

involve

a journal of mathematics

Asymptotic expansion of Warlimont functions
on Wright semigroups

Marco Aldi and Hanqiu Tan



Asymptotic expansion of Warlimont functions on Wright semigroups

Marco Aldi and Hanqiu Tan

(Communicated by Kenneth S. Berenhaut)

We calculate full asymptotic expansions of prime-independent multiplicative functions on additive arithmetic semigroups that satisfy a strong form of Knopfmacher's axioms. When applied to the semigroup of unlabeled graphs, our method yields detailed asymptotic information on how graphs decompose into connected components. As a second class of examples, we discuss polynomials in several variables over a finite field.

1. Introduction

Let G_n be the number of unlabeled graphs with n vertices and let G_n^+ be the number of connected unlabeled graphs with n vertices. Using the fact that the sequences $\{G_n\}$ and $\{G_n^+\}$ are related by the identity

$$\sum_{n=0}^{\infty} G_n x^n = \prod_{m=1}^{\infty} (1 - x^m)^{-G_m^+}, \quad (1)$$

Wright [1967] proved that $G_n \sim G_n^+$; i.e., almost all graphs are connected. As observed in that paper and further clarified in [Warlimont 2001], this asymptotic result is intimately related to the fact that the power series at the right-hand side of (1) has trivial convergence radius. Armed with a full asymptotic expansion for G_n [Wright 1969], Wright [1970] further improved this result by constructing a sequence $\{\omega_s\}$ of polynomials such that, for any fixed positive integer R , the asymptotic relation

$$G_n^+ = G_n + \sum_{s=1}^{R-1} \omega_s(n) G_{n-s} + O(n^R G_{n-R}) \quad (2)$$

holds in the limit $n \rightarrow \infty$.

In the context of abstract analytic number theory [Knopfmacher 1975], Knopfmacher [1976] (see also [Flajolet and Sedgewick 2009; Burris 2001] for the more

MSC2010: primary 05A16; secondary 05C30, 11T06.

Keywords: arithmetical semigroups, asymptotic enumeration, graph enumeration.

general setting of weighted decomposable combinatorial structures) observed that (1) is a particular case of an Euler product type of identity which holds for arbitrary additive arithmetical semigroups and that the methods of [Wright 1967] can be used to study the distribution of certain arithmetical functions on additive arithmetical semigroups in which almost all elements are prime. For instance, if d_2 is the *divisor function* that to each unlabeled graph g assigns the number of ways to write g as a disjoint union of an ordered pair of graphs then

$$\lim_{n \rightarrow \infty} \frac{1}{G_n} \sum d_2(g) = 2 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{G_n} \sum (d_2(g) - 2)^2 = 0, \quad (3)$$

where both sums are taken over all graphs g with n vertices.

The goal of the present paper is to investigate Knopfmacher's suggestion [1976] that restricting to arithmetical semigroups in which the total number of elements is related to the number of prime elements by a formula analogous to (2) might lead to a strengthening of (3). To illustrate our results with an example, consider again the divisor function d_2 on the semigroup of graphs. We prove that for every positive integer M , there exists a sequence $\{\tau_s(n)\}$ of polynomials such that, for any fixed positive integer R , the asymptotic relation

$$\frac{1}{G_n} \sum (d_2(g) - 2)^M = 2^M \sum_{s=1}^{R-1} \tau_s(n) 2^{-sn} + O(n^{2R-1} 2^{-Rn}) \quad (4)$$

holds in the limit $n \rightarrow \infty$. Clearly, (3) can be recovered by setting $M = 1$ and $M = 2$ in (4) and taking the limit as $n \rightarrow \infty$. More generally, we show that (4) is a particular case of a formula that holds if d_2 is replaced by an arbitrary *Warlimont function*, i.e., a multiplicative prime-independent function whose restriction to power of primes grows in a prescribed way. Even more generally, the semigroup of graphs can be replaced by any *Wright semigroup*, which we define to be an additive arithmetical semigroup subject to a growth condition introduced in [Wright 1970]. Examples of Wright semigroups include the semigroup of unlabeled graphs with an even number of edges and the semigroup of polynomials in at least two variables over a finite field.

The paper is organized as follows. Section 2 contains the main technical results used in the rest of the paper. We work with triples of sequences related by a generalization of (1) that were introduced in [Warlimont 1993]. The main result is Theorem 5 which can be thought of as a generalization of [Wright 1970], modeled after the way in which [Warlimont 1993] generalizes [Wright 1967]. In Section 3, after introducing the key notions of Wright semigroup and of Warlimont function, we provide asymptotic formulas for moments of Warlimont functions in terms of the number of elements of given degree in the underlying (not necessarily Wright) semigroup. In the special case of Wright semigroup, we construct full

asymptotic expansions generalizing (4). We illustrate our results in Section 4 by calculating asymptotic expansion of some of the arithmetical functions considered in [Knopfmacher 1976] on three examples of Wright semigroups: the semigroup of all unlabeled graphs, the semigroup of unlabeled graphs with an even number of edges and the semigroup of nonzero polynomials (up to scaling) in at least two variables over a finite field.

2. Warlimont triples

Definition 1. A *Warlimont triple* is a triple $(\{T_n\}, \{t_n\}, \{a_n\})$ of sequences of non-negative real numbers related by the identity

$$\sum_{n=0}^{\infty} T_n x^n = \prod_{m=1}^{\infty} \left(\sum_{k=0}^{\infty} a_k x^{km} \right)^{t_m} \quad (5)$$

of formal power series and such that

- (i) $T_0 = a_0 = 1$,
- (ii) $a_1 > 0$,
- (iii) $t_m \in \mathbb{Z}$ for all m and $t_m > 0$ for all but finitely many m .

In order for the three sequences to be all indexed by nonnegative integers, we set $t_0 = 0$.

Lemma 2. Let $(\{T_n\}, \{t_n\}, \{a_n\})$ be a Warlimont triple and consider the sequences $\{v_n\}$, $\{\beta_n\}$ and $\{b_n\}$ defined the recursion formulas

$$v_n = T_n - \sum_{s=1}^{n-1} \frac{s}{n} v_s T_{n-s}, \quad (6)$$

$$\beta_n = - \sum_{s=0}^{n-1} \beta_s T_{n-s}, \quad (7)$$

$$b_n = na_n - \sum_{s=1}^{n-1} b_s a_{n-s}, \quad (8)$$

with initial conditions $v_1 = T_1$, $\beta_0 = 1$, $b_1 = a_1$. Then for all n :

- (i) $v_n = \sum_{d|n} (d/n) t_d b_{n/d}$, where the sum is over all integers $1 < d \leq n$ that divide n .
- (ii) $\beta_n = - \sum_{s=1}^n (s/n) v_s \beta_{n-s}$.
- (iii) For every positive integer R

$$\sum_{s=0}^{R-1} \beta_s T_{n-s} = v_n + \frac{1}{n} \sum_{r=0}^{R-1} \beta_r \sum_{s=R-r}^{n-R} s v_s T_{n-r-s}.$$

Proof. Using formal term-by-term differentiation it is easy to show that (6) and (7) are equivalent to the formal identities

$$\log\left(\sum_{n=0}^{\infty} T_n x^n\right) = \sum_{m=1}^{\infty} v_m x^m, \tag{9}$$

$$\log\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{s=1}^{\infty} \frac{b_s}{s} x^s, \tag{10}$$

respectively. Taking the formal logarithm of (5) and substituting (9), (10), we obtain

$$\sum_{m=1}^{\infty} v_m x^m = \sum_{r,s=1}^{\infty} t_r \frac{b_s}{s} x^{rs}$$

from which (i) easily follows. Since (7) is equivalent to the identity

$$\left(\sum_{s=0}^{\infty} \beta_s x^s\right) \left(\sum_{n=0}^{\infty} T_n x^n\right) = 1$$

of formal power series, taking formal logarithms yields

$$\log\left(\sum_{s=0}^{\infty} \beta_s x^s\right) = -\sum_{m=1}^{\infty} v_m x^m.$$

Comparing with (6) and (9) proves (ii). It follows from (ii) that

$$\sum_{u=0}^{R-1} \sum_{r=0}^u \beta_r ((n-u)v_{n-u} T_{u-r} + (u-r)v_{u-r} T_{n-u}) = v_n - \sum_{u=0}^{R-1} u \beta_u T_{n-u}$$

and thus

$$\begin{aligned} & \sum_{s=0}^{R-1} \beta_s T_{n-s} - v_n \\ &= \frac{1}{n} \sum_{u=0}^{R-1} \sum_{r=0}^u \beta_r \left(\frac{n-r}{R-r} T_{n-r} - (n-u)v_{n-u} T_{u-r} - (u-r)v_{u-r} T_{n-u} \right) \\ &= \frac{1}{n} \sum_{r=0}^{R-1} \beta_r \left((n-r) T_{n-r} - \sum_{s=0}^{R-r-1} (n-r-s) T_s v_{n-r-s} + \sum_{s=0}^{R-r-1} s v_s T_{n-r-s} \right) \\ &= \frac{1}{n} \sum_{r=0}^{R-1} \beta_r \sum_{s=R-r}^{n-R} s v_s T_{n-r-s}, \end{aligned}$$

where the last line follows from applying (6) to T_{n-r} for each $r \in \{0, 1, \dots, R-1\}$. \square

Lemma 3. *Let $(\{T_n\}, \{t_n\}, \{a_n\})$ be a Warlimont triple. Then*

$$a_1 \sum_{s=0}^{\lfloor n/2 \rfloor} T_s t_{n-s} \leq T_n$$

for all n .

Proof. Since

$$\prod_{m=N+1}^{\infty} \left(\sum_{k=0}^{\infty} a_k x^{km} \right)^{t_m} \in 1 + x^{N+1} \mathbb{R}[[x]]$$

for every integer $N \geq 0$, we have

$$\sum_{n=0}^N T_n x^n \in \prod_{m=1}^N \left(\sum_{k=0}^{\infty} a_k x^{km} \right)^{t_m} + x^{N+1} \mathbb{R}[[x]]$$

and thus

$$\sum_{n=0}^N T_n x^n \in \left(\sum_{s=0}^{\lfloor N/2 \rfloor} T_s x^s \right) \prod_{m=\lfloor N/2 \rfloor + 1}^N \left(\sum_{k=0}^{\infty} a_k x^{km} \right)^{t_m} + x^{N+1} \mathbb{R}[[x]]. \tag{11}$$

On the other hand, by assumption t_m is a nonnegative integer for all m and thus by the binomial theorem

$$\left(\sum_{k=0}^{\infty} a_k x^{km} \right)^{t_m} \in 1 + a_1 t_m x^m + x^{2m} \mathbb{R}[[x]]. \tag{12}$$

Since the sequences $\{a_k\}$, $\{t_m\}$ and $\{T_n\}$ are nonnegative, the lemma follows by substituting (12) into (11) and comparing coefficients. \square

Lemma 4. *Let $(\{T_n\}, \{t_n\}, \{a_n\})$ be a Warlimont triple such that $\log(a_n) = O(n)$. Then for every nonnegative integer R*

$$|v_n - a_1 t_n| = \begin{cases} O(T_{n-R}) & \text{if } T_{n-1} = o(T_n), \\ O(t_{n-R}) & \text{if } t_{n-1} = o(t_n). \end{cases}$$

Proof. Assume $T_{n-1} = o(T_n)$. Since $\log(a_n) = O(n)$, there exists $r > 1$ such that $a_n \leq r^n$ for all n . By induction on the definition of $\{b_n\}$, we obtain $|b_n| \leq (3r)^n$ for all n . Moreover, since $T_{n-1} = o(T_n)$ and condition (iii) in the definition of Warlimont triple implies $T_n > 0$ for all but finitely many n , there exists a positive integer N such that $0 < T_n \leq (3r)^{-2} T_{n+1}$ for all $n \geq N$. If

$$C = \max \left\{ 1, \frac{(3r)^{2N} T_0}{T_N}, \frac{(3r)^{2N} T_1}{T_{N+1}}, \dots, \frac{(3r)^{2N} T_{N-1}}{T_{2N-1}} \right\}$$

then for any $n > 0$ and for any $m \geq N$ we obtain

$$T_n \leq C(3r)^{-2N} T_{n+N} \leq C(3r)^{-2(N+1)} T_{n+N+1} \leq \dots \leq C(3r)^{-2m} T_{n+m}.$$

Therefore, using Lemmas 2 and 3

$$|v_n - a_1 t_n| \leq \sum_{d/n} \frac{d}{n} t_d |b_{n/d}| \leq \sum_{d/n} T_d (3r)^{n/d} \leq C T_{n-R} \sum_{d/n} (3r)^{-n+2R+2d} = O(T_{n-R}).$$

The proof for the case $t_{n-1} = o(t_n)$ is similar and left to the reader. □

Theorem 5. *Let $(\{T_n\}, \{t_n\}, \{a_n\})$ be a Warlimont triple such that $\log(a_n) = O(n)$ and let R be a fixed positive integer. Then the following are equivalent:*

(i) $T_{n-1} = o(T_n)$ and

$$\sum_{s=R}^{n-R} T_s T_{n-s} = O(T_{n-R}).$$

(ii) $T_{n-1} = o(T_n)$ and

$$a_1 t_n = \sum_{s=0}^{R-1} \beta_s T_{n-s} + O(T_{n-R}).$$

(iii) $t_{n-1} = o(t_n)$ and

$$T_n = a_1 \sum_{s=0}^{R-1} T_s t_{n-s} + O(t_{n-R}).$$

(iv) $t_{n-1} = o(t_n)$ and

$$\sum_{s=R}^{n-R} t_s t_{n-s} = O(t_{n-R}).$$

Proof. Assume (i) holds. Using Lemmas 3 and 4 and $T_{n-1} = o(T_n)$, we obtain

$$|v_n| \leq |v_n - a_1 t_n| + a_1 t_n = O(T_n).$$

Therefore, there exist an integer $N > R$ and a constant $C > 0$ such that $|v_n| \leq C T_n \leq C T_{n+r}$ for all $n \geq N$ and for all $r \in \{0, \dots, R - 1\}$. Combining this observation with Lemma 2 yields

$$\begin{aligned} \left| v_n - \sum_{s=0}^{R-1} \beta_s T_{n-s} \right| &\leq \sum_{r=0}^{R-1} \beta_r \sum_{s=R-r}^{n-R} \frac{s}{n} |v_s| T_{n-s} \\ &\leq \sum_{r=0}^{R-1} \beta_r \left(\sum_{s=R-r}^{N-1} |v_s| T_{n-r-s} + C \sum_{s=N}^{n-R} T_s T_{n-r-s} \right) \\ &\leq \sum_{r=0}^{R-1} \beta_r \left(\sum_{s=R-r}^{N-1} |v_s| T_{n-r-s} + C^2 \sum_{s=R}^{n-R} T_s T_{n-s} \right) \\ &= O(T_{n-R}), \end{aligned}$$

and thus

$$\left| a_1 t_n - \sum_{s=0}^{R-1} \beta_s T_{n-s} \right| \leq |a_1 t_n - v_n| + \left| v_n - \sum_{s=0}^{R-1} \beta_s T_{n-s} \right| = O(T_{n-R}).$$

Hence, (i) implies (ii). Assume (ii) holds. Then

$$\begin{aligned} a_1 \sum_{s=0}^{R-1} T_s t_{n-s} &= \sum_{s=0}^{R-1} T_s \sum_{r=0}^{R-1-s} \beta_r T_{n-s-r} + O(T_{n-R}) \\ &= \sum_{s=0}^{R-1} \sum_{u=s}^{R-1} T_s \beta_{u-s} T_{n-u} + O(T_{n-R}) \\ &= \sum_{u=0}^{R-1} T_{n-u} \sum_{s=0}^u T_s \beta_{u-s} + O(T_{n-R}) \\ &= T_n + O(T_{n-R}), \end{aligned}$$

where the last equality is obtained using the definition of the sequence $\{\beta_n\}$. In particular, setting $R = 1$ we obtain $a_1 t_n - T_n = O(T_{n-1}) = o(T_n)$, which implies $a_1 t_n \sim T_n$ and hence $o(t_{n-1}) = O(T_{n-1}) = o(T_n) = o(t_n)$. Therefore, (ii) implies (iii). Assume (iii) holds. By Lemma 3

$$a_1^2 \sum_{s=R}^{\lfloor n/2 \rfloor} t_s t_{n-s} \leq a_1 \sum_{s=R}^{\lfloor n/2 \rfloor} T_s t_{n-s} \leq T_n - a_1 \sum_{s=0}^{R-1} T_s t_{n-s} = O(t_{n-R}).$$

This proves (iv) since

$$\sum_{s=R}^{n-R} t_s t_{n-s} = 2 \sum_{s=R}^{\lfloor n/2 \rfloor} t_s t_{n-s} + O(t_{n-R}).$$

Finally, assume (iv) holds. Lemma 4 implies $|v_n - a_1 t_n| = O(t_{n-R}) = o(t_n)$ and thus $v_n \sim a_1 t_n$. This implies that there exist an integer $N \geq R$ and constants $c, C > 0$ such that

$$0 < c t_n \leq v_n \leq C t_n \tag{13}$$

for all $n \geq N$. As a consequence,

$$v_{n-1} = O(t_{n-1}) = o(t_n) = o(v_n) \tag{14}$$

and

$$\sum_{j=N}^{n-N} v_{n-j} v_j \leq C^2 \sum_{j=R}^{n-R} t_{n-j} t_j = O(t_{n-R}) = O(v_{n-R}). \tag{15}$$

For each $n \geq N$, let

$$M_n = \max \left\{ \frac{T_j}{v_j} \mid N \leq j \leq n \right\}.$$

By Lemma 2, we obtain

$$|T_n - v_n| \leq \sum_{j=1}^{n-1} |v_{n-j}| T_j \leq \sum_{j=1}^{N-1} v_{n-j} T_j + M_{n-1} \left(\sum_{j=N}^{n-N} v_{n-j} v_j + \sum_{j=n-N+1}^{n-1} |v_{n-j}| v_j \right) = o(v_n)(1 + M_{n-1}),$$

where (14) and (15) were used to obtain the last equality. Hence there exists $N_1 \geq N$ such that for all $n \geq N_1$

$$\frac{T_n}{v_n} \leq \frac{3 + M_{n-1}}{2}$$

and thus

$$M_n = \max \left\{ M_{n-1}, \frac{T_n}{v_n} \right\} \leq \max \{ M_{n-1}, 3 \}.$$

This shows that the sequence $\{M_n\}$ is bounded; i.e., there exists a constant $K > 0$ such that $T_n \leq K v_n$ for all $n \geq N$. Therefore, using (13) and Lemma 3, we obtain

$$T_{n-1} = O(v_{n-1}) = O(t_{n-1}) = o(t_n) = o(T_n).$$

Moreover (14) yields

$$\sum_{s=R}^{n-R} T_{n-s} T_s \leq 2 \sum_{s=R}^N T_s T_{n-s} + K^2 \sum_{s=N}^{n-N} v_{n-s} v_s = O(T_{n-R}) + O(v_{n-R}) = O(T_{n-R}).$$

This concludes the proof that (iv) implies (i) and the theorem is proved. □

Remark 6. Let $(\{T_n\}, \{t_n\}, \{a_n\})$ be a Warlimont triple that satisfies the equivalent conditions of Theorem 5 for some $R > 2$. Then

$$\sum_{s=R-1}^{n-R+1} T_s T_{n-s} = \sum_{s=R}^{n-R} T_s T_{n-s} + 2T_{R-1} T_{n-R+1} = O(T_{n-R}) + O(T_{n-R+1}) = O(T_{n-R+1})$$

and thus $(\{T_n\}, \{t_n\}, \{a_n\})$ satisfies the equivalent conditions of Theorem 5 for any fixed positive integer less than or equal to R . In particular, $t_{n-1} = o(t_n)$ and $T_n \sim a_1 t_n$.

3. Warlimont functions and Wright semigroups

Definition 7. An *additive arithmetical semigroup* is a pair $(G, +, \partial)$ consisting of an abelian semigroup $(G, +)$ with identity and a semigroup homomorphism $\partial : (G, +) \rightarrow (\mathbb{Z}_{\geq 0}, +)$ such that

- (i) the cardinality G_n of the preimage $\partial^{-1}(n)$ is finite for all n ,
- (ii) G is freely generated by $G^+ \subseteq G$.

We denote by G_n^+ the cardinality of the set $\partial^{-1}(n) \cap G^+$.

Remark 8. Let $(G, +, \partial)$ be an additive arithmetical semigroup. As pointed out in [Knopfmacher 1976; Warlimont 1993], $(\{G_n\}, \{G_n^+\}, \{1\})$ is a Warlimont triple.

Definition 9. A *Wright semigroup* is an additive arithmetical semigroup $(G, +, \partial)$ satisfying

$$\log(G_n) = \alpha n^{a+1} + \beta n \log(n) + \gamma n + O(n^b) \quad (16)$$

for some real numbers $\alpha, \beta, \gamma, a, b$ such that $\alpha > 0$ and $0 < b < a$.

Definition 10. Let R be a positive integer. We say that an additive arithmetical semigroup $(G, +, \partial)$ satisfies *axiom* \mathcal{W}_R if $G_{n-1} = o(G_n)$ and

$$\sum_{s=R}^{n-R} G_s G_{n-s} = O(G_{n-R}).$$

Remark 11. Let $(G, +, \partial)$ be an additive arithmetical semigroup that satisfies axiom \mathcal{W}_R for some positive integer R . Combining Remarks 6 and 8, we conclude that $(G, +, \partial)$ satisfies axiom $\mathcal{W}_{R'}$ for any positive integer $R' \leq R$. In particular, $G_n \sim G_n^+$ and $G_{n-1}^+ = o(G_n^+)$; i.e., the additive arithmetical semigroup $(G, +, \partial)$ satisfies both axiom \mathcal{G}_1 and axiom \mathcal{G}_2 as defined in [Knopfmacher 1976]. Notice that the combination of Axioms \mathcal{G}_1 and \mathcal{G}_2 is slightly weaker than axiom \mathcal{W}_1 since $\sum_{s=1}^{n-1} G_s G_{n-s} = o(G_n)$ does not necessarily imply $\sum_{s=1}^{n-1} G_s G_{n-s} = O(G_{n-1})$.

Proposition 12. *Every Wright semigroup satisfies axiom \mathcal{W}_R for every positive integer R .*

Proof. This is a straightforward consequence of the definitions and Theorem 7 of [Wright 1970]. \square

Definition 13. Let $(G, +, \partial)$ be an additive arithmetical semigroup. A function $F : G \rightarrow \mathbb{R}$ is *multiplicative* if $F(g_1 + g_2) = F(g_1)F(g_2)$ for all $g_1, g_2 \in G$ coprime. We say that F is *prime-independent* if there exists a sequence $\{F_n^+\}$ such that $F_n^+ = F(np)$ for every $p \in G^+$ and every positive integer n . For every function $F : G \rightarrow \mathbb{R}$, we denote by $\{F_n\}$ the sequence defined by setting

$$F_n = \sum_{\partial(g)=n} F(g)$$

for each nonnegative integer n . A *Warlimont function* is a nonnegative multiplicative prime-independent function such that $\log(F_n^+) = O(n)$ and $F_1^+ > 0$. The *normalization* of a Warlimont function F is the (not necessarily multiplicative) function $\tilde{F} : G \rightarrow \mathbb{R}$ such that $\tilde{F}(g) = F(g)/F_1^+$ for all $g \in G$.

Example 14. Let $(G, +, \partial)$ be an additive arithmetical semigroup and let $F : G \rightarrow \mathbb{R}$ be such that $F(g) = 1$ for all $g \in G$. Then F is a Warlimont function and $F_n = \tilde{F}_n = G_n$ for all n .

Example 15. Let $(G, +, \partial)$ be an additive arithmetical semigroup and, for each $k \geq 2$, consider the *generalized divisor function* $d_k : G \rightarrow \mathbb{R}$ that to each $g \in G$ assigns the number $d_k(g)$ of k -tuples $(g_1, \dots, g_k) \in G^k$ such that $g = g_1 + \dots + g_k$. Then d_k is multiplicative, prime-independent and $(d_k)_n^+ = \binom{n+k-1}{k-1}$ for each integer $n \geq 1$. Therefore, d_k is Warlimont.

Example 16. Let $(G, +, \partial)$ be an additive arithmetical semigroup and consider the *unitary divisor function* $d_* : G \rightarrow \mathbb{R}$ that to each $g \in G$ assigns the number $d_*(g)$ of coprime pairs (g_1, g_2) such that $g = g_1 + g_2$. Then d_* is multiplicative, prime-independent and $(d_*)_n^+ = 2$ for each integer $n \geq 1$. Therefore d_* is Warlimont.

Example 17. Let $(G, +, \partial)$ be an additive arithmetical semigroup and consider the *prime divisor function* $B : G \rightarrow \mathbb{R}$ such that $B(k_1 p_1 + k_2 p_2 + \dots + k_r p_r) = k_1 k_2 \dots k_r$ for any $p_1, \dots, p_r \in G$ primes and k_1, \dots, k_r positive integers. Then B is multiplicative, prime-independent and $B_n^+ = n$ for each integer $n \geq 1$. Therefore, B is Warlimont.

Remark 18. Let F be a Warlimont function on an additive arithmetical semigroup $(G, +, \partial)$. Then the function $F^m : G \rightarrow \mathbb{R}$ such that $F^m(g) = (F(g))^m$ for all $g \in G$ is again a Warlimont function for every integer $m \geq 1$ since

$$\log((F^m)_n^+) = m \log(F_n^+) = O(n).$$

Moreover, $\widetilde{F}^m = (\widetilde{F})^m$.

Remark 19. Let F be a Warlimont function on an additive arithmetical semigroup $(G, +, \partial)$. Then, as observed in [Warlimont 1993], $(\{F_n\}, \{G_n^+\}, \{F_n^+\})$ is a Warlimont triple.

Theorem 20. *Let $(G, +, \partial)$ be an additive arithmetical semigroup that satisfies axiom \mathcal{W}_R and let F be a Warlimont function on G . Then for every positive integer M there exist constants ξ_1, \dots, ξ_{R-1} such that*

$$\sum_{\partial(g)=n} (\widetilde{F}(g) - 1)^M = \sum_{s=1}^{R-1} \xi_s G_{n-s} + O(G_{n-R}). \tag{17}$$

Proof. By Remark 19 and Example 14, $(\{G_n\}, \{G_n^+\}, \{1\})$ and $(\{F_n\}, \{G_n^+\}, \{F_n^+\})$ are both Warlimont triples. Since $\{G_n\}$ satisfies axiom \mathcal{W}_R , it follows from Theorem 5 applied to the Warlimont triple $(\{G_n\}, \{G_n^+\}, \{1\})$ that $G_{n-1}^+ = o(G_n^+)$,

$$\sum_{s=R}^{n-R} G_s^+ G_{n-s}^+ = O(G_{n-R}^+). \tag{18}$$

Moreover, if $\{\beta_n\}$ is the sequence defined recursively by setting $\beta_0 = 1$ and

$$\beta_n = - \sum_{s=0}^{n-1} \beta_s G_{n-s} \quad (19)$$

for every positive integer n , then

$$G_{n-s}^+ = \sum_{r=0}^{R-1} \beta_r G_{n-s-r} + O(G_{n-s-R}) \quad (20)$$

for all $s \geq 0$. In particular, we can apply Theorem 5 to the Warlimont triple $(\{F_n\}, \{G_n^+\}, \{F_n^+\})$ and obtain

$$F_n = F_1^+ \sum_{s=0}^{R-1} F_s G_{n-s}^+ + O(G_{n-R}^+). \quad (21)$$

Since by definition $G_n^+ \leq G_n$ for all n and $G_{n-s-R} = o(G_{n-R})$ for all $s > 0$, substituting (20) into (21) yields

$$\tilde{F}_n = \sum_{s=0}^{R-1} \left(\sum_{r=0}^s \beta_r F_{s-r} \right) G_{n-s} + O(G_{n-R}). \quad (22)$$

Using the binomial theorem and Remark 18 we obtain

$$\begin{aligned} \sum_{\partial(g)=n} (\tilde{F}(g) - 1)^M &= (-1)^M \sum_{\partial(g)=n} \sum_{m=0}^M (-1)^m \binom{M}{m} \tilde{F}^m(g) \\ &= (-1)^M \sum_{m=0}^M (-1)^m \binom{M}{m} (\tilde{F}^m)_n. \end{aligned} \quad (23)$$

Applying (22) to the Warlimont function F^m and substituting into the last line of (23) (after an obvious rearrangement) yields

$$\sum_{\partial(g)=n} (\tilde{F}(g) - 1)^M = \sum_{s=0}^{R-1} \xi_s G_{n-s} + O(G_{n-R}), \quad (24)$$

with

$$\begin{aligned} \xi_s &= (-1)^M \sum_{m=0}^M (-1)^m \binom{M}{m} \sum_{r=0}^s \beta_r (F^m)_{s-r} \\ &= (-1)^M \sum_{m=1}^M (-1)^m \binom{M}{m} \sum_{r=0}^s \beta_r (F^m)_{s-r} \end{aligned} \quad (25)$$

for all $s \in \{0, \dots, R-1\}$ where the second equality follows from (19) and Example 14. This implies (17) since, combining Remarks 18 and 19, $(F^m)_0 = 1$

for all m and thus

$$\xi_0 = (-1)^M \sum_{m=0}^M (-1)^m \binom{M}{m} \beta_0(F^m)_0 = 0. \quad \square$$

Definition 21. Let F be a Warlimont function on an additive arithmetical semigroup $(G, +, \partial)$ and let M be a positive integer. We define the *normalized M -th moments of F* to be the functions $\mu_{F,M} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$ defined by

$$\mu_{F,M}(n) = \frac{1}{G_n} \sum_{\partial(g)=n} (\tilde{F}(g) - 1)^M \tag{26}$$

for all $n \geq 0$.

Remark 22. Let F be a Warlimont function on an additive arithmetical semigroup $(G, +, \partial)$. The average value of F on $\partial^{-1}(n)$ is given by

$$\frac{F_n}{G_n} = F_1^+ (1 + \mu_{F,1}(n)).$$

The higher normalized moments can be thought of as capturing the deviation of F from F_1^+ . For instance, if $\mu_{F,1}(n) = o(1)$, then

$$\frac{1}{G_n} \sum_{\partial(g)=n} (F(g) - F_1^+)^2 = (F_1^+)^2 \mu_{F,2}(n)$$

can be thought of as an asymptotic measure of the variance of F on $\partial^{-1}(n)$.

Corollary 23. Let F be a Warlimont function on an additive arithmetical semigroup $(G, +, \partial)$ that satisfies axiom W_1 . Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{F_n}{G_n} &= F_1^+, \\ \lim_{n \rightarrow \infty} \frac{1}{G_n} \sum_{\partial(g)=n} (F(g) - F_1^+)^2 &= 0. \end{aligned}$$

Proof. Combining Remark 22 and Theorem 20 (with $R = M = 1$), we obtain

$$\frac{F_n}{G_n} = F_1^+ (1 + \mu_{F,1}(n)) = F_1^+ + o(1).$$

Similarly,

$$\frac{1}{G_n} \sum_{\partial(g)=n} (F(g) - F_1^+)^2 = (F_1^+)^2 \left(\xi_1 \frac{G_{n-1}}{G_n} + O\left(\frac{G_{n-1}}{G_n}\right) \right) = o(1). \quad \square$$

Remark 24. A slightly stronger (see Remark 11) version of Corollary 23 is proved in [Knopfmacher 1976] for particular choices of F . A sharper result is given in [Warlimont 1993] where it is shown that the assumption $G_{n-1} = O(G_n)$ (which is part of axiom W_1) is unnecessary.

Theorem 25. Let F be a Warlimont function on a Wright semigroup $(G, +, \partial)$ with α, a, b as in Definition 9 and let $q = e^{\alpha(a+1)}$:

(i) For every positive integer M there exists a sequence $\{\lambda_s\}$ of functions $\lambda_s: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$ such that $\log(\lambda_s(n)) = O(n^{a-1} + n^b)$ and, for every fixed positive integer R , the asymptotic relation

$$\mu_{F,M}(n) = \sum_{s=1}^{R-1} \lambda_s(n) q^{-sn^a} + O(\lambda_R(n) q^{-Rn^a}) \quad (27)$$

holds in the limit $n \rightarrow \infty$.

(ii) Assume further that there exist constants $0 \leq d_2 \leq d_1$ and a sequence $\{\psi_s\}$ of polynomials such that $\deg(\psi_s) \leq d_1 s - d_2$ for all $s \geq 1$ and, for every fixed positive integer R , the asymptotic relation

$$\frac{G_{n-1}}{G_n} = \sum_{s=1}^{R-1} \psi_s(n) q^{-ns} + O(n^{d_1 R - d_2} q^{-Rn}) \quad (28)$$

holds in the limit $n \rightarrow \infty$. Then there exists a sequence $\{\tau_s\}$ of polynomials such that $\deg(\tau_s) \leq d_1 s - d_2$ and, for every positive integer R , the asymptotic relation

$$\mu_{F,M}(n) = \sum_{s=1}^{R-1} \tau_s(n) q^{-sn} + O(n^{d_1 R - d_2} q^{-Rn}) \quad (29)$$

holds in the limit $n \rightarrow \infty$.

Proof. Let ξ_s be defined by (25) for all $s \geq 1$. By Proposition 12 and Theorem 20, we obtain

$$\mu_{F,M}(n) = \sum_{s=1}^{R-1} \xi_s \frac{G_{n-s}}{G_n} + O\left(\frac{G_{n-R}}{G_n}\right) \quad (30)$$

for every fixed integer $R > 0$. Since

$$\log\left(\frac{G_{n-s}}{G_n}\right) = \alpha((n-s)^{a+1} - n^{a+1}) + O(n^b) = -\alpha(a+1)sn^a + O(n^{a-1} + n^b),$$

in order to prove (i) it suffices to choose λ_s such that

$$\lambda_s(n) = \xi_s q^{sn^a} \frac{G_{n-s}}{G_n}$$

for all $n \geq s \geq 1$. Using (28) repeatedly and induction on t , for every fixed positive integer R , we obtain

$$\frac{G_{n-t}}{G_n} = \frac{G_{n-1}}{G_n} \cdots \frac{G_{n-t}}{G_{n-t+1}} = \sum_{s=t}^{R-1} v_{s,t}(n) q^{-sn} + O(n^{DR} q^{-Rn}), \quad (31)$$

where

$$v_{s,t}(n) = \sum_{i_1+\dots+i_t=s} \psi_{i_1}(n)\psi_{i_2}(n-1)\cdots\psi_{i_t}(n-t+1)q^{i_2+2i_3+\dots+(t-1)i_t} \quad (32)$$

is a polynomial in n of degree at most $d_1s - d_2t$ for all $1 \leq t \leq s$. Substituting (31) into (30) yields, for every fixed positive integer R ,

$$\begin{aligned} \mu_{F,M}(n) &= \sum_{t=1}^{R-1} \xi_t \sum_{s=t}^{R-1} v_{s,t}(n)q^{-ns} + O(n^{DR}q^{-nR}) \\ &= \sum_{s=1}^{R-1} \left(\sum_{t=1}^s \xi_t v_{s,t}(n) \right) q^{-sn} + O(n^{DR}q^{-nR}), \end{aligned}$$

which proves (ii) upon setting

$$\tau_s(n) = \sum_{t=1}^s \xi_t v_{s,t}(n) \quad (33)$$

for all s, n . □

Remark 26. Comparison of (28) and (16) shows that the assumptions of (ii) in Theorem 25 require in particular that (16) holds with $a = 1$.

4. Examples

4.1. Graphs. Let $(G, +)$ be the semigroup of (simple, unlabeled) graphs with semigroup operation $+$ given by disjoint union. If ∂ is the map that to each graph g assigns the cardinality of its set of vertices, then $(G, +, \partial)$ is an additive arithmetical semigroup and $g \in G^+$ if and only if the graph g is connected. As proved in [Wright 1969], there exists a sequence $\{\varphi_s\}$ of polynomials such that φ_s has degree $2s$ for every s and, for every fixed positive integer R , the asymptotic relation

$$G_n = \frac{2^{\binom{n}{2}}}{n!} \left(\sum_{s=0}^{R-1} \varphi_s(n)2^{-sn} + O(n^{2R}2^{-Rn}) \right) \quad (34)$$

holds in the limit $n \rightarrow \infty$. The polynomials φ_s can be calculated explicitly, the first few being

$$\begin{aligned} \varphi_0(n) &= 1, \\ \varphi_1(n) &= 2n^2 - 2n, \\ \varphi_2(n) &= 8n^4 - \frac{128}{3}n^3 + 72n^2 - \frac{112}{3}n, \\ \varphi_3(n) &= \frac{256}{3}n^6 - \frac{3712}{3}n^5 + \frac{20672}{3}n^4 - \frac{54272}{3}n^3 + 21952n^2 - 9600n. \end{aligned}$$

In particular,

$$\log(G_n) = \log(\sqrt{2})n^2 - n \log(n) + (1 - \log(\sqrt{2}))n + O(n^b)$$

for any $b > 0$ and thus $(G, +, \partial)$ is a Wright semigroup. Moreover, using (34) and expanding the denominator as a geometric series we obtain, for every fixed $R > 0$,

$$\begin{aligned} \frac{G_{n-1}}{G_n} &= 2n2^{-n} \left(\sum_{s=0}^{R-1} 2^s \varphi_s(n-1) 2^{-sn} \right) \sum_{r=0}^{R-1} \left(- \sum_{s=1}^{R-1} \varphi_s(n) 2^{-sn} \right)^r + O(n^{2R+1} 2^{-(R+1)s}) \\ &= \sum_{s=1}^{R-1} \psi_s(n) 2^{-sn} + O(n^{2R-1} 2^{-Rn}), \end{aligned}$$

where the ψ_s are polynomials of degree $\deg(\psi_s) = 2s - 1$ which can be explicitly calculated in terms of the polynomials φ_s in (34). For instance

$$\psi_1(n) = n,$$

$$\psi_2(n) = 4n^3 - 20n^2 + 16n,$$

$$\psi_3(n) = 40n^5 - 464n^4 + 1768n^3 - 2624n^2 + 1280n,$$

$$\psi_4(n) = \frac{3248}{3}n^7 - 24176n^6 + \frac{630608}{3}n^5 - 908496n^4 + \frac{6137792}{3}n^3 - 2250240n^2 + 925696n.$$

Substitution into (32) yields $v_{s,1}(n) = \psi_s(n)$ for all s and

$$v_{2,2}(n) = 8n^2 - 8n,$$

$$v_{3,2}(n) = 48n^4 - 352n^3 + 688n^2 - 384n,$$

$$v_{3,3}(n) = 64n^3 - 192n^2 + 128n,$$

$$v_{4,2}(n) = 864n^6 - 13472n^5 + 77216n^4 - 203488n^3 + 245376n^2 - 106496n,$$

$$v_{4,3}(n) = 896n^5 - 9728n^4 + 35200n^3 - 50944n^2 + 24576n,$$

$$v_{4,4}(n) = 1024n^4 - 6144n^3 + 11264n^2 - 6144n.$$

Inspection of graphs with up to four vertices shows that $G_1 = 1$, $G_2 = 2$, $G_3 = 4$ and $G_4 = 11$. Substitution into (19) yields $\beta_1 = \beta_2 = \beta_3 = -1$ and $\beta_4 = -4$.

Example 27. Consider the Warlimont function d_2 from Example 15. When specialized to the semigroup of graphs, d_2 counts the number of ways of writing a given graph as the disjoint union of two graphs. The order is taken into account, so that if g_1 is not isomorphic to g_2 , then $g = g_1 + g_2$ and $g = g_2 + g_1$ count as two distinct decompositions. Moreover, decompositions in which one of the components is the empty graph are allowed. Combining Remark 22 and Theorem 25 we obtain (4). In particular, setting $M = 1$ yields a full asymptotic expansion for the average of d_2 of the form

$$\frac{1}{G_n} \sum_{\partial(g)=n} d_2(g) = 2 + 2 \sum_{s=1}^{R-1} \tau_s(n) 2^{-sn} + O(n^{2R-1} 2^{-Rn}),$$

valid for every fixed positive integer R , where the $\tau_s(n)$ are polynomials of degree $2s - 1$. For instance, direct inspection of graphs with up to four vertices yields

$(d_2)_1 = 2, (d_2)_2 = 5, (d_2)_3 = 12$ and $(d_2)_4 = 34$. Substituting into (25) and then into (33) we obtain

$$\tau_1(n) = 2n,$$

$$\tau_2(n) = 4n^3 - 4n^2,$$

$$\tau_3(n) = 40n^5 - 368n^4 + 1320n^3 - 2016n^2 + 1024n,$$

$$\tau_4(n) = \frac{3248}{3}n^7 - 22448n^6 + \frac{560528}{3}n^5 - 781712n^4 + \frac{5136512}{3}n^3 - 1839360n^2 + 743424n.$$

4.2. Graphs with an even number of edges. Let $(G, +)$ be the semigroup of (simple, unlabeled) graphs with an even number of edges and semigroup operation $+$ given by disjoint union. If ∂ is the map that to each graph g assigns the cardinality of its set of vertices, then $(G, +, \partial)$ is an additive arithmetical semigroup. G^+ consists of graphs g with an even number of edges that cannot be written as the disjoint union of two nonempty graphs with an even number of edges. While G is a subsemigroup of the semigroup of all unlabeled graphs, not all graphs in G^+ are connected. For instance, while $2K_1$ is not connected, it is nevertheless prime in the semigroup of graphs with even edges. As pointed out in [Aldi 2019], for every fixed positive integer R , the asymptotic relation

$$G_n = \frac{2^{\binom{n}{2}}}{2n!} \left(\sum_{s=0}^{R-1} \varphi_s(n) 2^{-sn} + O(n^{2R} 2^{-Rn}) \right)$$

holds in the $n \rightarrow \infty$ limit, where the polynomials $\varphi_s(n)$ coincide with those of Section 4.1. In particular, $(G, +, \partial)$ is a Wright semigroup and, for every fixed positive integer R ,

$$\frac{G_{n-1}}{G_n} = \sum_{s=1}^{R-1} \psi_s(n) 2^{-sn} + O(n^{2R-1} 2^{-Rn}),$$

where the polynomials $\psi_s(n)$ coincide with those calculated in Section 4.1. Inspection of graphs with up to four vertices shows that $G_1 = G_2 = 1, G_3 = 2$ and $G_4 = 6$. Substitution into (19) yields $\beta_1 = -1, \beta_2 = 0, \beta_3 = -1$ and $\beta_4 = -3$.

Example 28. Consider the Warlimont function d_* from Example 16. Combining Remark 22 and Theorem 25 we obtain a full asymptotic expansion for the second moment of d_* about 2:

$$\frac{1}{G_n} \sum_{\partial(g)=n} (d_*(g) - 2)^2 = 4 \sum_{s=1}^{R-1} \tau_s(n) 2^{-sn} + O(n^{2R-1} 2^{-Rn})$$

for every fixed positive integer R , where the $\tau_s(n)$ are polynomials of degree $2s - 1$. To calculate these explicitly for small values of s , we first observe by direct calculation that $(d_*)_1 = 2, (d_*)_2 = 2, (d_*)_3 = 4, (d_*)_4 = 14$ as well as $(d_*^2)_1 = 4, (d_*^2)_2 = 4, (d_*^2)_3 = 8, (d_*^2)_4 = 36$. Substitution into (25) (upon setting $M = 2$) and

then into (33) yields

$$\begin{aligned} \tau_1(n) &= 2n, \\ \tau_2(n) &= 4n^3 - 20n^2 + 16n, \\ \tau_3(n) &= 40n^5 - 464n^4 + 1832n^3 - 2816n^2 + 1408n, \\ \tau_4(n) &= \frac{3248}{3}n^7 - 24176n^6 + \frac{633296}{3}n^5 - 906960n^4 + \frac{6040640}{3}n^3 - 2177280n^2 + 882688n. \end{aligned}$$

4.3. Polynomials over a finite field. Consider the field \mathbb{F}_q with q elements and let G be the set of nonzero polynomials in $\mathbb{F}_q[x_1, \dots, x_k]$ modulo the equivalence relation such that $f \sim g$ if and only if $f = \lambda g$ for some $\lambda \in \mathbb{F}_q$. G has a natural structure of additive semigroup with semigroup operation $+$ given by multiplication of polynomials. If ∂ is the semigroup homomorphism that to each polynomial $f \in G$ assigns its total degree, then $(G, +, \partial)$ is an additive arithmetical semigroup and G^+ is the set of equivalent classes of irreducible polynomials in $\mathbb{F}_q[x_1, \dots, x_k]$. Since

$$G_n = \frac{q^{\binom{n+k}{k}} - q^{\binom{n-1+k}{k}}}{q - 1} \tag{35}$$

for every n ,

$$\log(G_n) = \log(q) \frac{n^k}{k!} + O(n^{k-1})$$

for every $k \geq 2$. On the other hand if $k = 1$, then $\log(G_n) = \log(q)n$ for every n . Hence (G, \cdot, ∂) is a Wright semigroup if and only if $k \geq 2$. If $k = 2$ then for every fixed positive integer R

$$\frac{G_{n-1}}{G_n} = q^{-n-1} \frac{1 - q^{-n}}{1 - q^{-n-1}} = \sum_{s=1}^{R-1} \psi_s(n) q^{-sn} + O(q^{-Rn}),$$

where $\psi_1(n) = q^{-1}$ and $\psi_s(n) = q^{-s}(1 - q)$ for all $s \geq 2$. By Theorem 25, each $\mu_{F,M}$ admits an asymptotic expansion as a power series in q^{-n} with constant coefficients. For instance, substitution into (32) yields

$$\begin{aligned} v_{2,1}(n) &= q^{-2} - q^{-1}, & v_{2,2}(n) &= q^{-1}, \\ v_{3,1}(n) &= q^{-3} - q^{-2}, & v_{3,2}(n) &= q^{-2} - 1, & v_{3,3}(n) &= 1. \end{aligned}$$

Example 29. We further specialize to the case where G is the semigroup of nonzero polynomials in two variables over the field with two elements. By Theorem 25, there exist constants τ_s such that for every fixed positive integer R the average of the Warlimont function B (as defined in Example 17) on polynomials of degree n is

$$\frac{B_n}{G_n} = 1 + \sum_{s=1}^{R-1} \tau_s 2^{-sn} + O(2^{-Rn}). \tag{36}$$

Since $B_1 = 6$, $B_2 = 62$ and $B_3 = 1002$, substituting (35) into (19) and then into (25) shows that in particular $\tau_1 = 0$, $\tau_2 = 3$ and $\tau_3 = \frac{3}{2}$.

Example 30. If $k > 2$, then by Remark 26 we are in the second part of Theorem 25. Nevertheless, the asymptotic behavior of Warlimont functions can be described using (27) as follows. Consider for instance the Warlimont function B of Example 17 on the semigroup of polynomials in three variables with coefficients in \mathbb{F}_q . Since

$$G_1 = -\beta_1 = (B^m)_1$$

for all m , substitution in (25) yields $\xi_1 = 0$ and thus

$$\frac{1}{G_n} \sum_{\partial(g)=n} (B(g) - 1)^M = O\left(\frac{G_{n-2}}{G_n}\right) = O(q^{-n^2-2n})$$

for all M .

Acknowledgments

Part of this work was carried out during the summer of 2016 at VCU and supported by a UROP Summer Research Fellowship.

References

- [Aldi 2019] M. Aldi, “Arithmetical semirings”, *Discrete Math.* **342**:7 (2019), 2035–2047. MR Zbl
- [Burris 2001] S. N. Burris, *Number theoretic density and logical limit laws*, Math. Surveys and Monographs **86**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl
- [Flajolet and Sedgewick 2009] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge Univ. Press, 2009. MR Zbl
- [Knopfmacher 1975] J. Knopfmacher, *Abstract analytic number theory*, North-Holland Math. Library **12**, North-Holland, Amsterdam, 1975. MR Zbl
- [Knopfmacher 1976] J. Knopfmacher, “Arithmetical properties of finite graphs and polynomials”, *J. Combinatorial Theory Ser. B* **20**:3 (1976), 205–215. MR Zbl
- [Warlimont 1993] R. Warlimont, “A relationship between two sequences and arithmetical semigroups”, *Math. Nachr.* **164** (1993), 201–217. MR Zbl
- [Warlimont 2001] R. Warlimont, “About the radius of convergence of the zeta function of an additive arithmetical semigroup”, *Quaest. Math.* **24**:3 (2001), 355–362. MR Zbl
- [Wright 1967] E. M. Wright, “A relationship between two sequences”, *Proc. London Math. Soc.* (3) **17** (1967), 296–304. MR Zbl
- [Wright 1969] E. M. Wright, “The number of graphs on many unlabelled nodes”, *Math. Ann.* **183** (1969), 250–253. MR Zbl
- [Wright 1970] E. M. Wright, “Asymptotic relations between enumerative functions in graph theory”, *Proc. London Math. Soc.* (3) **20** (1970), 558–572. MR Zbl

Received: 2017-07-08 Revised: 2019-01-08 Accepted: 2019-04-02

maldi2@vcu.edu

*Department of Mathematics and Applied Mathematics,
Virginia Commonwealth University, Richmond, VA,
United States*

tanh4@vcu.edu

*Virginia Commonwealth University, Richmond, VA,
United States*

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

BOARD OF EDITORS

Colin Adams	Williams College, USA	Robert B. Lund	Clemson University, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Gaven J. Martin	Massey University, New Zealand
Martin Bohner	Missouri U of Science and Technology, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of N Carolina, Chapel Hill, USA	Frank Morgan	Williams College, USA
Pietro Cerone	La Trobe University, Australia	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Scott Chapman	Sam Houston State University, USA	Zuhair Nashed	University of Central Florida, USA
Joshua N. Cooper	University of South Carolina, USA	Ken Ono	Univ. of Virginia, Charlottesville
Jem N. Corcoran	University of Colorado, USA	Yuval Peres	Microsoft Research, USA
Toka Diagana	Howard University, USA	Y.-F. S. Pétermann	Université de Genève, Switzerland
Michael Dorff	Brigham Young University, USA	Jonathon Peterson	Purdue University, USA
Sever S. Dragomir	Victoria University, Australia	Robert J. Plemmons	Wake Forest University, USA
Joel Foisy	SUNY Potsdam, USA	Carl B. Pomerance	Dartmouth College, USA
Errin W. Fulp	Wake Forest University, USA	Vadim Ponomarenko	San Diego State University, USA
Joseph Gallian	University of Minnesota Duluth, USA	Bjorn Poonen	UC Berkeley, USA
Stephan R. Garcia	Pomona College, USA	József H. Przytycki	George Washington University, USA
Anant Godbole	East Tennessee State University, USA	Richard Rebarber	University of Nebraska, USA
Ron Gould	Emory University, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Javier Rojo	Oregon State University, USA
Jim Haglund	University of Pennsylvania, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Johnny Henderson	Baylor University, USA	Hari Mohan Srivastava	University of Victoria, Canada
Glenn H. Hurlbert	Virginia Commonwealth University, USA	Andrew J. Sterge	Honorary Editor
Charles R. Johnson	College of William and Mary, USA	Ann Trenk	Wellesley College, USA
K. B. Kulasekera	Clemson University, USA	Ravi Vakil	Stanford University, USA
Gerry Ladas	University of Rhode Island, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
David Larson	Texas A&M University, USA	John C. Wierman	Johns Hopkins University, USA
Suzanne Lenhart	University of Tennessee, USA	Michael E. Zieve	University of Michigan, USA
Chi-Kwong Li	College of William and Mary, USA		

PRODUCTION

Silvio Levy, Scientific Editor


Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2019 is US \$195/year for the electronic version, and \$260/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2019 Mathematical Sciences Publishers

involve

2019 vol. 12 no. 7

Asymptotic expansion of Warlimont functions on Wright semigroups MARCO ALDI AND HANQIU TAN	1081
A systematic development of Jeans' criterion with rotation for gravitational instabilities KOHL GILL, DAVID J. WOLLKIND AND BONNI J. DICHONE	1099
The linking-unlinking game ADAM GIAMBRONE AND JAKE MURPHY	1109
On generalizing happy numbers to fractional-base number systems ENRIQUE TREVIÑO AND MIKITA ZHYLINSKI	1143
On the Hadwiger number of Kneser graphs and their random subgraphs ARRAN HAMM AND KRISTEN MELTON	1153
A binary unrelated-question RRT model accounting for untruthful responding AMBER YOUNG, SAT GUPTA AND RYAN PARKS	1163
Toward a Nordhaus–Gaddum inequality for the number of dominating sets LAUREN KEOUGH AND DAVID SHANE	1175
On some obstructions of flag vector pairs (f_1, f_{04}) of 5-polytopes HYE BIN CHO AND JIN HONG KIM	1183
Benford's law beyond independence: tracking Benford behavior in copula models REBECCA F. DURST AND STEVEN J. MILLER	1193
Closed geodesics on doubled polygons IAN M. ADELSTEIN AND ADAM Y. W. FONG	1219
Sign pattern matrices that allow inertia \mathbb{S}_n ADAM H. BERLINER, DEREK DEBLIECK AND DEEPAK SHAH	1229
Some combinatorics from Zeckendorf representations TYLER BALL, RACHEL CHAISER, DEAN DUSTIN, TOM EDGAR AND PAUL LAGARDE	1241



1944-4176(2019)12:7;1-Z