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Positive solutions to singular second-order boundary value problems for dynamic equations

Curtis Kunkel and Alex Lancaster

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We study singular second-order boundary value problems with mixed boundary conditions on an infinitely discrete time scale. We prove the existence of a positive solution by means of a lower and upper solutions method and the Brouwer fixed-point theorem, in conjunction with perturbation methods used to approximate regular problems.

1. Introduction

This paper continues the work done previously by Kunkel [2008], where he studied a singular second-order boundary value problem in purely discrete time scales of nonuniform step size. Although similar throughout most of the time scale, this result is different in the fact that the time scale itself has a limit point at the right-side boundary condition, forcing a nearly continuous behavior at that end. If this limiting condition were not present, the result would be trivial using [Kunkel 2008], but as it stands, this result continues to expand the work of that paper to another type of time scale.

More specifically, [Kunkel 2008] dealt with the discrete boundary value problem

$$u^{\Delta\Delta}(t_{i-1}) + f(t_i, u(t_i), u^{\Delta}(t_{i-1})) = 0, \quad t \in \mathbb{T}^{\circ},$$
$$u^{\Delta}(t_0) = u(t_{n+1}) = 0,$$

where $\mathbb{T}^{\circ}$ is the discrete interval of nonuniform step size $[t_1, t_n] := \{t_1, t_2, \dots, t_n\}$ and $f(t, x, y)$ is singular in x. This work was an extension of a previous result by Rachůnková and Rachůnek [2006], where they studied a singular second-order boundary value problem for the discrete p-Laplacian, $\phi_p(x) = |x|^{p-2}x,\ p > 1$.

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In particular, Rachůnková and Rachůnek dealt with the discrete boundary value problem

$$\Delta(\phi_p(\Delta u(t-1))) + f(t, u(t), \Delta u(t-1)) = 0, \quad t \in [1, T+1],$$
$$\Delta u(0) = u(T+2) = 0,$$

in which $f(t, x, y)$ was singular in x.

Combine these works with [Kunkel 2006], which deals with the continuous boundary value problem

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1),$$
$$u'(0) = u(1) = 0,$$

where $f(t, x)$ is singular in x, and you have a similar boundary value problem scenario across time scales ranging from being entirely continuous to varying degrees of discrete. This result fits between these two ends of the time scale continuum being a discrete interval with a continuous point, the ultimate goal of which would be to create a unifying theorem for this type of problem across all types of time scales (forthcoming).

The methods in this paper rely heavily on lower and upper solution methods in conjunction with an application of the Brouwer fixed-point theorem [Zeidler 1986]. We consider only the singular second-order boundary value problem, while letting our function range over an infinitely discrete interval of nonuniform step size, included in which is the limit point. We will provide definitions of appropriate lower and upper solutions. The lower and upper solutions will be applied to nonsingular perturbations of our nonlinear problem, ultimately giving rise to our boundary value problem by passing to the limit.

Lower and upper solutions have been used extensively in establishing solutions of boundary value problems for finite difference equations. Representative works include [Bao et al. 2012; Henderson and Kunkel 2006; Precup 2016].

Singular boundary value problems have also received a good deal of attention. Representative works include [Agarwal and O'Regan 1999; Precup 2016; Rachůnková and Rachůnek 2009].

2. Preliminaries

We now state some definitions used throughout the remainder of the paper, many of which can be found in [Bohner and Peterson 2001; Kelley and Peterson 1991]. Some definitions are required prior to the introduction of the problem we intend to solve.

Definition 2.1. For $i = 1, 2, 3, \ldots$, let $t_i = 1 - 1/i$. Define the time scale

$$\mathbb{T} = \{t_i\}_{i=1}^{\infty} \cup \{1\}.$$

We conveniently make note of the standard notation for both forward and backward jump operators on time scales of this nature.

Definition 2.2. The forward step operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The backward step operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

Definition 2.3. For the function $u : \mathbb{T} \to \mathbb{R}$, define the delta derivative $u^\Delta : \mathbb{T} \to \mathbb{R}$ by

$$u^\Delta(t) := \frac{u(\sigma(t)) - u(t)}{\mu(t)},$$

where $\mu(t) := \sigma(t) - t$. Note that μ is the graininess function.

Having introduced these definitions, we can now consider the following second-order dynamic equation, which will be our focus throughout this paper:

$$u^{\Delta\Delta}(t_{i-1}) + f(t_i, u(t_i), u^\Delta(t_{i-1})) = 0, \quad t_{i-1} \in \mathbb{T}, \tag{1}$$

satisfying the mixed boundary conditions,

$$u^\Delta(0) = u(1) = 0. \tag{2}$$

Our goal is to prove the existence of a positive solution to this problem (1), (2), where f has a specific type of singularity as explained below.

Definition 2.4. Define a solution to problem (1), (2) to mean a function $u : \mathbb{T} \to \mathbb{R}$ such that u satisfies (1) on $\mathbb{T}$ and also satisfies the boundary conditions (2). If $u(t) > 0$ for $t \in \mathbb{T}$, except possibly at the boundary conditions, we call u a positive solution to problem (1), (2).

Definition 2.5. Let $D \subseteq \mathbb{R}^2$. We say f is continuous on $\mathbb{T} \times D$ if $f(\cdot, x, y)$ is defined on $\mathbb{T}$ for each $(x, y) \in D$ and if $f(t, \cdot, \cdot)$ is continuous on D for each $t \in \mathbb{T}$.

Definition 2.6. Let $D \subseteq \mathbb{R}^2$. Let $f : \mathbb{T} \times D \to \mathbb{R}$. If $D = \mathbb{R}^2$, then we call (1), (2) a regular problem. If $D \subsetneq \mathbb{R}^2$ and f has singularities on the boundary of D, then we call (1), (2) a singular problem.

We assume the following throughout this paper:

(A) $D = [0, \infty) \times \mathbb{R}$.

(B) f is continuous on $\mathbb{T} \times D$.

(C) $f(t, x, y)$ has a singularity at $x = 0$; i.e., $\limsup_{x\to 0^+} |f(t, x, y)| = \infty$ for $t \in \mathbb{T}$ and $y \in \mathbb{R}$.

3. Lower and upper solutions method

For the purpose of establishing a lower and upper solutions method to be used in solving our pre-existing singular problem, we first consider the following regular problem:

$$u^{\Delta\Delta}(t_{i-1}) + h(t_i, u(t_i), u^{\Delta}(t_{i-1})) = 0, \quad t_{i-1} \in \mathbb{T}, \tag{3}$$

where h is continuous on $\mathbb{T} \times \mathbb{R}^2$ and the same boundary conditions (2) are satisfied. Now, (3), (2) is clearly a regular problem and it is our current goal to establish a lower and upper solutions method as a means to establish an existence result. To this end, we first must define what is meant by a lower and an upper solution.

Definition 3.1. Let $\alpha : \mathbb{T} \to \mathbb{R}$. We call α a lower solution of problem (3), (2) if

$$\alpha^{\Delta\Delta}(t_{i-1}) + h(t_i, \alpha(t_i), \alpha^{\Delta}(t_{i-1})) \geq 0, \quad t_{i-1} \in \mathbb{T}, \tag{4}$$

satisfying

$$\alpha^{\Delta}(0) \geq 0, \quad \alpha(1) \leq 0. \tag{5}$$

Definition 3.2. Let $\beta : \mathbb{T} \to \mathbb{R}$. We call β an upper solution of problem (3), (2) if

$$\beta^{\Delta\Delta}(t_{i-1}) + h(t_i, \beta(t_i), \beta^{\Delta}(t_{i-1})) \leq 0, \quad t_{i-1} \in \mathbb{T}, \tag{6}$$

satisfying

$$\beta^{\Delta}(0) \leq 0, \quad \beta(1) \geq 0. \tag{7}$$

Theorem 3.3 (lower and upper solutions method). *Let α and β be lower and upper solutions of the regular problem* (3), (2), *respectively*, *where $\alpha \leq \beta$ on $\mathbb{T}$. Let $h(t, x, y)$ be continuous on $\mathbb{T} \times \mathbb{R}^2$ and nonincreasing in its y-variable. Then* (3), (2) *has a solution u satisfying*

$$\alpha(t) \leq u(t) \leq \beta(t), \quad t \in \mathbb{T}.$$

Proof. We proceed with this proof through a sequence of steps involving modifications of the function h.

Step 1: For $t_{i-1} \in \mathbb{T}$ and $(x, y) \in \mathbb{R}^2$, define

$$\tilde{h}\Big(t_i, x, \frac{x-y}{\mu(t_{i-1})}\Big) = \begin{cases} h\Big(t_i, \beta(t_i), \dfrac{\beta(t_i) - S(t_{i-1}, y)}{\mu(t_{i-1})}\Big) - \dfrac{x - \beta(t_i)}{x - \beta(t_i) + 1}, & x > \beta(t_i), \\ h\Big(t_i, x, \dfrac{x - S(t_{i-1}, y)}{\mu(t_{i-1})}\Big), & \alpha(t_i) \leq x \leq \beta(t_i), \\ h\Big(t_i, \alpha(t_i), \dfrac{\alpha(t_i) - S(t_{i-1}, y)}{\mu(t_{i-1})}\Big) + \dfrac{\alpha(t_i) - x}{\alpha(t_i) - x + 1}, & x < \alpha(t_i), \end{cases} \tag{8}$$

where,

$$S(t_{i-1}, y) = \begin{cases} \beta(t_{i-1}), & y > \beta(t_{i-1}), \\ y, & \alpha(t_{i-1}) \le y \le \beta(t_{i-1}), \\ \alpha(t_{i-1}), & y < \alpha(t_{i-1}). \end{cases}$$

Given this construction, $\tilde{h}$ is continuous on $\mathbb{T} \times \mathbb{R}^2$ and there exists $M > 0$ so that

$$|\tilde{h}(t, x, y)| \le M$$

for all $t \in \mathbb{T}$ and $(x, y) \in \mathbb{R}^2$.

We now study the auxiliary equation

$$u^{\Delta\Delta}(t_{i-1}) + \tilde{h}(t_i, u(t_i), u^{\Delta}(t_{i-1})) = 0, \quad t_{i-1} \in \mathbb{T}, \tag{9}$$

satisfying boundary conditions (2). Our immediate goal is to prove the existence of a solution to problem (9), (2).

Step 2: For this existence result, we lay the foundation to use the Brouwer fixed-point theorem. To this end, define

$$E = \{u : \mathbb{T} \to \mathbb{R} : u^{\Delta}(0) = u(1) = 0\}.$$

Also, define

$$\|u\| = \max\{|u(t)| : t \in \mathbb{T}\}.$$

Given E and $\|\cdot\|$, we say E is a Banach space. Further, we define an operator $\mathcal{T} : E \to E$ by

$$(\mathcal{T}u)(t_k) = \sum_{j=k}^{\infty} \mu(t_j) \sum_{i=2}^{j} \mu(t_{i-1}) \tilde{h}(t_i, u(t_i), u^{\Delta}(t_{i-1})). \tag{10}$$

$\mathcal{T}$ is a continuous operator. Moreover, from the bounds placed on $\tilde{h}$ in Step 1 and from (10), if $r > M$, then $T(\overline{B(r)}) \subseteq \overline{B(r)}$, where $B(r) := \{u \in E : \|u\| < r\}$. Hence, by the Brouwer fixed-point theorem [Zeidler 1986], there exists $u \in \overline{B(r)}$ such that $u = \mathcal{T}u$.

Step 3: We now show that u is a fixed point of $\mathcal{T}$ if and only if u is a solution to the problem (9), (2).

To this end, let us first assume that u solves the problem (9), (2). Then, since the boundary conditions (2) are satisfied, $u \in E$.

It is convenient for the first part of this subproof to consider a relabeling of the points in $\mathbb{T}$ as follows: let $\tau_\infty = \lim_{i\to\infty} \tau_i := t_1 = 0$, let $\tau_0 := 1$, and, for each $i > 0$, let there exist some $j > 0$ so that $t_i = \tau_j$, $t_{i+1} = \tau_{j-1}$, etc. Using this notation, we then consider

$$u^{\Delta}(\tau_1) = \frac{u(\tau_0) - u(\tau_1)}{\mu(\tau_1)} = \frac{-u(\tau_1)}{\mu(\tau_1)},$$

and we have

$$u(\tau_1) = -\mu(\tau_1)u^\Delta(\tau_1).$$

Also,

$$u^\Delta(\tau_2) = \frac{u(\tau_1) - u(\tau_2)}{\mu(\tau_2)} = \frac{-\mu(\tau_1)u^\Delta(\tau_1) - u(\tau_2)}{\mu(\tau_2)},$$

and we have

$$u(\tau_2) = -\mu(\tau_1)u^\Delta(\tau_1) - \mu(\tau_2)u^\Delta(\tau_2).$$

Continuing in this manner, we have, for $m > 0$,

$$u(\tau_m) = -\sum_{i=1}^{m} \mu(\tau_i)u^\Delta(\tau_i). \tag{11}$$

And, given our relabeling between the τ's and the t's, we can conclude that for each $m > 0$ there exists some $k > 0$ such that

$$u(\tau_m) = u(t_k) = -\sum_{i=k}^{\infty} \mu(\tau_i)u^\Delta(\tau_i).$$

We also have

$$u^{\Delta\Delta}(t_1) = \frac{u^\Delta(t_2) - u^\Delta(t_1)}{\mu(t_1)} = \frac{u^\Delta(t_2) - u^\Delta(0)}{\mu(t_1)} = \frac{u^\Delta(t_2)}{\mu(t_1)},$$

and from (9) we have $u^{\Delta\Delta}(t_1) = -\tilde{h}(t_2, u(t_2), u^\Delta(t_1))$, which yields

$$u^\Delta(t_2) = -\mu(t_1)\tilde{h}(t_2, u(t_2), u^\Delta(t_1)).$$

Similarly, we have

$$u^{\Delta\Delta}(t_2) = \frac{u^\Delta(t_3) - u^\Delta(t_2)}{\mu(t_2)} = -\tilde{h}(t_3, u(t_3), u^\Delta(t_2)),$$

and via substitution of $u^\Delta(t_2)$ and simply solving for $u^\Delta(t_3)$, we have

$$u^\Delta(t_3) = -\mu(t_1)\tilde{h}(t_2, u(t_2), u^\Delta(t_1)) - \mu(t_2)\tilde{h}(t_3, u(t_3), u^\Delta(t_2)).$$

Continuing in this manner, we conclude that

$$u^\Delta(t_j) = -\sum_{i=2}^{j} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^\Delta(t_{i-1})). \tag{12}$$

By substituting (12) into (11), we see that for $k > 0$

$$\begin{aligned} u(t_k) &= -\sum_{j=k}^{\infty} \mu(t_j)\Big(-\sum_{i=2}^{j} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^\Delta(t_{i-1}))\Big) \\ &= \sum_{j=k}^{\infty} \mu(t_j) \sum_{i=2}^{j} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^\Delta(t_{i-1})) = (\mathcal{T}u)(t_k). \end{aligned}$$

We now assume that u is a fixed point of $\mathcal{T}$, i.e., $u = \mathcal{T}u$. Then,

$$u(t_k) = (\mathcal{T}u)(t_k) = \sum_{j=k}^{\infty} \mu(t_j) \sum_{i=2}^{j} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^{\Delta}(t_{i-1})).$$

Also,

$$\begin{aligned}
u^{\Delta}(t_{k-1}) &= \frac{u(t_k) - u(t_{k-1})}{\mu(t_{k-1})} \\
&= \frac{\left(\sum_{j=k}^{\infty} \mu(t_j) \sum_{i=2}^{j} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^{\Delta}(t_{i-1}))\right)}{\mu(t_{k-1})} \\
&\qquad - \frac{\left(\sum_{j=k-1}^{\infty} \mu(t_j) \sum_{i=2}^{j} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^{\Delta}(t_{i-1}))\right)}{\mu(t_{k-1})} \\
&= -\frac{\mu(t_{k-1}) \sum_{i=2}^{k-1} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^{\Delta}(t_{i-1}))}{\mu(t_{k-1})} \\
&= -\sum_{i=2}^{k-1} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^{\Delta}(t_{i-1})),
\end{aligned}$$

and

$$\begin{aligned}
&u^{\Delta\Delta}(t_{k-1}) \\
&= \frac{u^{\Delta}(t_k) - u^{\Delta}(t_{k-1})}{\mu(t_{k-1})} \\
&= \frac{\left(-\sum_{i=2}^{k} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^{\Delta}(t_{i-1}))\right)}{\mu(t_{k-1})} - \frac{\left(-\sum_{i=2}^{k-1} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^{\Delta}(t_{i-1}))\right)}{\mu(t_{k-1})} \\
&= -\frac{\mu(t_{k-1})\tilde{h}(t_k, u(t_k), u^{\Delta}(t_{k-1}))}{\mu(t_{k-1})} \\
&= -\tilde{h}(t_k, u(t_k), u^{\Delta}(t_{k-1})).
\end{aligned}$$

Thus, u solves (9).

We need now only consider the boundary conditions (2) in order to complete Step 3 of this proof. To this end, we recall the construction of the time scale $\mathbb{T}$ and notice the following based on what was just derived as the formula for u^{Δ}:

$$u^{\Delta}(0) = u^{\Delta}(t_1) = -\sum_{i=2}^{1} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^{\Delta}(t_{i-1})) = 0.$$

We now turn our attention over to $t = 1$ and recall from the construction of $\mathbb{T}$ that $t_{\infty} = 1$. Also note that standard convention when discussing time scales of this sort is $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t, or for our purposes $\mu(t) = 0$ if $\mathbb{T}$ has a maximum t. As such,

$$u(1) = u(t_\infty) = \sum_{j=\infty}^{\infty} \mu(t_j) \sum_{i=2}^{j} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^\Delta(t_{i-1}))$$
$$= \mu(t_\infty) \sum_{i=2}^{\infty} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^\Delta(t_{i-1}))$$
$$= \mu(1) \sum_{i=2}^{\infty} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^\Delta(t_{i-1}))$$
$$= 0 \cdot \sum_{i=2}^{\infty} \mu(t_{i-1})\tilde{h}(t_i, u(t_i), u^\Delta(t_{i-1})) = 0.$$

Therefore, we get that u solves (9), (2) and Step 3 is complete.

Step 4: The remaining piece we need to show is that solutions of (9), (2) satisfy

$$\alpha(t) \le u(t) \le \beta(t), \quad t \in \mathbb{T}.$$

To this end, without loss of generality consider the case of obtaining $u(t) \le \beta(t)$, and let $v(t) = u(t) - \beta(t)$. For the purpose of establishing a contradiction, assume that $\max\{v(t) : t \in \mathbb{T}\} := v(l) > 0$. From (2) and (7), we see that l must be an interior point in $\mathbb{T}$; i.e., $l := t_j \in \mathbb{T} \setminus \{0, 1\}$. With t_j necessarily being an interior point, t_{j-1} and t_{j+1} are well-defined, and we have

$$v(t_{j-1}) \le v(t_j) \quad \text{and} \quad v(t_{j+1}) \le v(t_j).$$

Consequently,

$$v^\Delta(t_{j-1}) \ge 0 \quad \text{and} \quad v^\Delta(t_j) \le 0.$$

Further, we now know also that

$$v^{\Delta\Delta}(t_{j-1}) = \frac{v^\Delta(t_j) - v^\Delta(t_{j-1})}{\mu(t_{j-1})} \le 0.$$

Therefore,

$$u^{\Delta\Delta}(t_{j-1}) - \beta^{\Delta\Delta}(t_{j-1}) \le 0. \tag{13}$$

On the other hand, since h is nonincreasing in its third variable, we have from (9) and (8) that

$$u^{\Delta\Delta}(t_{j-1}) - \beta^{\Delta\Delta}(t_{j-1}) = -\tilde{h}(t_j, u(t_j), u^\Delta(t_{j-1})) - \beta^{\Delta\Delta}(t_{j-1})$$
$$= -\left(\tilde{h}(t_j, \beta(t_j), \beta^\Delta(t_{j-1})) - \frac{u(t_j) - \beta(t_j)}{u(t_j) - \beta(t_j) + 1}\right) - \beta^{\Delta\Delta}(t_{j-1})$$
$$= -\tilde{h}(t_j, \beta(t_j), \beta^\Delta(t_{j-1})) + \frac{v(t_j)}{v(t_j)+1} - \beta^{\Delta\Delta}(t_{j-1})$$
$$\ge \beta^{\Delta\Delta}(t_{j-1}) + \frac{v(l)}{v(l)+1} - \beta^{\Delta\Delta}(t_{j-1}) = \frac{v(l)}{v(l)+1} > 0.$$

Hence we have a contradiction to (13) and we conclude that $\max\{v(t) : t \in \mathbb{T}\} \le 0$. Thus, $v(t) \le 0$ for all $t \in \mathbb{T}$, or rather

$$u(t) \le \beta(t) \quad \text{for all } t \in \mathbb{T}.$$

A similar argument shows that $\alpha(t) \le u(t)$ for all $t \in \mathbb{T}$.

Thus, our conclusion holds and the proof is complete. □

4. Main result

In this section, we make use of Theorem 3.3 to obtain positive solutions to the singular problem (1), (2). In particular, in applying Theorem 3.3, we deal with a sequence of regular perturbations of (1), (2). Ultimately, we obtain a desired solution by passing to the limit on a sequence of solutions for the perturbations.

Theorem 4.1. *Assume conditions* (A), (B), *and* (C) *hold*, *along with the following*:

(D) *There exists* $c \in (0, \infty)$ *so that* $f(t, c, y) \le 0$ *for all* $t \in \mathbb{T}$ *and* $y \in \mathbb{R}$.

(E) $f(t, x, y)$ *is nonincreasing in its* y*-variable for all* $t \in \mathbb{T}$ *and* $x \in (0, c)$.

(F) $\lim_{x\to 0^+} f(t, x, y) = \infty$ *for* $t \in \mathbb{T}$ *and* $y \in (-c, c)$.

Then, (1), (2) *has a solution* u *satisfying*

$$0 < u(t) \le c, \quad t \in \mathbb{T} \setminus \{1\}.$$

Proof. We proceed through this proof via a sequence of steps.

Step 1: For $k > 0$, $t \in \mathbb{T}$, and $y \in \mathbb{R}$, define

$$f_k(t, x, y) = \begin{cases} f(t, |x|, y) & \text{if } |x| \ge 1/k, \\ f(t, 1/k, y) & \text{if } |x| < 1/k. \end{cases}$$

Then, f_k is continuous on $\mathbb{T} \times \mathbb{R}^2$.

Assumption (F) implies that there exists k_0 such that, for all $k \ge k_0$,

$$f_k(t, 0, y) = f\Big(t, \frac{1}{k}, y\Big) > 0 \quad \text{for all } t \in \mathbb{T},\ y \in \mathbb{R}.$$

We now consider

$$u^{\Delta\Delta}(t_{i-1}) + f_k(t_i, u(t_i), u^{\Delta}(t_{i-1})) = 0, \quad t \in \mathbb{T}. \tag{14}$$

Now, let $\alpha(t) = 0$ and $\beta(t) = c$. Then, for each $k \ge k_0$, α and β are lower and upper solutions of (14), (2), respectively. Also, $\alpha(t) \le \beta(t)$ for $t \in \mathbb{T}$. Thus, by Theorem 3.3, for each $k \ge k_0$, there exists a solution u_k to each problem (14), (2) that satisfies $0 \le u_k(t) \le c$ for $t \in \mathbb{T}$.

Consequently, for all $t_i \in \mathbb{T}$,

$$|u^{\Delta}(t_i)| \le c \cdot \mu(t_{i-1}). \tag{15}$$

Step 2: For $k \geq k_0$, let $\delta \in (0, 1)$ and consider the time scale $\mathbb{T}_1 := \mathbb{T} \cap [0, \delta]$. Since u_k solves (14), we get from work similar to that in the proof of Theorem 3.3 that

$$u_k^\Delta(t_j) = -\sum_{i=2}^{j} \mu(t_{i-1}) f_k(t_i, u(t_i), u^\Delta(t_{i-1})). \tag{16}$$

We use this version of u_k^Δ as follows:

By assumption (F), there exists $\varepsilon_1 \in (0, 1/k_0)$ such that for all $k \geq 1/\varepsilon_1$

$$f_k(t_2, x, y) > c, \quad x \in (0, \varepsilon_1), \; y \in (-c, c). \tag{17}$$

For the sake of establishing a contradiction, assume that for $k \geq 1/\varepsilon_1$ we have $u_k(t_2) < \varepsilon_1$. Then, by (16) and (17),

$$\begin{aligned} u_k^\Delta(t_2) &= -\sum_{i=2}^{2} \mu(t_{i-1}) f_k(t_i, u(t_i), u^\Delta(t_{i-1})) \\ &= -\mu(t_1) f_k(t_2, u(t_2), u^\Delta(t_1)) < -\mu(t_1) \cdot c. \end{aligned}$$

However, this contradicts (15). Thus, $u_k(t_2) \geq \varepsilon_1$ for all $k \geq 1/\varepsilon_1$.

Continuing, also by assumption (F), there now exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that for all $k \geq 1/\varepsilon_2$

$$f_k(t_3, x, y) > c + m_1, \quad x \in (0, \varepsilon_2), \; y \in (-c, c), \tag{18}$$

where $m_1 = \max\{|f_k(t_2, x, y)| : x \in [\varepsilon_1, c], \, y \in (-c, c)\}$. For the sake of establishing a contradiction, assume that for $k \geq 1/\varepsilon_2$ we have $u_k(t_3) < \varepsilon_2$. Then, by (16) and (18),

$$\begin{aligned} u_k^\Delta(t_3) &= -\sum_{i=2}^{3} \mu(t_{i-1}) f_k(t_i, u(t_i), u^\Delta(t_{i-1})) \\ &= -\mu(t_1) f_k(t_2, u(t_2), u^\Delta(t_1)) - \mu(t_2) f_k(t_3, u(t_3), u^\Delta(t_2)) \\ &\leq \mu(t_1) \cdot m_1 - \mu(t_2)(c + m_1) < \mu(t_2) \cdot c. \end{aligned}$$

However, this contradicts (15). Thus, $u_k(t_3) \geq \varepsilon_2$ for all $k \geq 1/\varepsilon_2$.

We continue in this manner, proceeding across the interval $\mathbb{T}_1$ for $j = 3, 4, 5, \ldots, l-1$ and we create a nested sequence of epsilons, $0 < \varepsilon_{l-1} < \cdots < \varepsilon_2 < \varepsilon_1$, where $u_k(t_j) \geq \varepsilon_{j-1}$ when $k \geq 1/\varepsilon_{j-1}$.

Continuing, by assumption (F), there exists $\varepsilon_l \in (0, \varepsilon_{l-1})$ such that for all $k \geq 1/\varepsilon_l$

$$f_k(t_{l+1}, x, y) > c + \sum_{i=1}^{l-1} m_i, \quad x \in [\varepsilon_i, c], \; y \in (-c, c), \tag{19}$$

where $m_i = \max\{|f_k(t_{i+1}, x, y)| : x \in [\varepsilon_i, c], \, y \in (-c, c)\}$. For the sake of establishing a contradiction, assume that for $k \geq 1/\varepsilon_l$ we have $u_k(t_{l+1}) < \varepsilon_l$. Then, by

(16) and (19),

$$\begin{aligned} u_k^{\Delta}(t_{l+1}) &= -\sum_{i=2}^{l+1} \mu(t_{i-1}) f_k(t_i, u(t_i), u^{\Delta}(t_{i-1})) \\ &= -\mu(t_1) f_k(t_2, u(t_2), u^{\Delta}(t_1)) - \mu(t_2) f_k(t_3, u(t_3), u^{\Delta}(t_2)) \\ &\qquad - \cdots - \mu(t_l) f_k(t_{l+1}, u(t_{l+1}), u^{\Delta}(t_l)) \\ &\leq \mu(t_1) m_1 - \cdots - \mu(t_{l-1}) m_{l-1} - \mu(t_l)\left(c + \sum_{i=1}^{l-1} m_i\right) \\ &< \mu(t_l) \cdot c. \end{aligned}$$

However, this contradicts (15). Thus, $u_k(t_{l+1}) \geq \varepsilon_l$ for all $k \geq 1/\varepsilon_l$.

Now, recall that we are on the interval $\mathbb{T}_1$, which just so happens to be an interval with a finite number of points included in its time scale. Call the largest of these points t_M, and based on the previous argument, there exists $\varepsilon_{m-1} > 0$ so that $u_k(t_M) \geq \varepsilon_{m-1}$ for $k \geq 1/\varepsilon_{m-1}$. Choose $\varepsilon = \varepsilon_{m-1}/2$ and note that

$$0 < \varepsilon \leq u_k(t) \leq c \quad \text{for all } t \in \mathbb{T}_1,\ k \geq \frac{1}{\varepsilon}.$$

We now need only discuss what happens when

$$t \in \mathbb{T}_2 := \mathbb{T} \setminus \mathbb{T}_1 = \mathbb{T} \cap [\delta, 1].$$

Note that for each δ as $\delta \to 1$, via previous arguments, we have for sufficiently large k that $u_k(t) > 0,\ t \in \mathbb{T}_2$. Also note that for sufficiently large k, as $\delta \to 1$, we have $u_k(t) \geq 0$. This leads to the fact that for sufficiently large k, we get $u_k(t) > 0$ for $t \in \mathbb{T} \setminus \{1\}$ and, as our boundary condition states, $u_k(1) = 0$.

We now choose a subsequence $\{u_{k_n}(t)\} \subseteq \{u_k(t)\}$ so that

$$\lim_{n\to\infty} u_{k_n}(t) = u(t), \qquad t \in \mathbb{T},$$

and note that $u(t) \in E$, where E is defined as in the proof of Theorem 3.3.

Moreover, (16) yields, for sufficiently large n,

$$u_{k_n}^{\Delta}(t_j) = -\sum_{i=2}^{j} \mu(t_{i-1}) f(t_i, u_{k_n}(t_i), u_{k_n}^{\Delta}(t_{i-1})),$$

and so letting $n \to \infty$ and from the continuity of f we get

$$u^{\Delta}(t_j) = -\sum_{i=2}^{j} \mu(t_{i-1}) f(t_i, u(t_i), u^{\Delta}(t_{i-1})).$$

Consequently,

$$u^{\Delta\Delta}(t_{i-1}) = -f(t_i, u(t_i), u^{\Delta}(t_{i-1})). \qquad \square$$

5. Example

Let $\mathbb{T}$ be as given in Definition 2.1. Let $\alpha \in [0, \infty)$, $c, \beta \in (0, \infty)$, and $a : \mathbb{T} \to \mathbb{R}$. Then, by Theorem 4.1, the problem

$$u^{\Delta\Delta}(t_{i-1}) + \big(a(t_i) + (u(t_i))^{\alpha} + (u(t_i))^{-\beta}\big)(c - u(t_i)) - (u^{\Delta}(t_{i-1}))^3 = 0, \quad t_{i-1} \in \mathbb{T},$$

along with the boundary conditions (2), has a solution u satisfying the desired inequality.

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