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Sidon sets and 2-caps in $\mathbb{F}_3^n$

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For each natural number d, we introduce the concept of a d-cap in $\mathbb{F}_3^n$. A set of points in $\mathbb{F}_3^n$ is called a d-cap if, for each $k = 1, 2, \dots, d$, no $k+2$ of the points lie on a k-dimensional flat. This generalizes the notion of a cap in $\mathbb{F}_3^n$. We prove that the 2-caps in $\mathbb{F}_3^n$ are exactly the Sidon sets in $\mathbb{F}_3^n$ and study the problem of determining the size of the largest 2-cap in $\mathbb{F}_3^n$.

1. Introduction

Throughout, let $\mathbb{F}_q$ denote the field with q elements and let $\mathbb{F}_q^n$ denote n-dimensional affine space over $\mathbb{F}_q$. A *cap* in $\mathbb{F}_3^n$ is a collection of points such that no three are collinear. Although this definition is geometric, there is an equivalent definition that is arithmetic: a set of points C is a cap in $\mathbb{F}_3^n$ if and only if C contains no three-term arithmetic progressions.

Here, we consider natural generalizations of caps in $\mathbb{F}_3^n$. For $d \in \mathbb{N}$, we call a set of points a *d-cap* if, for each $k = 1, 2, \dots, d$, no $k+2$ of the points lie on a k-dimensional flat. With this definition, a 1-cap corresponds to the usual definition of a cap. We also remark that if C is a set of points in $\mathbb{F}_3^n$, then the points of C are in general linear position if and only if C is an $(n-1)$-cap.

Let $r(1, \mathbb{F}_3^n)$ denote the maximal size of a 1-cap in $\mathbb{F}_3^n$. In general, it is a difficult problem to determine $r(1, \mathbb{F}_3^n)$ — in fact, the exact answer is known only when $n \le 6$. Table 1 lists the best known upper and lower bounds on $r(1, \mathbb{F}_3^n)$ for $n \le 10$ [Versluis 2017]. It is also known that in dimension $n \le 6$, maximal 1-caps are equivalent up to affine transformation [Edel et al. 2002; Pellegrino 1970; Potechin 2008].

The asymptotic bounds on $r(1, \mathbb{F}_3^n)$ are well-studied. Edel [2004] showed that

$$\limsup_{n\to\infty} \frac{\log_3(r(1, \mathbb{F}_3^n))}{n} \ge 0.724851$$

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dimension	1	2	3	4	5	6	7	8	9	10
lower bound	2	4	9	20	45	112	236	496	1064	2240
upper bound	2	4	9	20	45	112	291	771	2070	5619

Table 1. The best known bounds for the size of a maximal 1-cap in $\mathbb{F}_3^n$.

dimension	1	2	3	4	5	6	7	8	n even	n odd
lower bound	2	3	5	9	13	27	33	81	$3^{n/2}$	$3^{(n-1)/2}+1$
upper bound	2	3	5	9	13	27	47	81	$3^{n/2}$	$\lceil 3^{n/2} \rceil$

Table 2. Bounds for the size of a maximal 2-cap in $\mathbb{F}_3^n$.

and consequently that $r(1, \mathbb{F}_3^n)$ is $\Omega(2.2174^n)$ (using Hardy and Littlewood's Ω notation). In more recent breakthrough work Ellenberg and Gijswijt [2017] (adapting a method of Croot, Lev, and Pach in [Croot et al. 2017]) proved that $r(1, \mathbb{F}_3^n)$ is $o(2.756^n)$.

In this paper, we focus on the study of 2-caps in $\mathbb{F}_3^n$. We show that there is an equivalent arithmetic formulation of the definition of a 2-cap. In particular, the 2-caps in $\mathbb{F}_3^n$ are exactly the Sidon sets in $\mathbb{F}_3^n$, which are important objects in combinatorial number theory (we refer the interested reader to the survey [O'Bryant 2004]). Using this definition, we are able to compute the exact maximal size of a 2-cap in $\mathbb{F}_3^n$ when n is even. We also examine 2-caps in low dimension when n is odd, in particular considering dimensions $n = 3$, 5, and 7.

Table 2 lists the bounds we obtain for the size of a maximal 2-cap in $\mathbb{F}_3^n$. The values in dimension 3, 5, and 7 are given by Theorems 3.9 and 3.10, and Proposition 3.12, respectively. The bounds for even dimension follow from Theorem 3.4. The upper bound in odd dimension n follows from Proposition 3.3 and the lower bound is given by adding one affinely independent point to the construction in dimension $n-1$. Knowing the exact value in even dimension also allows us to conclude that asymptotically, the maximal size of a 2-cap in $\mathbb{F}_3^n$ is $\Theta(3^{n/2})$.

2. Preliminaries

In this section, we establish basic notation, definitions, and background. The set of natural numbers is denoted by $\mathbb{N} = \{1, 2, 3, \dots\}$. Throughout, d and n will always denote natural numbers. An element $\boldsymbol{a} \in \mathbb{F}_3^n$ will be written as a row vector $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$, with each $a_i \in \{0, 1, 2\}$. We will sometimes order the vectors of $\mathbb{F}_3^n$ lexicographically — i.e., by regarding them as ternary strings. We use the notation $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ to denote the n standard basis vectors in an n-dimensional vector space.

A k-dimensional affine subspace of a vector space is called a *k-dimensional flat*. In particular, a 1-dimensional flat is also called a *line*. In the affine space $\mathbb{F}_3^n$, every line consists of the points $\{\boldsymbol{a}, \boldsymbol{a}+\boldsymbol{b}, \boldsymbol{a}+2\boldsymbol{b}\}$ for some $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{F}_3^n$, where $\boldsymbol{b} \neq \boldsymbol{0}$. Hence, the lines in $\mathbb{F}_3^n$ correspond to three-term arithmetic progressions. It is easy to see that three distinct points in $\mathbb{F}_3^n$ are collinear if and only if they sum to $\boldsymbol{0}$. Likewise, a 2-dimensional flat is called a *plane*. Any three noncollinear points determine a unique plane. For $\boldsymbol{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{F}_3^k$ with $k < n$, the subset of $\mathbb{F}_3^n$ whose first k entries are $a_1, a_2, \ldots, a_k$ is an $(n-k)$-dimensional flat which we call *the $\boldsymbol{a}$-affine subspace* of $\mathbb{F}_3^n$.

Two subsets C and D of a vector space are called *affinely equivalent* if there exists an invertible affine transformation T such that $T(C) = D$. It is clear that affine equivalence determines an equivalence relation on the power set of a vector space. Given a set of points X in a vector space, its affine span is given by the set of all affine combinations of points of X. A set X is called *affinely independent* if no proper subset of X has the same affine span as X. Equivalently, $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ is affinely independent if and only if $\{\boldsymbol{x}_1 - \boldsymbol{x}_0, \boldsymbol{x}_2 - \boldsymbol{x}_0, \ldots, \boldsymbol{x}_n - \boldsymbol{x}_0\}$ is linearly independent.

Definition 2.1. A subset C of $\mathbb{F}_3^n$ is called a *d-cap* if, for each $k = 1, 2, \ldots, d$, no $k+2$ points of C lie on a k-dimensional flat. Equivalently, C is a d-cap if and only if any subset of C of size at most $d+2$ is affinely independent. A d-cap is called *complete* if it is not a proper subset of another d-cap and is called *maximal* if it is of the largest possible cardinality.

As mentioned in the Introduction, a 1-cap is a classical cap. We will denote the size of a maximal d-cap in $\mathbb{F}_3^n$ by $r(d, \mathbb{F}_3^n)$. We remark that since invertible affine transformations preserve affine independence, the image of a d-cap under an invertible affine transformation is again a d-cap. As a warm-up, we prove some basic facts about maximal d-caps in $\mathbb{F}_3^n$.

Lemma 2.2. *We have that $r(d, \mathbb{F}_3^n) \geq n+1$ with equality if $n \leq d$.*

Proof. The set $\{\boldsymbol{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is an affinely independent subset of $\mathbb{F}_3^n$ of size $n+1$ and hence is a d-cap for any $d \in \mathbb{N}$. Therefore, $r(d, \mathbb{F}_3^n) \geq n+1$.

Now suppose $n \leq d$. Since, by definition, a d-cap must be an n-cap, we have that $r(d, \mathbb{F}_3^n) \leq r(n, \mathbb{F}_3^n)$. A maximal affinely independent set in $\mathbb{F}_3^n$ has size $n+1$ so $r(n, \mathbb{F}_3^n) \leq n+1$, and so $r(d, \mathbb{F}_3^n) = n+1$. □

Corollary 2.3. *When $n \leq d$, all maximal d-caps in $\mathbb{F}_3^n$ are affinely equivalent.*

Proof. By Lemma 2.2, when $n \leq d$, a maximal d-cap in $\mathbb{F}_3^n$ is a maximal affinely independent set, i.e., an affine basis of $\mathbb{F}_3^n$. All affine bases in an affine space are equivalent up to affine transformation. □

Lemma 2.4. *For fixed d, $r(d, \mathbb{F}_3^n)$ is a nondecreasing function of n and for fixed n, $r(d, \mathbb{F}_3^n)$ is a nonincreasing function of d.*

Proof. Since $\mathbb{F}_3^{n-1}$ is an affine subspace of $\mathbb{F}_3^n$, a d-cap in $\mathbb{F}_3^{n-1}$ naturally embeds as a d-cap in $\mathbb{F}_3^n$. Hence $r(d, \mathbb{F}_3^{n-1}) \leq r(d, \mathbb{F}_3^n)$ so the first statement follows. The second statement follows since, by definition, a d-cap in $\mathbb{F}_3^n$ must be a $(d-1)$-cap. Hence, $r(d-1, \mathbb{F}_3^n) \geq r(d, \mathbb{F}_3^n)$. □

3. 2-caps in $\mathbb{F}_3^n$

We now restrict our attention to the study of 2-caps in $\mathbb{F}_3^n$. Our first observation is that in $\mathbb{F}_3^n$, the definition of a 2-cap is equivalent to the definition of a Sidon set.

Definition 3.1. Let G be an abelian group. A subset $A \subseteq G$ is called a *Sidon set* if, whenever $a+b=c+d$ with $a, b, c, d \in A$, the pair (a, b) is a permutation of the pair (c, d).

Theorem 3.2. *A subset C of $\mathbb{F}_3^n$ is a 2-cap if and only if it is a Sidon set.*

Proof. First suppose that C is not a 2-cap. Then C contains three points which are collinear or C contains four points which are coplanar. If C contains three distinct collinear points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ then $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\boldsymbol{0}$ and hence $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{c}+\boldsymbol{c}$ so C is not a Sidon set.

Suppose therefore that no three points in C are collinear. Then C contains four coplanar points, say $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$. Every set of three distinct noncollinear points in $\mathbb{F}_3^n$ lies on a unique 2-dimensional flat. In particular, the 2-dimensional flat F containing $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is given by

$F =$

$\boldsymbol{a}$	$\boldsymbol{b}$	$-\boldsymbol{a}-\boldsymbol{b}$
$\boldsymbol{c}$	$-\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$	$\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}$
$-\boldsymbol{a}-\boldsymbol{c}$	$\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}$	$-\boldsymbol{b}-\boldsymbol{c}$

and since we assumed that no three points in C are collinear, we must have that $\boldsymbol{d}=-\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$, $\boldsymbol{d}=\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}$ or $\boldsymbol{d}=\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}$. In the first case, $\boldsymbol{a}+\boldsymbol{d}=\boldsymbol{b}+\boldsymbol{c}$, in the second case, $\boldsymbol{b}+\boldsymbol{d}=\boldsymbol{a}+\boldsymbol{c}$, and in the third case $\boldsymbol{c}+\boldsymbol{d}+\boldsymbol{a}+\boldsymbol{b}$. In any case, C is not a Sidon set.

Conversely, suppose that C is not a Sidon set. Then either C contains three distinct points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ such that $\boldsymbol{a}+\boldsymbol{a}=\boldsymbol{b}+\boldsymbol{c}$, or C contains four distinct points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ such that $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{c}+\boldsymbol{d}$. In the first case, $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\boldsymbol{0}$ so C contains a line. In the second case, $\boldsymbol{d}=\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}$, so $\boldsymbol{d}$ lies in the plane determined by $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, and hence the four points are coplanar. In either case, C is not a 2-cap. □

Since, in $\mathbb{F}_3^n$, 2-caps correspond to Sidon sets, we will use the terms interchangeably throughout. We obtain an upper bound on $r(2, \mathbb{F}_3^n)$ by an easy counting argument; see [Cilleruelo et al. 2010, Corollary 2.2].

Proposition 3.3. *For any $n \in \mathbb{N}$,*

$$r(2, \mathbb{F}_3^n) \cdot (r(2, \mathbb{F}_3^n) - 1) \leq 3^n - 1.$$

Proof. Suppose $C \subset \mathbb{F}_3^n$ is a 2-cap and hence, by Theorem 3.2, a Sidon set. For $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in C$, if $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{c} - \boldsymbol{d}$ then $\{\boldsymbol{a}, \boldsymbol{d}\} = \{\boldsymbol{c}, \boldsymbol{b}\}$ and so we have either $\boldsymbol{a} = \boldsymbol{b}$, or else $\boldsymbol{a} = \boldsymbol{c}$ and $\boldsymbol{b} = \boldsymbol{d}$. Therefore, the set $\{\boldsymbol{a} - \boldsymbol{b} : \boldsymbol{a}, \boldsymbol{b} \in C,\ \boldsymbol{a} \neq \boldsymbol{b}\}$ has size $|C|(|C| - 1)$. Since these differences are nonzero, we have

$$|C|(|C| - 1) \leq 3^n - 1. \qquad \square$$

Even dimension.

Theorem 3.4. *If n is even, then $r(2, \mathbb{F}_3^n) = 3^{n/2}$.*

Proof. First we will show the lower bound, $r(2, \mathbb{F}_3^n) \geq 3^{n/2}$. Since $\mathbb{F}_3^n$ is additively isomorphic to $\mathbb{F}_3^{n/2} \times \mathbb{F}_3^{n/2}$, it suffices to construct a Sidon set of size $3^{n/2}$ in $\mathbb{F}_3^{n/2} \times \mathbb{F}_3^{n/2}$. As vector spaces over $\mathbb{F}_3$, $\mathbb{F}_3^{n/2}$ is isomorphic to $\mathbb{F}_{3^{n/2}}$, the finite field with $3^{n/2}$ elements. Hence, it suffices to construct a Sidon set of size $3^{n/2}$ in $\mathbb{F}_{3^{n/2}} \times \mathbb{F}_{3^{n/2}}$ This follows easily from the following claim; for a proof, see [Cilleruelo 2012, Example 1].

Claim. *Let q be an odd prime power and $\mathbb{F}_q$ be the finite field of order q. Then the set $\{(x, x^2) : x \in \mathbb{F}_q\}$ is a Sidon set in $\mathbb{F}_q \times \mathbb{F}_q$.*

It is clear that the set $\{(x, x^2) : x \in \mathbb{F}_{3^{n/2}}\}$ has size $3^{n/2}$, so we have $r(2, \mathbb{F}_3^n) \geq 3^{n/2}$. For the upper bound, let $C \subset \mathbb{F}_3^n$ be a 2-cap. Since n is even, $3^{n/2}$ is an integer, and if $|C| \geq 3^{n/2} + 1$, this contradicts Proposition 3.3. Therefore, $r(2, \mathbb{F}_3^n) \leq 3^{n/2}$. $\square$

Corollary 3.5. *As $n \to \infty$, $r(2, \mathbb{F}_3^n)$ is $\Theta(3^{n/2})$.*

The construction above can be leveraged into the following partitioning theorem.

Theorem 3.6. *When n is even, there is a partition of $\mathbb{F}_3^n$ into maximal 2-caps.*

This serves as an analogue to similar results for 1-caps in $\mathbb{F}_3^n$. It is well known that $\mathbb{F}_3^3$ can be partitioned into three maximal 1-caps of size 9. It is possible to partition $\mathbb{F}_3^2$ into a single point and two disjoint maximal 1-caps of size 4. Finally, [Follett et al. 2014, Theorem 3.3] shows that $\mathbb{F}_3^4$ can be partitioned into a single point and four disjoint maximal 1-caps of size 20.

Proof of Theorem 3.6. Since translations of Sidon sets are also Sidon sets, for each $a \in \mathbb{F}_{3^{n/2}}$ the set $S_a := \{(x, x^2 + a) : x \in \mathbb{F}_{3^{n/2}}\}$ is a maximal 2-cap. Since $(x, x^2 + a) = (y, y^2 + b)$ implies $x = y$ and hence $a = b$, we have that S_a and S_b

are disjoint for $a \neq b$. Therefore, as a ranges over $\mathbb{F}_{3^{n/2}}$ the sets S_a cover 3^n points and thus there is the claimed partition. □

Question 3.7. By Corollary 2.3, all maximal 2-caps in $\mathbb{F}_3^2$ are affinely equivalent. Is this true in $\mathbb{F}_3^n$ when n is even?

We remark that when $n = 4$, a computer program verified that all maximal 2-caps sum to $\mathbf{0}$. If a set of nine points sums to $\mathbf{0}$ in $\mathbb{F}_3^4$, then its image under any affine transformation will likewise sum to $\mathbf{0}$, so this is a necessary condition for all maximal 2-caps in $\mathbb{F}_3^4$ to be affinely equivalent.

Odd dimension.

Lemma 3.8. *If $C = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$ is a 2-cap of size 4 in $\mathbb{F}_3^n$ then $D = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}\}$ is a 2-cap of size 5.*

Proof. First we note that the points of D are distinct since if, without loss of generality, $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d} = \boldsymbol{a}$, this implies that $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are collinear, which is impossible since C is a 2-cap.

Now, suppose for contradiction that D is not a 2-cap, so there exist some $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w} \in D$ with $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{z}+\boldsymbol{w}$. Since C is a 2-cap, we may assume that $\boldsymbol{x} = \boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}$. Without loss of generality, we then have that one of the following occurs:

(1) $(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})+\boldsymbol{a} = \boldsymbol{b}+\boldsymbol{c}$. Then $\boldsymbol{a} = \boldsymbol{d}$, which is impossible since C has size 4.

(2) $(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})+\boldsymbol{a} = 2\boldsymbol{b}$. Then $\boldsymbol{a}+\boldsymbol{b} = \boldsymbol{c}+\boldsymbol{d}$, which is impossible since C is a 2-cap.

(3) $2(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}) = \boldsymbol{b}+\boldsymbol{c}$. Then $\boldsymbol{a}+\boldsymbol{d} = \boldsymbol{b}+\boldsymbol{c}$, which is impossible since C is a 2-cap.

(4) $2(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}) = 2\boldsymbol{a}$. Then $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are collinear, which is impossible since C is a 2-cap.

Hence, D is a 2-cap. □

Theorem 3.9. *In $\mathbb{F}_3^3$, a maximal 2-cap has size 5; that is, $r(2, \mathbb{F}_3^3) = 5$. Further, all complete 2-caps are maximal and all maximal 2-caps are affinely equivalent.*

Proof. Since $\{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is an affinely independent set in $\mathbb{F}_3^3$, by Lemma 3.8 $\{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3\}$ is a 2-cap in $\mathbb{F}_3^3$. Hence, $r(2, \mathbb{F}_3^3) \geq 5$. But by Proposition 3.3, $r(2, \mathbb{F}_3^3) < 6$ and hence $r(2, \mathbb{F}_3^3) = 5$.

Let C be any complete 2-cap in $\mathbb{F}_3^3$. Since $\mathbb{F}_3^3$ is a 3-dimensional affine space, if $|C| \leq 3$, then $\mathbb{F}_3^3$ contains a point which is affinely independent from the points of C, so C cannot be complete. Hence, $|C| \geq 4$. But if $|C| = 4$ then by Lemma 3.8, C is not complete. Hence, $|C| = 5$, and any complete 2-cap in $\mathbb{F}_3^3$ is already maximal.

For the final claim, suppose C is a maximal 2-cap in $\mathbb{F}_3^3$. Pick any four points in C. Since these points are affinely independent, there exists an invertible affine transformation mapping these points to the set $\{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. Hence, we need only show that all maximal 2-caps containing $\{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ are affinely equivalent.

It is easy to verify that there are exactly five such maximal 2-caps, namely

$$\begin{aligned} C_1 &= \{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, (1,1,1)\}, & C_4 &= \{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, (2,2,1)\},\\ C_2 &= \{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, (1,2,2)\}, & C_5 &= \{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, (2,2,2)\}.\\ C_3 &= \{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, (2,1,2)\}, \end{aligned}$$

It suffices to exhibit an invertible affine transformation T_i mapping C_1 to C_i for $i = 2, 3, 4, 5$. We provide these T_i explicitly, writing $T_i(\boldsymbol{x}) = A_i\boldsymbol{x} + \boldsymbol{b}_i$ for an invertible matrix A_i and $\boldsymbol{b}_i \in \mathbb{F}_3^3$:

$$A_2 = \begin{bmatrix} 1 & 0 & 0\\ 2 & 1 & 0\\ 2 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{b}_2 = \begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 1 & 0 & 2\\ 0 & 1 & 2\\ 0 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{b}_4 = \begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0 & 2\\ 0 & 0 & 1\\ 0 & 1 & 2 \end{bmatrix} \text{ and } \boldsymbol{b}_3 = \begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}, \qquad A_5 = \begin{bmatrix} 2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2 \end{bmatrix} \text{ and } \boldsymbol{b}_5 = \begin{bmatrix} 2\\ 2\\ 2 \end{bmatrix}. \quad \square$$

Theorem 3.10. *A maximal* 2*-cap in* $\mathbb{F}_3^5$ *has size* 13; *that is*, $r(2, \mathbb{F}_3^5) = 13$.

Proof. Let C be a maximal 2-cap in $\mathbb{F}_3^5$. By Theorem 3.4, $r(2, \mathbb{F}_3^4) = 9$ so by Lemma 2.4 we may assume that $|C| \geq 9$. We will apply a sequence of affine transformations to C to conclude that lexicographically, the first points in C are $\{\mathbf{0}, \boldsymbol{e}_5, \boldsymbol{e}_4, \boldsymbol{e}_3, \boldsymbol{e}_3 + \boldsymbol{e}_4 + \boldsymbol{e}_5, \boldsymbol{e}_2\}$ or $\{\mathbf{0}, \boldsymbol{e}_5, \boldsymbol{e}_4, \boldsymbol{e}_3, \boldsymbol{e}_2\}$.

Given any four affinely independent points, there exists an invertible affine transformation mapping them to $\mathbf{0}$, $\boldsymbol{e}_5$, $\boldsymbol{e}_4$, and $\boldsymbol{e}_3$, so without loss of generality we may assume that C contains the subset $\{\mathbf{0}, \boldsymbol{e}_5, \boldsymbol{e}_4, \boldsymbol{e}_3\}$. These points all lie in the $(0, 0)$-affine subspace of $\mathbb{F}_3^5$. Since $r(2, \mathbb{F}_3^3) = 5$, the $(0, 0)$-affine subspace contains four points or five points of C. If it contains five points, then by Theorem 3.9, we may apply an affine transformation (using a block matrix) and assume that the fifth point is $\boldsymbol{e}_3 + \boldsymbol{e}_4 + \boldsymbol{e}_5$.

Consider any other point $\boldsymbol{a} \in C$. Since $\boldsymbol{a}$ is not in the $(0, 0)$-affine subspace of $\mathbb{F}_3^5$, $\{\mathbf{0}, \boldsymbol{e}_5, \boldsymbol{e}_4, \boldsymbol{e}_3, \boldsymbol{a}\}$ is an affinely independent set so there exists an affine transformation T fixing $\mathbf{0}$, $\boldsymbol{e}_5$, $\boldsymbol{e}_4$, and $\boldsymbol{e}_3$ and mapping $\boldsymbol{a}$ to $\boldsymbol{e}_2$. Notice that if T is given by multiplication by the invertible matrix A followed by addition by $\boldsymbol{b} \in \mathbb{F}_3^5$, we have

$$T(\boldsymbol{e}_3+\boldsymbol{e}_4+\boldsymbol{e}_5) = A(\boldsymbol{e}_3+\boldsymbol{e}_4+\boldsymbol{e}_5)+\boldsymbol{b} = T(\mathbf{0})+T(\boldsymbol{e}_3)+T(\boldsymbol{e}_4)+T(\boldsymbol{e}_5) = \boldsymbol{e}_3+\boldsymbol{e}_4+\boldsymbol{e}_5,$$

so T fixes $\boldsymbol{e}_3 + \boldsymbol{e}_4 + \boldsymbol{e}_5$.

Hence, up to affine equivalence, we may assume that the lexicographically earliest points in C are $\{\mathbf{0}, \boldsymbol{e}_5, \boldsymbol{e}_4, \boldsymbol{e}_3, \boldsymbol{e}_3 + \boldsymbol{e}_4 + \boldsymbol{e}_5, \boldsymbol{e}_2\}$ or $\{\mathbf{0}, \boldsymbol{e}_5, \boldsymbol{e}_4, \boldsymbol{e}_3, \boldsymbol{e}_2\}$. A computer program was used to enumerate all possible complete 2-caps beginning with these sets of points. This verified that $r(2, \mathbb{F}_3^5) = 13$. The C++ code for the program is available on Won's professional website. □

Remark 3.11. The maximal 2-cap in $\mathbb{F}_3^5$ that is lexicographically earliest is explicitly given by the points

$$(0,0,0,0,0),\quad (0,0,0,0,1),\quad (0,0,0,1,0),\quad (0,0,1,0,0),\quad (0,0,1,1,1),$$
$$(0,1,0,0,0),\quad (0,1,1,1,2),\quad (0,2,1,2,0),\quad (0,2,2,1,2),\quad (1,0,0,0,0),$$
$$(1,0,1,2,1),\quad (2,0,1,0,2),\quad (2,2,0,2,2).$$

We conclude by giving bounds on $r(2, \mathbb{F}_3^7)$.

Proposition 3.12. *One has that* $33 \le r(2, \mathbb{F}_3^7) \le 47$.

Proof. The upper bound on $r(2, \mathbb{F}_3^7)$ is a consequence of Proposition 3.3. For the lower bound, we constructed a 2-cap of size 33 by first embedding a maximal 2-cap in $\mathbb{F}_3^6$ as a 2-cap C of size 27 in $\mathbb{F}_3^7$. We then used a computer program to enumerate all complete 2-caps containing C as a subset. The largest of these complete 2-caps has size 33. The lexicographically earliest one is given by the points

$$(0,0,0,0,0,0,0),\quad (0,0,0,1,0,0,1),\quad (0,0,0,2,0,0,1),$$
$$(0,0,1,0,1,0,0),\quad (0,0,1,1,1,2,1),\quad (0,0,1,2,1,1,1),$$
$$(0,0,2,0,1,0,0),\quad (0,0,2,1,1,1,1),\quad (0,0,2,2,1,2,1),$$
$$(0,1,0,0,1,2,0),\quad (0,1,0,1,0,2,1),\quad (0,1,0,2,2,2,1),$$
$$(0,1,1,0,2,1,1),\quad (0,1,1,1,1,0,2),\quad (0,1,1,2,0,2,2),$$
$$(0,1,2,0,2,0,2),\quad (0,1,2,1,1,1,0),\quad (0,1,2,2,0,2,0),$$
$$(0,2,0,0,1,2,0),\quad (0,2,0,1,2,2,1),\quad (0,2,0,2,0,2,1),$$
$$(0,2,1,0,2,0,2),\quad (0,2,1,1,0,2,0),\quad (0,2,1,2,1,1,0),$$
$$(0,2,2,0,2,1,1),\quad (0,2,2,1,0,2,2),\quad (0,2,2,2,1,0,2),$$
$$(1,0,0,0,0,0,0),\quad (1,0,0,0,0,0,1),\quad (2,0,0,1,0,2,0),$$
$$(2,0,0,1,1,0,1),\quad (2,0,0,1,1,1,2),\quad (2,0,0,1,1,2,2).$$
□

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