c_{n+\ell}$ and $j_n = j_{n+\ell}$ for some $n, \ell \in \mathbb{Z}_{>0}$ then $\{(c_n, j_n)\}$ is preperiodic, which by (4-7) makes $\{|f^n(\alpha)|\}$ also preperiodic.

We summarize the discussion above in the following result.

Algorithm 4.7. Fix $j, k \in \mathbb{Z}_{>0}$ and fix $a_0, b_0 \in \mathbb{Z}$ such that a_0 is odd, b_0 is even, $0 \le a_0 < 2^j$, and $0 \le b_0 < 2^k$. Consider $\alpha, \beta \in \mathbb{Z}_2$ such that $\alpha \equiv a_0 \pmod{2^j}$ and $\beta \equiv b_0 \pmod{2^k}$. Let $f = f_{\alpha,\beta}$, $c_0 = 4a_0$, and $j_0 = j+2$. We recursively define sequences $\{c_n\}$ and $\{j_n\}$ as follows: if $j_n \ge 2$ and $c_n \equiv 2 \pmod 4$ or if $j_n \ge 3$ and $c_n \equiv 4 \pmod 8$, define

$$j_{n+1} = \max\left\{ j' \in \mathbb{Z}_{\ge 0} : 4f\left(\frac{c_n + 2^{j_n} p_n}{4}\right) \equiv c \pmod{2^{j'}} \text{ for all } p_n \in \mathbb{Z}_2 \text{ for some fixed } c \in \mathbb{Z} \right\}$$

and define c_{n+1} to be the unique integer such that

$$c_{n+1} \equiv 4f\left(\frac{c_n + 2^{j_n} p_n}{4}\right) \pmod{2^{j_{n+1}}}$$

and $0 \le c_{n+1} < 2^{j_{n+1}}$; otherwise terminate $\{c_n\}$ and $\{j_n\}$ at index n. Then for $n \in \mathbb{Z}_{>0}$ for which c_n and j_n are defined, we have:

(a) If $j_n > 0$ and $c_n \ne 0$ then $|f^n(\alpha)| = 2^{2-e}$, where $e \in \mathbb{Z}_{\ge 0}$ such that $e+1 \le j_n$ and $c_n \equiv 2^e \pmod{2^{e+1}}$.

(b) If $j_n > 0$ and c_n is odd then $f \notin \mathcal{M}_{3,2}$.

(c) If $j_n \ge 3$ and $c_n \equiv 0 \pmod 8$ then $f \in \mathcal{M}_{3,2}$.

(d) If $c_n = c_{n+\ell}$ and $j_n = j_{n+\ell}$ for some $\ell \in \mathbb{Z}_{>0}$ then $f \in \mathcal{M}_{3,2}$.

We implemented Algorithm 4.7 in Mathematica [Bate et al. 2018]. To do so, we fixed an upper bound $N \in \mathbb{Z}_{>0}$ and computed c_n and j_n (when possible) for $1 \le n \le N$. For the computations in this section, we took $N = 50$. Of course, it may happen that we fail to classify f. This happens when $c_n = 0$ and $j_n \le 2$ or when $|f^n(\alpha)| \in \{1, 2\}$ for $1 \le n \le N$. Failure to classify in the former case is often due to the fact that $f_{\alpha_1,\beta_1} \notin \mathcal{M}_{3,2}$ and $f_{\alpha_2,\beta_2} \in \mathcal{M}_{3,2}$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}_2$ such that $\alpha_1 \equiv \alpha_2 \equiv a_0 \pmod{2^j}$ and $\beta_1 \equiv \beta_2 \equiv b_0 \pmod{2^k}$ (i.e., membership of $f_{\alpha,\beta}$ in $\mathcal{M}_{3,2}$ is ill-defined for $\alpha \equiv a_0 \pmod{2^j}$ and $\beta \equiv b_0 \pmod{2^k}$). In the latter case, we can sometimes gain resolution by choosing larger N, but doing so may be futile since there may be $f \in \mathcal{M}_{3,2}$ with $|f^n(\alpha)| \in \{1, 2\}$ for all $n \in \mathbb{Z}_{>0}$ which do not have preperiodicity in $\{|f^n(\alpha)|\}$ detectable by (4-9) or simply lack any preperiodicity at all.

We performed this computation for $\alpha, \beta \pmod{2^7}$; that is, we performed this computation for $j = k = 7$ and all $a_0, b_0 \in \mathbb{Z}$ with a_0 is odd, b_0 is even, and $0 \le a_0, b_0 < 2^7$. Figure 1 depicts these results for $0 \le a_0, b_0 < 30$. In Figure 1, the first column lists values for a_0, and the first row lists values for b_0. For given a_0 and b_0, the corresponding entry in the table is colored red if $f_{\alpha,\beta} \notin \mathcal{M}_{3,2}$, green if $f_{\alpha,\beta} \in \mathcal{M}_{3,2}$, and white if we cannot determine whether $f_{\alpha,\beta}$ is in $\mathcal{M}_{3,2}$.

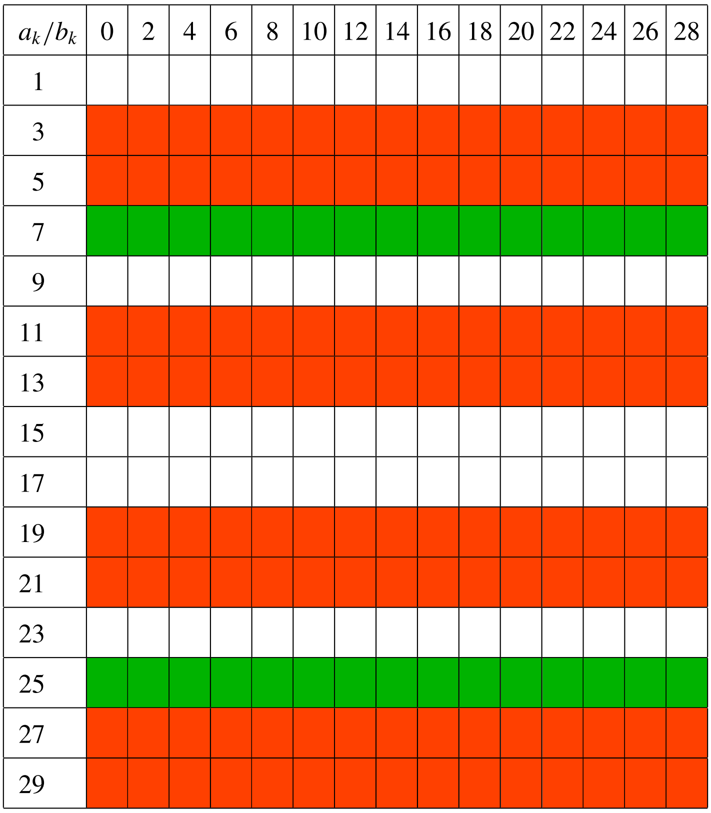

Figure 1. Partial analysis of $\mathcal{M}_{3,2}$ for α, $\beta \pmod{2^7}$.

Figure 1 suggests that the classification of $f_{\alpha,\beta}$ is identical to the classification of $f_{\alpha,0}$ for any α, $\beta \pmod{2^7}$. For $N = 50$ this is indeed the case, and appears to hold for larger N as well. Similar analysis for α, $\beta \pmod{2^j}$ for $j \le 9$ reveals the same type of conformity, but for $j \ge 10$ this pattern ceases. As an example of this, consider Figure 2, where we give a partial depiction of the results of this computation for α, $\beta \pmod{2^{15}}$ (we restrict to $0 \le a_0, b_0 < 110$). As before, an entry is colored red if $f_{\alpha,\beta} \notin \mathcal{M}_{3,2}$ and white if we cannot determine whether $f_{\alpha,\beta}$ is in $\mathcal{M}_{3,2}$. But instead of simply coloring an entry green if $f_{\alpha,\beta} \in \mathcal{M}_{3,2}$, we color it cyan (light blue) if $\lim_{n\to\infty} f^n_{\alpha,\beta}(\alpha) = 0$, yellow if $|f^n_{\alpha,\beta}(\alpha)| = \frac{1}{2}$ for all $n \ge M$ for some $M \in \mathbb{Z}_{>0}$, and blue if $\{|f^n_{\alpha,\beta}(\alpha)|\}$ exhibits the preperiodicity detected by (4-9). Delineation between these first two cases can be achieved by applying Algorithm 4.7(a) and Lemma 4.4.

Figure 2 suggests that if $\lim_{n\to\infty} f^n_{\alpha,0}(\alpha) = 0$ then $\lim_{n\to\infty} f^n_{\alpha,\beta}(\alpha) = 0$ for all $\beta \pmod{2^{15}}$. It also suggests that if $|f^n_{\alpha,0}| = \frac{1}{2}$ for all $n \ge M$, for some $M \in \mathbb{Z}_{>0}$, then the same is true for $f_{\alpha,\beta}$ for all $\beta \pmod{2^{15}}$. We know of no evidence that shows

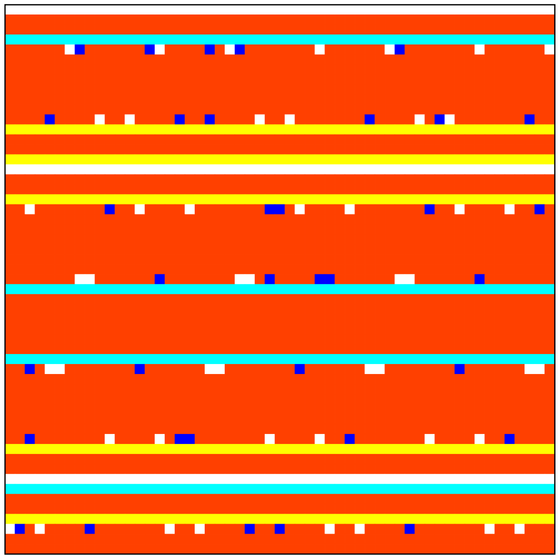

Figure 2. Partial analysis of $\mathcal{M}_{3,2}$ for $\alpha, \beta \pmod{2^{15}}$.

these observations do not persist in general. The differences which arise between $f_{\alpha,\beta}$ and $f_{\alpha,0}$ in Figure 2 occur in part because in some cases $f_{\alpha,\beta} \in \mathcal{M}_{3,2}$ due to (4-9) when $f_{\alpha,0} \notin \mathcal{M}_{3,2}$ (and vice versa). There exist other discrepancies between $f_{\alpha,\beta}$ and $f_{\alpha,0}$ due to the failure of our algorithm to find a classification. As mentioned earlier, we suspect these failures are in part due to a form of preperiodicity in $\{|f^n(\alpha)|\}$ not detected by (4-9). We will discuss other means of detecting preperiodicity, at least for $\{|f^n_{\alpha,0}(\alpha)|\}$, in Theorem 5.7 and the subsequent discussion.

Having discussed the case where $\alpha, \beta \in \mathbb{Z}_2$ with $\alpha + \beta$ odd, we now consider, as promised, the case where $\alpha = a/2,\ \beta = b/2$ for odd $a, b \in \mathbb{Z}_2$ such that $a + b \equiv 0 \pmod 4$.

Lemma 4.8. *Let* $f = f_{\alpha,\beta}$ *with* $\alpha = a/2,\ \beta = b/2$, *and odd* $a, b \in \mathbb{Z}_2$ *such that* $a + b \equiv 0 \pmod 4$. *If* $x \in \mathbb{Z}_2$ *then* $|f(x)| = 4|x|$.

Proof. Let $x \in \mathbb{Z}_2$. Thus $|x| = 2^{-k}$ for some $k \in \mathbb{Z}_{\geq 0}$. Since

$$\left| x^2 - \frac{3(a+b)}{4} x \right| \leq 1 \quad \text{and} \quad |3\alpha\beta| = \left| \frac{3ab}{4} \right| = 4,$$

by (2-1)

$$|f(x)| = |x| \left| x^2 - \frac{3(a+b)}{4} x + \frac{3ab}{4} \right| = 4|x|. \qquad \square$$

The following proposition follows from Lemmas 4.8 and 4.3.

Proposition 4.9. *Let $f = f_{\alpha,\beta}$ with $\alpha = a/2$, $\beta = b/2$, and odd $a, b \in \mathbb{Z}_2$ such that $a+b \equiv 0 \pmod 4$. If $|f^m(\alpha)| = 4^k$ for some $m \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}$ then $\{f^n(\alpha)\}$ is unbounded; hence $f \notin \mathcal{M}_{3,2}$.*

One might hope that we can obtain results similar to Algorithm 4.7 for $\alpha = a/2$, $\beta = b/2$ for odd $a, b \in \mathbb{Z}_2$ such that $a+b \equiv 0 \pmod 4$. However, the definition and resulting analysis of $\{c_n\}$ and $\{j_n\}$ given in Algorithm 4.7 fundamentally depend upon Proposition 4.6. By Lemma 4.8 we cannot have a nice analogue of Proposition 4.6(b), although Proposition 4.9 is an analogue of Proposition 4.6(a). Hence any algorithm that classifies such $f_{\alpha,\beta}$ would likely deviate from Algorithm 4.7 in some significant ways. We will not pursue this matter in this paper.

5. Analysis of $\mathcal{M}_{3,2} \cap \{f_{\alpha,0} : \alpha \in \mathbb{Q}_2\}$

In this section we consider $\mathcal{M}_{3,2} \cap \{f_{\alpha,0} : \alpha \in \mathbb{Q}_2\}$, where

$$f(x) = f_{\alpha,0}(x) = x^2\left(x - \frac{3\alpha}{2}\right) \in \mathcal{P}_{3,2}. \tag{5-1}$$

Restricting our attention to this set is worthwhile since, as was pointed out in Section 4, the dynamics of $f = f_{\alpha,0}$ appear representative of $f_{\alpha,\beta}$ for $\alpha \pmod{2^j}$ and $\beta \pmod{2^k}$. Although we could adapt Algorithm 4.7 slightly to analyze $\{f^n(\alpha)\}$ for fixed $\alpha \pmod{2^j}$, we will instead use Algorithm 5.3 which is more computationally efficient and produces results that bring greater clarity to the structure of $\mathcal{M}_{3,2} \cap \{f_{\alpha,0} : \alpha \in \mathbb{Q}_2\}$.

Anderson's critical radius bound, mentioned at the start of Section 4, together with Proposition 4.2(a,c) show that $f_{\alpha,0} \notin \mathcal{M}_{3,2}$ for $|\alpha| > 1$ and $f_{\alpha,0} \in \mathcal{M}_{3,2}$ for $|\alpha| < 1$. Therefore we restrict our attention to odd $\alpha \in \mathbb{Z}_2$. Rather than focusing on $\{|f^n(\alpha)|\}$ directly, we will instead analyze the sequence $\{x_n\}$ defined in the following lemma, which will serve as a proxy for our study of $\{|f^n(\alpha)|\}$.

Lemma 5.1. *Let*

$$x_n = -\frac{f^{n+1}(\alpha)}{2\alpha f^n(\alpha)^2} \tag{5-2}$$

for $n \in \mathbb{Z}_{>0}$. Then $x_1 = (3+\alpha^2)/4$ and

$$4x_n = \alpha^2 x_{n-1}(4x_{n-1} - 3)^2 + 3 \tag{5-3}$$

for $n \in \mathbb{Z}_{>1}$.

Proof. Since $f^{n+1}(\alpha) = f^n(\alpha)^2(f^n(\alpha) - 3\alpha/2)$,

$$x_n = -\frac{1}{2\alpha}\left(f^n(\alpha) - \frac{3\alpha}{2}\right) = \frac{3}{4} - \frac{f^n(\alpha)}{2\alpha}, \tag{5-4}$$

and so in particular,

$$x_1 = \frac{3}{4} - \frac{f(\alpha)}{2\alpha} = \frac{3}{4} - \frac{\alpha^2 \cdot (-\alpha/2)}{2\alpha} = \frac{3+\alpha^2}{4}.$$

For $n \in \mathbb{Z}_{>1}$, we have by (5-4) and (5-1) that

$$\begin{aligned} 4x_n &= 3 - \frac{2}{\alpha} f^n(\alpha) = 3 - \frac{2}{\alpha}\left(f^{n-1}(\alpha)^2\left(f^{n-1}(\alpha) - \frac{3\alpha}{2}\right)\right) \\ &= 3 - \frac{2}{\alpha} f^{n-1}(\alpha)^3 + 3 f^{n-1}(\alpha)^2 = 3 - \alpha^2\left(\frac{2f^{n-1}(\alpha)}{\alpha}\right)^2\left(\frac{f^{n-1}(\alpha)}{2\alpha} - \frac{3}{4}\right) \\ &= \alpha^2\left(\frac{3}{4} - \frac{f^{n-1}(\alpha)}{2\alpha}\right)\left(\frac{2f^{n-1}(\alpha)}{\alpha}\right)^2 + 3 = \alpha^2 x_{n-1}(4x_{n-1}-3)^2 + 3. \qquad \square \end{aligned}$$

Suppose

$$|f^k(\alpha)| \in \{1, 2\} \quad \text{for } 1 \le k \le n, \tag{5-5}$$

for some $n \in \mathbb{Z}_{>0}$. Such n must exist for the simple reason that by (5-1), $|f(\alpha)| = 2$. By (5-1),

$$|f^k(\alpha)| = 1 \quad \Longrightarrow \quad |f^{k+1}(\alpha)| = 2; \tag{5-6}$$

hence $|f^k(\alpha)| = |f^{k+1}(\alpha)| = 1$ is impossible. Thus by (5-2),

$$|x_k| = 2\left|\frac{f^{k+1}(\alpha)}{f^k(\alpha)^2}\right| = \begin{cases} 4 & \text{if } |f^k(\alpha)| = 1,\ |f^{k+1}(\alpha)| = 2, \\ \frac{1}{2} & \text{if } |f^k(\alpha)| = 2,\ |f^{k+1}(\alpha)| = 1, \\ 1 & \text{if } |f^k(\alpha)| = 2,\ |f^{k+1}(\alpha)| = 2 \end{cases} \tag{5-7}$$

for $k < n$. We wish to describe the possible values for $|x_n|$. If $|f^{n+1}(\alpha)| \in \{1, 2\}$ then by the same reasoning as in (5-7), we know that $|x_n| \in \left\{\frac{1}{2}, 1, 4\right\}$. If $|f^{n+1}(\alpha)| \ge 4$ then in actuality $|f^{n+1}(\alpha)| = 4$. To see why this is the case, notice that if $|f^n(\alpha)| \le 2$ then by (5-1) $|f^{n+1}(\alpha)| \le 4$. If $|f^{n+1}(\alpha)| = 4$ then by (5-6) $|f^n(\alpha)| = 2$, and so

$$|x_n| = 2\left|\frac{f^{n+1}(\alpha)}{f^n(\alpha)^2}\right| = 2. \tag{5-8}$$

Likewise, if $|f^{n+1}(\alpha)| \le \frac{1}{2}$ then by (5-6), $|f^n(\alpha)| = 2$, and so

$$|x_n| = 2\left|\frac{f^{n+1}(\alpha)}{f^n(\alpha)^2}\right| \le \frac{1}{4}. \tag{5-9}$$

These observations, along with Proposition 4.6, prove the following:

Proposition 5.2. *Let $f = f_{\alpha,0}$ with odd $\alpha \in \mathbb{Z}_2$. Let $\{x_n\}$ be as defined in* (5-2). *Let $n \in \mathbb{Z}_{>0}$ such that* (5-5) *holds, or equivalently, such that $|x_k| \in \left\{\frac{1}{2}, 1, 4\right\}$ for $1 \le k < n$:*

(a) *If $|x_n| = 2$ then $f \notin \mathcal{M}_{3,2}$.*

(b) *If $|x_n| \le \frac{1}{4}$ then $f \in \mathcal{M}_{3,2}$.*

We also have the following converse to Proposition 5.2(a): if $f = f_{\alpha,0} \notin \mathcal{M}_{3,2}$ then there exists $n \in \mathbb{Z}_{>0}$ such that $|x_k| \in \left\{\frac{1}{2}, 1, 4\right\}$ for $k < n$ and $|x_n| = 2$; this follows from (5-7) and (5-8) since if $f_{\alpha,0} \notin \mathcal{M}_{3,2}$ then there exists $n \in \mathbb{Z}_{>0}$ such that $|f^k(\alpha)| \in \{1, 2\}$ for $k \leq n$ and $|f^{n+1}(\alpha)| = 4$. Therefore since x_n is only dependent upon α^2 by Lemma 5.1, we have then that either both $f_{\alpha,0}, f_{-\alpha,0} \notin \mathcal{M}_{3,2}$ or both $f_{\alpha,0}, f_{-\alpha,0} \in \mathcal{M}_{3,2}$. Because of this, Algorithm 5.3 (which we are about to describe) will consider $\alpha^2 \pmod{2^m}$ for fixed $m \in \mathbb{Z}_{>0}$ rather than $\alpha \pmod{2^j}$ for fixed $j \in \mathbb{Z}_{>0}$

Fix $m \in \mathbb{Z}_{>0}$ and fix odd $d \in \mathbb{Z}$ such that d is a quadratic residue modulo 2^m and $0 \leq d < 2^m$; it is well known that d is a quadratic residue modulo 2^m if and only if $d \equiv 1 \pmod 8$. Consider odd $\alpha \in \mathbb{Z}_2$ such that $\alpha^2 \equiv d \pmod{2^m}$, or equivalently, $\alpha^2 = d + 2^m p$, where $p \in \mathbb{Z}_2$ is indeterminate. Let

$$g(x) = \tfrac{1}{4}(\alpha^2 x(4x-3)^2 + 3). \tag{5-10}$$

By Lemma 5.1, $x_n = g(x_{n-1}) = g^{n-1}(x_1)$, where $x_1 = (3+\alpha^2)/4$. Recall that in Section 4, we defined sequences $\{c_n\}$ and $\{j_n\}$ from which we could often determine the value of $|f^n(\alpha)|$. We employ the same approach here, defining (possibly finite) sequences $\{d_n\}$ and $\{m_n\}$ in a completely analogous way so as to determine the value of $|x_n| = |g^{n-1}(x_1)|$. Since we covered the aforementioned case in detail, we will give a more abbreviated treatment here, trusting that the reader has a solid grasp of our conventions regarding indeterminates.

We recursively define $\{d_n\}$ and $\{m_n\}$ as follows. Suppose $4g((d_n+2^{m_n}p_n)/4) \in \mathbb{Z}_2$; this is true for $n = 1$ if we let $d_1 = 3 + d$, $m_1 = m$, and $p_1 = p$. Let

$$m_{n+1} = \max\left\{ m' \in \mathbb{Z}_{\geq 0} : 4g\left(\frac{d_n + 2^{m_n} p_n}{4}\right) \equiv d' \pmod{2^{m'}} \text{ for some fixed } d' \in \mathbb{Z} \right\}. \tag{5-11}$$

Thus $4g((d_n + 2^{m_n} p_n)/4) = d_{n+1} + 2^{m_{n+1}} p_{n+1}$ for indeterminate $p_{n+1} \in \mathbb{Z}_2$ and unique $d_{n+1} \in \mathbb{Z}$ such that $0 \leq d_{n+1} < 2^{m_{n+1}}$. One can then show that

$$4x_{n+1} = 4g\left(\frac{d_n + 2^{m_n} p_n}{4}\right) = d_{n+1} + 2^{m_{n+1}} p_{n+1}. \tag{5-12}$$

If $4g((d_n + 2^{m_n} p_n)/4) \notin \mathbb{Z}_2$ then we terminate $\{d_n\}$ and $\{m_n\}$ at index n.

If $m_n > 0$ and $d_n \neq 0$, then there exists $e \in \mathbb{Z}_{\geq 0}$ such that $d_n \equiv 2^e \pmod{2^{e+1}}$ and $e + 1 \leq m_n$. Just as in (4-7),

$$d_n \equiv 2^e \pmod{2^{e+1}} \iff |x_n| = 2^{2-e}. \tag{5-13}$$

Thus by Proposition 5.2, $f \notin \mathcal{M}_{3,2}$ if $e = 1$ and $f \in \mathcal{M}_{3,2}$ if $e \geq 4$. We terminate $\{d_n\}$ and $\{m_n\}$ at index n in both of these cases. If $d_n = 0$ then just as in (4-8),

$$d_n \equiv 0 \pmod{2^{m_n}} \implies |x_n| \leq 2^{2-m_n}. \tag{5-14}$$

Therefore by Proposition 5.2(b), if $m_n \geq 4$ and $d_n = 0$ then $f \in \mathcal{M}_{3,2}$. If $m_n \leq 3$ and $d_n = 0$ then we cannot determine the value of $|x_n|$. Since we use Proposition 5.2 to determine whether f is in $\mathcal{M}_{3,2}$ and since the hypothesis of that proposition requires us to know $|x_k| \in \left\{\frac{1}{2}, 1, 4\right\}$ for all $k < n$ we can never afford to be ignorant of the value of $|x_n|$ when it comes to classifying f via $|x_{n+1}|$. In light of this, we terminate $\{d_n\}$ and $\{m_n\}$ at index n whenever $d_n = 0$.

The additional termination criteria that we just gave shows that we only compute d_{n+1} and m_{n+1} if

(i) $m_n \geq 1$ and $d_n \equiv 1 \pmod 2$ (i.e., $e = 0$),

(ii) $m_n \geq 3$ and $d_n \equiv 4 \pmod 8$ (i.e., $e = 2$), or

(iii) $m_n \geq 4$ and $d_n \equiv 8 \pmod{16}$ (i.e., $e = 3$).

One can show that in all of these cases, $4g((d_n + 2^{m_n} p_n)/4) \in \mathbb{Z}_2$. Thus the additional termination criteria makes it so we never need to check if $4g((d_n + 2^{m_n} p_n)/4) \in \mathbb{Z}_2$.

The preperiodicity condition (4-8) has the following analogue for d_n and m_n:

$$\text{if } d_n = d_{n+\ell} \text{ and } m_n = m_{n+\ell} \text{ for some } n, \ell \in \mathbb{Z}_{>0} \text{ then } f \in \mathcal{M}_{3,2}. \tag{5-15}$$

To see why this is true, first note that if $d_n = d_{n+\ell}$ and $m_n = m_{n+\ell}$ for some $n, \ell \in \mathbb{Z}_{>0}$ then $\{(d_n, m_n)\}$ is preperiodic. The termination criteria given above guarantees that each d_n and m_n satisfy either (i), (ii), or (iii). Thus by (5-13), if $\{(d_n, m_n)\}$ is preperiodic then $\{|x_n|\}$ is also preperiodic, with $|x_n| \in \left\{\frac{1}{2}, 1, 4\right\}$. Therefore by (5-7), $\{|f^n(\alpha)|\}$ is preperiodic.

We summarize the discussion above in the following result.

Algorithm 5.3. Fix $m \in \mathbb{Z}_{>0}$ and fix $d \in \mathbb{Z}$ such that $d \equiv 1 \pmod 8$ and $0 \leq d < 2^m$. Consider $\alpha \in \mathbb{Z}_2$ such that $\alpha^2 \equiv d \pmod{2^m}$. Let $f = f_{\alpha,0}$, $d_1 = 3 + d$ and $m_1 = m$. We recursively define sequences $\{d_n\}$ and $\{m_n\}$ as follows. If

(i) $m_n \geq 1$ and $d_n \equiv 1 \pmod 2$ (i.e., $e = 0$),

(ii) $m_n \geq 3$ and $d_n \equiv 4 \pmod 8$ (i.e., $e = 2$), or

(iii) $m_n \geq 4$ and $d_n \equiv 8 \pmod{16}$ (i.e., $e = 3$),

define

$$m_{n+1} = \max\left\{ m' \in \mathbb{Z}_{\geq 0} : 4g\left(\frac{d_n + 2^{m_n} p_n}{4}\right) \equiv d' \pmod{2^{m'}} \text{ for all } p_n \in \mathbb{Z}_2 \text{ for some fixed } d' \in \mathbb{Z} \right\}$$

and define d_{n+1} to be the unique integer such that

$$d_{n+1} \equiv 4g\left(\frac{d_n + 2^{m_n} p_n}{4}\right) \pmod{2^{m_{n+1}}}$$

and $0 \le d_{n+1} < 2^{m_{n+1}}$; otherwise terminate $\{d_n\}$ and $\{m_n\}$ at index n. Then for $n \in \mathbb{Z}_{>0}$ for which d_n and m_n are defined, we have:

(a) If $m_n > 0$ and $d_n \neq 0$ then $|x_n| = 2^{2-e}$, where $e \in \mathbb{Z}_{\geq 0}$ such that $e+1 \le m_n$ and $d_n \equiv 2^e \pmod{2^{e+1}}$.

(b) If $m_n \geq 2$ and $d_n \equiv 2 \pmod 4$ then $f \notin \mathcal{M}_{3,2}$.

(c) If $m_n \geq 4$ and $d_n \equiv 0 \pmod{16}$ then $f \in \mathcal{M}_{3,2}$.

(d) If $d_n = d_{n+\ell}$ and $m_n = m_{n+\ell}$ for some $\ell \in \mathbb{Z}_{>0}$ then $f \in \mathcal{M}_{3,2}$.

As we did with Algorithm 4.7, we implemented Algorithm 5.3 in Mathematica [Bate et al. 2018]. In the following computations, we computed d_n and m_n (when possible) for $1 \le n \le N$ with $N = 50$. As before, it may happen that our algorithm fails to classify f for the same reasons mentioned in Section 4.

It is well known that

$$\{\alpha^2 : \text{odd } \alpha \in \mathbb{Z}_2\} = \{x \in \mathbb{Z}_2 : x \equiv 1 \pmod 8\}. \tag{5-16}$$

If $x \equiv 1 \pmod 8$, then either $x \equiv 1 \pmod{16}$ or $x \equiv 1+8 \equiv 9 \pmod{16}$. More generally, if $x \equiv d \pmod{2^m}$ then either $x \equiv d \pmod{2^{m+1}}$ or $x \equiv d + 2^m \pmod{2^{m+1}}$. Topologically speaking, we are simply asserting that the disk

$$D(d, 2^{-m}) = \{x \in \mathbb{Q}_p : |x-d| \le 2^{-m}\} = \{x \in \mathbb{Q}_p : x \equiv d \pmod{2^m}\} \tag{5-17}$$

decomposes into the disjoint union of $D(d, 2^{-m-1})$ and $D(d+2^m, 2^{-m-1})$. These disks form a partial order under subset inclusion which can be visualized as a binary tree with $\{\alpha^2 : \text{odd } \alpha \in \mathbb{Z}_2\} = D(1, 2^{-3})$ as the root vertex, $D(1, 2^{-4})$ and $D(9, 2^{-4})$ as its child vertices, and so forth. We can think of Algorithm 5.3 as providing an algorithm for (possibly) determining if a given disk $D(d, 2^{-m})$ is contained entirely within or is entirely disjoint from $\mathcal{M}_{3,2}$.

In Figure 3 we display the results of this algorithm, along with supplemental classification afforded by Lemma 4.4 and Theorem 5.7, out to disks of radius 2^{-11}. A vertex with corresponding disk D is colored

(i) white if $D \cap \mathcal{M}_{3,2} \neq \varnothing$ and $D \not\subset \mathcal{M}_{3,2}$,

(ii) red if $D \cap \mathcal{M}_{3,2} = \varnothing$,

(iii) cyan if for all $\alpha \in D$ we have $\lim_{n\to\infty} f^n_{\alpha,0}(\alpha) = 0$,

(iv) yellow if for each $\alpha \in D$ there exists $M \in \mathbb{Z}_{>0}$ such that $|f^n_{\alpha,0}(\alpha)| = \frac{1}{2}$ for all $n \geq M$,

(v) green if D decomposes into disks of types (iii) and (iv),

(vi) blue if the preperiodicity condition (5-15) is satisfied by $f_{\alpha,0}$ for all $\alpha \in D$,

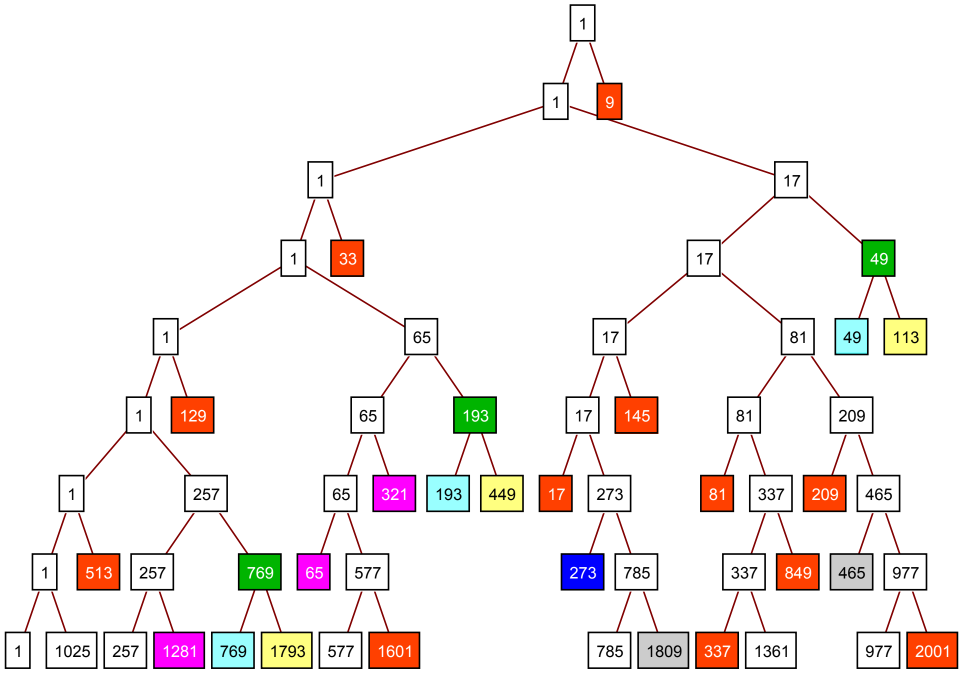

Figure 3. Analysis of $f_{\alpha,0}$ for $\alpha^2 \pmod{2^m}$ for $3 \le m \le 11$.

(vii) magenta if $D \subset D(1, 2^{-5})$ and if for each $\alpha \in D$ there exists $n \in \mathbb{Z}_{>0}$ such that $|f^n(\alpha)| = |f^{n+2}(\alpha)| = 2$ and $|f^{n+1}(\alpha)| = |f^{n+3}(\alpha)| = 1$ for $f = f_{\alpha,0}$ (in such a case, Theorem 5.7 proves $D \subset \mathcal{M}_{3,2}$), and

(viii) gray if we are unable to determine if $D \subset \mathcal{M}_{3,2}$, if $D \cap \mathcal{M}_{3,2} = \varnothing$, or if $D \cap \mathcal{M}_{3,2} \neq \varnothing$ and $D \not\subset \mathcal{M}_{3,2}$ (for the given N).

Disks of types (iii)–(vii) are all contained in $\mathcal{M}_{3,2}$, while disks of type (ii) are disjoint from $\mathcal{M}_{3,2}$. Disks of type (viii) occur when our algorithm is unable to classify a disk or any of the disks that it decomposes into. We can distinguish between disks of types (iii) and (iv) using Lemma 4.4 since we can determine $|f^n(\alpha)|$ from $|x_n|$. Indeed, by (5-2),

$$|f^{n+1}(\alpha)| = \tfrac{1}{2}\,|x_n|\,|f^n(\alpha)^2|. \tag{5-18}$$

We also recognize disks of type (vii) by examining $\{|f^n(\alpha)|\}$. As an aside, we mention that to our knowledge, there are no disks that are of both type (vi) and (vii). This indicates that the preperiodicity detected by Theorem 5.7 is of a subtler form than that of (5-15); we will discuss this distinction in more detail after Theorem 5.7.

Figure 4 shows the results of the computation above out to disks of radius 2^{-19}. There exists an obvious pattern of disks of types (ii), (iii), (iv), and (vii) along the left

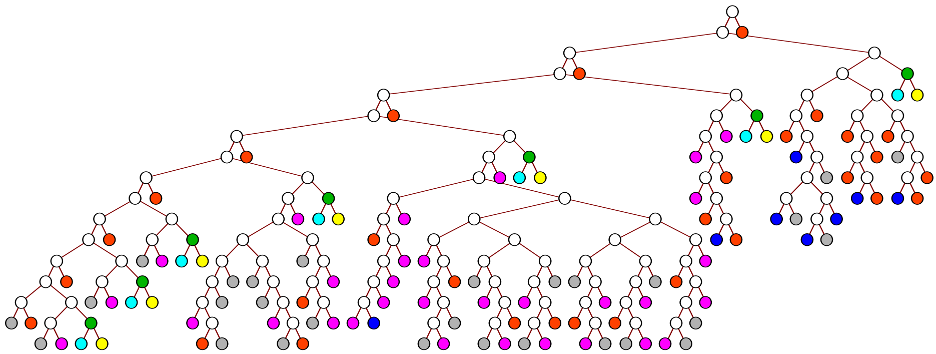

Figure 4. Analysis of $f_{\alpha,0}$ for $\alpha^2 \pmod{2^m}$ for $3 \le m \le 20$.

side of the tree. We prove that this pattern persists indefinitely in Theorem 5.5. As was pointed out in [Anderson 2013, §6], this pattern shows that the boundary of $\mathcal{M}_{3,2}$ at $\alpha = 1$ (and hence also $\alpha = -1$) has a degree of self-similarity. The disks of types (ii), (vi), and (vii) in the remainder of Figure 4 seem to indicate that there are other forms of self-similarity, although their exact characteristics seem difficult to quantify.

In Table 1 we list the values of $\{|f^n(\alpha)|\}$ obtained via Algorithm 5.3(a) and (5-18) for $\alpha^2 \pmod{2^m}$ for $3 \le m \le 11$. Table 1 has the following conventions:

- A listing terminates at index n if $|f^n(\alpha)| \ge 4$ or $|f^n(\alpha)| \le \frac{1}{2}$.
- If we know that $|f^n(\alpha)| \le \frac{1}{2}$ and nothing more, a listing terminates with $\frac{1}{2}^*$ at index n. A similar convention is used for $|f^n(\alpha)| \le \frac{1}{4}$.
- A listing ending with $[q_1 q_2 \cdots q_m]$ indicates $\{|f^n(\alpha)|\}$ has $q_1 q_2 \cdots q_m$ repeating indefinitely.
- A listing containing $(q_1 q_2 \cdots q_m)_k$ indicates $q_1 q_2 \cdots q_m$ repeats itself k times in $\{|f^n(\alpha)|\}$.
- If upon computing $\{|f^n_{\alpha_1,0}(\alpha_1)|\}$ and $\{|f^n_{\alpha_2,0}(\alpha_2)|\}$ for $\alpha_1^2 \equiv \alpha^2 \pmod{2^{m+1}}$ and $\alpha_2^2 \equiv \alpha^2 + 2^m \pmod{2^{m+1}}$ we find that $\{|f^n_{\alpha_1,0}(\alpha_1)|\}$ and $\{|f^n_{\alpha_2,0}(\alpha_2)|\}$ have in common a sequence which is longer than that obtained by computing $\{|f^n_{\alpha,0}(\alpha)|\}$, we give instead the sequence shared in common by $\{|f^n_{\alpha_1,0}(\alpha_1)|\}$ and $\{|f^n_{\alpha_2,0}(\alpha_2)|\}$ for $\{|f^n_{\alpha,0}(\alpha)|\}$. This means of determining $\{|f^n_{\alpha,0}(\alpha)|\}$ is applied recursively with the disks of radius 2^{-19} serving as the terminating cases.

Notice that the entries in Table 1 correspond to the vertices in Figure 3.[3] Of particular interest are the disks which our algorithm is unable to classify. In Figure 3 these occur for $D(465, 2^{-10})$ and $D(1809, 2^{-11})$. From Table 1, it seems very likely that $\{|f^n(\alpha)|\}$ is preperiodic for $\alpha^2 \in D(465, 2^{-10})$, although we do not present a

[3]The *Mathematica* notebook in [Bate et al. 2018] contains values of $|f^n(\alpha)|$ for vertices in Figure 4.

$\alpha^2 \pmod{2^m}$	$\lvert f^n(\alpha)\rvert$
$1 \pmod{2^3}$	22
$1 \pmod{2^4}$	22
$9 \pmod{2^4}$	224
$1 \pmod{2^5}$	222
$17 \pmod{2^5}$	22
$1 \pmod{2^6}$	222
$33 \pmod{2^6}$	2224
$17 \pmod{2^6}$	2212
$49 \pmod{2^6}$	$22\frac{1}{2}^*$
$1 \pmod{2^7}$	2222
$65 \pmod{2^7}$	222
$17 \pmod{2^7}$	22122
$81 \pmod{2^7}$	2212122
$49 \pmod{2^7}$	$22\frac{1}{4}^*$
$113 \pmod{2^7}$	$22\left[\frac{1}{2}\right]$
$1 \pmod{2^8}$	2222
$129 \pmod{2^8}$	22224
$65 \pmod{2^8}$	22212
$193 \pmod{2^8}$	$222\frac{1}{2}^*$
$17 \pmod{2^8}$	2212212
$145 \pmod{2^8}$	2212224
$81 \pmod{2^8}$	2212122
$209 \pmod{2^8}$	2212122
$1 \pmod{2^9}$	22222
$257 \pmod{2^9}$	2222
$65 \pmod{2^9}$	222122
$321 \pmod{2^9}$	22[21]
$193 \pmod{2^9}$	$222\frac{1}{4}^*$
$449 \pmod{2^9}$	$222\left[\frac{1}{2}\right]$
$17 \pmod{2^9}$	221221212224
$273 \pmod{2^9}$	2212212212
$81 \pmod{2^9}$	221212224
$337 \pmod{2^9}$	221212212
$209 \pmod{2^9}$	221212224
$465 \pmod{2^9}$	221212212
$1 \pmod{2^{10}}$	22222
$513 \pmod{2^{10}}$	222224
$257 \pmod{2^{10}}$	222212
$769 \pmod{2^{10}}$	$2222\frac{1}{2}^*$
$65 \pmod{2^{10}}$	22212[21]
$577 \pmod{2^{10}}$	2221222
$273 \pmod{2^{10}}$	[221]
$785 \pmod{2^{10}}$	$(221)_3212212$
$337 \pmod{2^{10}}$	221212212122
$849 \pmod{2^{10}}$	221212212224
$465 \pmod{2^{10}}$	$22121(221)_{11}2$
$977 \pmod{2^{10}}$	221212212122
$1 \pmod{2^{11}}$	222222
$1025 \pmod{2^{11}}$	22222
$257 \pmod{2^{11}}$	2222122
$1281 \pmod{2^{11}}$	222[21]
$769 \pmod{2^{11}}$	$2222\frac{1}{4}^*$
$1793 \pmod{2^{11}}$	$2222\left[\frac{1}{2}\right]$
$577 \pmod{2^{11}}$	222122212
$1601 \pmod{2^{11}}$	222122224
$785 \pmod{2^{11}}$	$(22122122121)_422122122$
$1809 \pmod{2^{11}}$	2212212212122121212212
$337 \pmod{2^{11}}$	22121221212224
$1361 \pmod{2^{11}}$	$(22121)_322$
$977 \pmod{2^{11}}$	$(22121)_322$
$2001 \pmod{2^{11}}$	22121221212224

Table 1. Values of $\lvert f^n(\alpha)\rvert$ for $\alpha^2 \pmod{2^m}$.

proof for this. In contrast, there appears to be no such preperiodicity in $\{|f^n(\alpha)|\}$ for $\alpha^2 \in D(1809, 2^{-11})$. In fact, closer inspection of α^2 in smaller disks within $D(1809, 2^{-11})$ results in $\{|f^n(\alpha)|\}$ which are composed of blocks of 21 and 221, but with no apparent preperiodicity emerging. We are uncertain whether such preperiodicity detection is simply beyond our computational limits or whether there is no preperiodicity to be found at all.

In addition to these observations, there are two interesting patterns displayed in Table 1 that are worth pointing out. The first is that if $\{|f^n(\alpha)|\}$ begins with a block of k 2's followed by a 1, then if a block of k 2's reappears later in the sequence, $f \notin \mathcal{M}_{3,2}$. The second is that if at any point $|f^n(\alpha)| = 1$ then $\lim_{n\to\infty} f^n(\alpha) \neq 0$. All numerical evidence we've seen confirms these two observations, but we've been unable to prove either.

The following lemma is used in the proof of Theorem 5.5.

Lemma 5.4. *Let $\alpha \in \mathbb{Z}_2$ odd and $f = f_{\alpha,0}$. If $\alpha^2 \equiv 1 + c \cdot 2^{2n} \pmod{2^{2n+\ell}}$ for $n \in \mathbb{Z}_{\geq 2}$, $c \in \mathbb{Z}$, and $\ell = 1, 2, 3$, then*

$$x_k \equiv 1 + c \cdot 5 \cdot 2^{2(n-k)} \pmod{2^{2(n-k)+\ell}} \tag{5-19}$$

for all k such that $2 \leq k \leq n$. Recall x_k is defined in (5-2).

Proof. Since $\alpha^2 \equiv 1 + c \cdot 2^{2n} \pmod{2^{2n+\ell}}$, we have $\alpha^2 \equiv 1 + c \cdot 2^{2n} + d \cdot 2^{2n+\ell}$ for some $d \in \mathbb{Z}_2$. Thus by Lemma 5.1,

$$\begin{aligned} x_1 = \frac{3+\alpha^2}{4} &= \frac{3 + 1 + c \cdot 2^{2n} + d \cdot 2^{2n+\ell}}{4} = 1 + c \cdot 2^{2(n-1)} + d \cdot 2^{2(n-1)+\ell} \\ &\equiv 1 + c \cdot 2^{2(n-1)} \pmod{2^{2(n-1)+\ell}}. \end{aligned} \tag{5-20}$$

Observe

$$\begin{aligned} (4x_1 - 3)^2 \equiv (4 + c \cdot 2^{2(n-1)+2} - 3)^2 &\equiv 1 + c \cdot 2^{2(n-1)+3} + c^2 \cdot 2^{4(n-1)+4} \\ &\equiv 1 \pmod{2^{2(n-1)+\ell}}. \end{aligned} \tag{5-21}$$

Since $\alpha^2 \equiv 1 + c \cdot 2^{2n} \pmod{2^{2(n-1)+\ell}}$, by Lemma 5.1, (5-20), and (5-21)

$$\begin{aligned} 4x_2 = \alpha^2 x_1 (4x_1 - 3)^2 + 3 &\equiv (1 + c \cdot 2^{2n})(1 + c \cdot 2^{2(n-1)}) + 3 \\ &\equiv 4 + c \cdot 2^{2(n-1)} + c \cdot 2^{2n} + c^2 \cdot 2^{2n+2(n-1)} \equiv 4 + c \cdot (1+4) 2^{2(n-1)} \\ &\equiv 4 + c \cdot 5 \cdot 2^{2(n-1)} \pmod{2^{2(n-1)+\ell}}; \end{aligned}$$

here we used the fact that $2n + 2(n-1) \geq 2(n-1) + \ell$ since $n \geq 2$. Therefore

$$x_2 \equiv 1 + c \cdot 5 \cdot 2^{2(n-2)} \pmod{2^{2(n-2)+\ell}}.$$

Thus (5-19) is satisfied for $k = 2$. Suppose (5-19) holds for some k such that $2 \le k < n$. Then

$$\begin{aligned}(4x_k - 3)^2 &\equiv (1 + c \cdot 5 \cdot 2^{2(n-k)+2})^2 \\ &\equiv 1 + c \cdot 5 \cdot 2^{2(n-k)+3} + c^2 \cdot 5^2 \cdot 2^{4(n-k)+4} \\ &\equiv 1 \pmod{2^{2(n-k)+\ell}}. \end{aligned} \tag{5-22}$$

Since $\alpha^2 \equiv 1 + c \cdot 2^{2n} \equiv 1 \pmod{2^{2(n-k)+\ell}}$ for $k \ge 2$, by Lemma 5.1 and (5-22)

$$4x_{k+1} = \alpha^2 x_k (4x_k - 3)^2 + 3 \equiv x_k + 3 \equiv 4 + c \cdot 5 \cdot 2^{2(n-k)} \pmod{2^{2(n-k)+\ell}}.$$

Thus $x_{k+1} \equiv 1 + c \cdot 5 \cdot 2^{2(n-(k+1))} \pmod{2^{2(n-(k+1))+\ell}}$. By induction, (5-19) holds for all k such that $2 \le k \le n$. □

Parts (a) and (b) of the following theorem were proved in a somewhat different form in [Anderson 2013, §6]. All available numerical evidence indicates the converses of parts (b) and (c) are true; however a proof of this has remained elusive.

Theorem 5.5. *Let $\alpha \in \mathbb{Z}_2$ odd and $f = f_{\alpha,0}$. Suppose $\alpha^2 \equiv 1 + c \cdot 2^{2n} \pmod{2^{2n+\ell}}$ for $n \in \mathbb{Z}_{\ge 2}$, $c \in \mathbb{Z}$, and $\ell = 1, 2, 3$. Then for all $k \le n$, we have $|f^k(\alpha)| = 2$. Furthermore*:

- (a) *If $\alpha^2 \equiv 1 + 2^{2n+1} \pmod{2^{2n+2}}$ then $f \notin \mathcal{M}_{3,2}$.*
- (b) *If $\alpha^2 \equiv 1 + 3 \cdot 2^{2n} \pmod{2^{2n+3}}$ then $\lim_{n\to\infty} f^n(\alpha) = 0$.*
- (c) *If $\alpha^2 \equiv 1 + 7 \cdot 2^{2n} \pmod{2^{2n+3}}$ then $|f^{n+j}(\alpha)| = \frac{1}{2}$ for all $j \in \mathbb{Z}_{>0}$.*
- (d) *If $\alpha^2 \equiv 1 + 5 \cdot 2^{2n} \pmod{2^{2n+3}}$ for $n \ge 3$ then*

$$|f^{n+2j}(\alpha)| = 2 \quad \text{and} \quad |f^{n+2j+1}(\alpha)| = 1$$

for all $j \in \mathbb{Z}_{\ge 0}$.

Proof of Theorem 5.5(a)–(c). By Lemma 5.4, $|x_k| = 1$ for all $k < n$. Since $|f(\alpha)| = 2$, by (5-7) $|f^k(\alpha)| = 2$ for all $k \le n$.

For part (a), $\alpha^2 \equiv 1 + c \cdot 2^{2n} \pmod{2^{2n+\ell}}$ for $c = 2$ and $\ell = 2$. By Lemma 5.4, $x_n \equiv 3 \pmod 4$; hence $|x_n| = 1$. Furthermore, by Lemma 5.1,

$$4x_{n+1} = \alpha^2 x_n (4x_n - 3)^2 + 3 \equiv 1 \cdot 3 \cdot 9^2 + 3 \equiv 2 \pmod 4.$$

Therefore $|x_{n+1}| = 2$. By Proposition 5.2(a), $f \notin \mathcal{M}_{3,2}$.

For part (b), $\alpha^2 \equiv 1 + c \cdot 2^{2n} \pmod{2^{2n+\ell}}$ for $c = 3$ and $\ell = 3$. By Lemma 5.4, $x_n \equiv 0 \pmod 8$. Thus $|x_n| \le \frac{1}{8}$. Since $|f^n(\alpha)| = 2$, by (5-18) $|f^{n+1}(\alpha)| \le \frac{1}{4}$. By Lemma 4.4(a), $\lim_{n\to\infty} f^n(\alpha) = 0$.

For part (c), $\alpha^2 \equiv 1 + c \cdot 2^{2n} \pmod{2^{2n+\ell}}$ for $c = 7$ and $\ell = 3$. By Lemma 5.4, $x_n \equiv 4 \pmod 8$. Thus $|x_n| = \frac{1}{4}$. Since $|f^n(\alpha)| = 2$, by (5-18) $|f^{n+1}(\alpha)| = \frac{1}{2}$. By Lemma 4.4(b), $|f^{n+j}(\alpha)| = \frac{1}{2}$ for $j \in \mathbb{Z}_{>0}$. □

We need the following lemma to prove part (d) of Theorem 5.5.

Lemma 5.6. *Let* $\alpha \in \mathbb{Z}_2$ *such that* $\alpha^2 \equiv 1 \pmod{32}$. *Let* $f = f_{\alpha,0}$. *If* $x_n \equiv 2 \pmod 8$ *then* $x_{n+2} \equiv 2 \pmod 8$. *Recall* x_n *is defined in* (5-2).

Proof. Suppose $x_n \equiv 2 \pmod 8$. Then $x_n = 2+8c_1$ for some $c_1 \in \mathbb{Z}_2$. By Lemma 5.1,

$$\begin{aligned}4x_{n+1} &= \alpha^2 x_n(4x_n-3)^2+3 \equiv (2+8c_1)(8+32c_1-3)^2+3\\ &\equiv (2+8c_1)\cdot 25+3 \equiv 21+8c_1 \pmod{32}.\end{aligned}$$

Thus there exists $c_2 \in \mathbb{Z}_2$ such that $x_{n+1} = (21+8c_1+32c_2)/4$. By Lemma 5.1,

$$\begin{aligned}4x_{n+2} &= \alpha^2 x_{n+1}(4x_{n+1}-3)^2+3\\ &= \alpha^2\, \tfrac{1}{4}\,(21+8c_1+32c_2)(18+8c_1+32c_2)^2+3\\ &= \alpha^2(21+8c_1+32c_2)(9+4c_1+16c_2)^2+3.\end{aligned}$$

By expanding and then reducing coefficients modulo 32, we find that

$$(21+8c_1+32c_2)(9+4c_1+16c_2)^2 \equiv 5+16(1+c_1)c_1 \equiv 5 \pmod{32};$$

here we used that $(1+c_1)c_1$ is even. Therefore

$$4x_{n+2} \equiv (21+8c_1)(9+4c_1+16c_2)^2+3 \equiv 8 \pmod{32}.$$

Thus $x_{n+2} \equiv 2 \pmod 8$. □

Proof of Theorem 5.5(d). Suppose $\alpha^2 \equiv 1+5\cdot 2^{2n} \pmod{2^{2n+3}}$ for $n \geq 3$. Then $\alpha^2 \equiv 1+c\cdot 2^{2n} \pmod{2^{2n+\ell}}$ for $c=5$ and $\ell=3$. By Lemma 5.4, $x_n \equiv 1+5^2 \equiv 2 \pmod 8$. Since $\alpha^2 \equiv 1+5\cdot 2^{2n} \equiv 1 \pmod{32}$, by Lemma 5.6 and induction $x_{n+2j} \equiv 2 \pmod 8$ for all $j \in \mathbb{Z}_{\geq 0}$, and hence $|x_{n+2j}| = \frac{1}{2}$ for all $j \in \mathbb{Z}_{\geq 0}$. Therefore by (5-6) and (5-7), $|x_{n+2j+1}| = 4$ for all $j \in \mathbb{Z}_{\geq 0}$. Thus by (5-7), $|f^{n+2j}(\alpha)| = 2$ and $|f^{n+2j+1}(\alpha)| = 1$ for all $j \in \mathbb{Z}_{\geq 0}$. □

Theorem 5.7. *Let* $\alpha \in \mathbb{Z}_2$ *such that* $\alpha^2 \equiv 1 \pmod{32}$. *Let* $f = f_{\alpha,0}$. *If for some* $n \in \mathbb{Z}_{>0}$ $|f^n(\alpha)| = |f^{n+2}(\alpha)| = 2$ *and* $|f^{n+1}(\alpha)| = |f^{n+3}(\alpha)| = 1$ *then*

$$|f^{n+2j}(\alpha)| = 2 \quad \text{and} \quad |f^{n+2j+1}(\alpha)| = 1$$

for all $j \in \mathbb{Z}_{\geq 0}$.

Proof. Since $|f^n(\alpha)| = |f^{n+2}(\alpha)| = 2$ and $|f^{n+1}(\alpha)| = |f^{n+3}(\alpha)| = 1$, by (5-7) $|x_n| = \frac{1}{2}$, $|x_{n+1}| = 4$, and $|x_{n+2}| = \frac{1}{2}$. Since $|x_n| = \frac{1}{2}$, we have $x_n = 2c$ for some odd $c \in \mathbb{Z}_2$. Therefore by Lemma 5.1,

$$\begin{aligned}4x_{n+2} &= \alpha^2 x_{n+1}(4x_{n+1}-3)^2+3\\ &= \alpha^2\, \tfrac{1}{4}(\alpha^2 x_n(4x_n-3)^2+3)((\alpha^2 x_n(4x_n-3)^2+3)-3)^2+3\\ &= \alpha^2\, \tfrac{1}{4}(\alpha^2(2c)(8c-3)^2+3)(\alpha^2(2c)(8c-3)^2)^2+3\end{aligned}$$

$$= \alpha^2(\alpha^2(2c)(8c-3)^2+3)(\alpha^2 c(8c-3)^2)^2+3$$
$$\equiv (2c(8c-3)^2+3)(c(8c-3)^2)^2+3 \equiv 3+3c^2+2c^3 \pmod{16};$$

in this last congruence we expanded the polynomial and reduced coefficients modulo 16. If $c \equiv 3 \pmod 4$ then $4x_{n+2} \equiv 84 \equiv 4 \pmod{16}$, and so $|x_{n+2}| = 1$, a contradiction. Therefore $c \equiv 1 \pmod 4$. Since $x_n = 2c$, we have $x_n \equiv 2 \pmod 8$. By Lemma 5.6 and induction, we find that $x_{n+2j} \equiv 2 \pmod 8$ for all $j \in \mathbb{Z}_{\geq 0}$. As the proof of Theorem 5.5(d) shows, from this we can then conclude that $|f^{n+2j}(\alpha)| = 2$ and $|f^{n+2j+1}(\alpha)| = 1$ for all $j \in \mathbb{Z}_{\geq 0}$. □

We reiterate that Theorem 5.7 detects preperiodicity in $\{|f^n(\alpha)|\}$ that (5-15) does not detect. Indeed, for $\alpha^2 \equiv 321 \pmod{2^9}$ we find that

$$\{d_n\} = \{324, 20, 8, 5, 8, 1, 0\} \quad \text{and} \quad \{m_n\} = \{9, 7, 5, 3, 4, 2, 2\},$$

which clearly does not satisfy (5-15). Yet from these sequences we find that $\{|f^n(\alpha)|\} = \{2, 2, 2, 1, 2, 1, 2, \ldots\}$, and so the hypothesis of Theorem 5.7 is fulfilled.

Lemma 5.6 is the key to the proofs of Theorem 5.5(d) and Theorem 5.7. We suspect there may be other results such as Lemma 5.6 where a congruence condition on α^2 forces a form of preperiodicity in x_n. If this is the case, then there may exist other results similar to Theorem 5.7 which allow for further detection of preperiodicity in $\{|f^n(\alpha)|\}$.

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