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Asymptotic expansion of Warlimont functions on Wright semigroups

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We calculate full asymptotic expansions of prime-independent multiplicative functions on additive arithmetic semigroups that satisfy a strong form of Knopfmacher's axioms. When applied to the semigroup of unlabeled graphs, our method yields detailed asymptotic information on how graphs decompose into connected components. As a second class of examples, we discuss polynomials in several variables over a finite field.

1. Introduction

Let G_n be the number of unlabeled graphs with n vertices and let G_n^+ be the number of connected unlabeled graphs with n vertices. Using the fact that the sequences $\{G_n\}$ and $\{G_n^+\}$ are related by the identity

$$\sum_{n=0}^{\infty} G_n x^n = \prod_{m=1}^{\infty} (1-x^m)^{-G_m^+}, \tag{1}$$

Wright [1967] proved that $G_n \sim G_n^+$; i.e., almost all graphs are connected. As observed in that paper and further clarified in [Warlimont 2001], this asymptotic result is intimately related to the fact that the power series at the right-hand side of (1) has trivial convergence radius. Armed with a full asymptotic expansion for G_n [Wright 1969], Wright [1970] further improved this result by constructing a sequence $\{\omega_s\}$ of polynomials such that, for any fixed positive integer R, the asymptotic relation

$$G_n^+ = G_n + \sum_{s=1}^{R-1} \omega_s(n) G_{n-s} + O(n^R G_{n-R}) \tag{2}$$

holds in the limit $n \to \infty$.

In the context of abstract analytic number theory [Knopfmacher 1975], Knopfmacher [1976] (see also [Flajolet and Sedgewick 2009; Burris 2001] for the more

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general setting of weighted decomposable combinatorial structures) observed that (1) is a particular case of an Euler product type of identity which holds for arbitrary additive arithmetical semigroups and that the methods of [Wright 1967] can be used to study the distribution of certain arithmetical functions on additive arithmetical semigroups in which almost all elements are prime. For instance, if d_2 is the *divisor function* that to each unlabeled graph g assigns the number of ways to write g as a disjoint union of an ordered pair of graphs then

$$\lim_{n\to\infty}\frac{1}{G_n}\sum d_2(g)=2 \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{G_n}\sum (d_2(g)-2)^2=0, \tag{3}$$

where both sums are taken over all graphs g with n vertices.

The goal of the present paper is to investigate Knopfmacher's suggestion [1976] that restricting to arithmetical semigroups in which the total number of elements is related to the number of prime elements by a formula analogous to (2) might lead to a strengthening of (3). To illustrate our results with an example, consider again the divisor function d_2 on the semigroup of graphs. We prove that for every positive integer M, there exists a sequence $\{\tau_s(n)\}$ of polynomials such that, for any fixed positive integer R, the asymptotic relation

$$\frac{1}{G_n}\sum (d_2(g)-2)^M = 2^M\sum_{s=1}^{R-1}\tau_s(n)2^{-sn}+O(n^{2R-1}2^{-Rn}) \tag{4}$$

holds in the limit $n\to\infty$. Clearly, (3) can be recovered by setting $M=1$ and $M=2$ in (4) and taking the limit as $n\to\infty$. More generally, we show that (4) is a particular case of a formula that holds if d_2 is replaced by an arbitrary *Warlimont function*, i.e., a multiplicative prime-independent function whose restriction to power of primes grows in a prescribed way. Even more generally, the semigroup of graphs can be replaced by any *Wright semigroup*, which we define to be an additive arithmetical semigroup subject to a growth condition introduced in [Wright 1970]. Examples of Wright semigroups include the semigroup of unlabeled graphs with an even number of edges and the semigroup of polynomials in at least two variables over a finite field.

The paper is organized as follows. Section 2 contains the main technical results used in the rest of the paper. We work with triples of sequences related by a generalization of (1) that were introduced in [Warlimont 1993]. The main result is Theorem 5 which can be thought of as a generalization of [Wright 1970], modeled after the way in which [Warlimont 1993] generalizes [Wright 1967]. In Section 3, after introducing the key notions of Wright semigroup and of Warlimont function, we provide asymptotic formulas for moments of Warlimont functions in terms of the number of elements of given degree in the underlying (not necessarily Wright) semigroup. In the special case of Wright semigroup, we construct full asymptotic expansions generalizing (4). We illustrate our results in Section 4 by calculating asymptotic expansion of some of the arithmetical functions considered in [Knopfmacher 1976] on three examples of Wright semigroups: the semigroup of all unlabeled graphs, the semigroup of unlabeled graphs with an even number of edges and the semigroup of nonzero polynomials (up to scaling) in at least two variables over a finite field.

2. Warlimont triples

Definition 1. A *Warlimont triple* is a triple $(\{T_n\}, \{t_n\}, \{a_n\})$ of sequences of non-negative real numbers related by the identity

$$\sum_{n=0}^{\infty} T_n x^n = \prod_{m=1}^{\infty} \left(\sum_{k=0}^{\infty} a_k x^{km} \right)^{t_m} \tag{5}$$

of formal power series and such that

(i) $T_0 = a_0 = 1$,

(ii) $a_1 > 0$,

(iii) $t_m \in \mathbb{Z}$ for all m and $t_m > 0$ for all but finitely many m.

In order for the three sequences to be all indexed by nonnegative integers, we set $t_0 = 0$.

Lemma 2. *Let $(\{T_n\}, \{t_n\}, \{a_n\})$ be a Warlimont triple and consider the sequences $\{v_n\}$, $\{\beta_n\}$ and $\{b_n\}$ defined the recursion formulas*

$$v_n = T_n - \sum_{s=1}^{n-1} \frac{s}{n} v_s T_{n-s}, \tag{6}$$

$$\beta_n = -\sum_{s=0}^{n-1} \beta_s T_{n-s}, \tag{7}$$

$$b_n = n a_n - \sum_{s=1}^{n-1} b_s a_{n-s}, \tag{8}$$

with initial conditions $v_1 = T_1,\ \beta_0 = 1,\ b_1 = a_1$. Then for all n:

(i) $v_n = \sum_{d\,|\,n} (d/n) t_d b_{n/d}$, *where the sum is over all integers $1 < d \le n$ that divide n.*

(ii) $\beta_n = -\sum_{s=1}^{n} (s/n) v_s \beta_{n-s}$.

(iii) *For every positive integer R*

$$\sum_{s=0}^{R-1} \beta_s T_{n-s} = v_n + \frac{1}{n} \sum_{r=0}^{R-1} \beta_r \sum_{s=R-r}^{n-R} s v_s T_{n-r-s}.$$

Proof. Using formal term-by-term differentiation it is easy to show that (6) and (7) are equivalent to the formal identities

$$\log\Big(\sum_{n=0}^{\infty} T_n x^n\Big) = \sum_{m=1}^{\infty} v_m x^m, \tag{9}$$

$$\log\Big(\sum_{n=0}^{\infty} a_n x^n\Big) = \sum_{s=1}^{\infty} \frac{b_s}{s} x^s, \tag{10}$$

respectively. Taking the formal logarithm of (5) and substituting (9), (10), we obtain

$$\sum_{m=1}^{\infty} v_m x^m = \sum_{r,s=1}^{\infty} t_r \frac{b_s}{s} x^{rs}$$

from which (i) easily follows. Since (7) is equivalent to the identity

$$\Big(\sum_{s=0}^{\infty} \beta_s x^s\Big)\Big(\sum_{n=0}^{\infty} T_n x^n\Big) = 1$$

of formal power series, taking formal logarithms yields

$$\log\Big(\sum_{s=0}^{\infty} \beta_s x^s\Big) = -\sum_{m=1}^{\infty} v_m x^m.$$

Comparing with (6) and (9) proves (ii). It follows from (ii) that

$$\sum_{u=0}^{R-1}\sum_{r=0}^{u} \beta_r((n-u)v_{n-u}T_{u-r} + (u-r)v_{u-r}T_{n-u}) = v_n - \sum_{u=0}^{R-1} u\beta_u T_{n-u}$$

and thus

$$\begin{aligned}
&\sum_{s=0}^{R-1} \beta_s T_{n-s} - v_n \\
&\qquad = \frac{1}{n}\sum_{u=0}^{R-1}\sum_{r=0}^{u} \beta_r \Big(\frac{n-r}{R-r} T_{n-r} - (n-u)v_{n-u}T_{u-r} - (u-r)v_{u-r}T_{n-u}\Big) \\
&\qquad = \frac{1}{n}\sum_{r=0}^{R-1} \beta_r \Big((n-r)T_{n-r} - \sum_{s=0}^{R-r-1}(n-r-s)T_s v_{n-r-s} + \sum_{s=0}^{R-r-1} s v_s T_{n-r-s}\Big) \\
&\qquad = \frac{1}{n}\sum_{r=0}^{R-1} \beta_r \sum_{s=R-r}^{n-R} s v_s T_{n-r-s},
\end{aligned}$$

where the last line follows from applying (6) to T_{n-r} for each $r \in \{0, 1, \ldots, R-1\}$. □

Lemma 3. *Let* $(\{T_n\}, \{t_n\}, \{a_n\})$ *be a Warlimont triple. Then*

$$a_1 \sum_{s=0}^{\lfloor n/2 \rfloor} T_s t_{n-s} \le T_n$$

for all n.

Proof. Since

$$\prod_{m=N+1}^{\infty} \Big(\sum_{k=0}^{\infty} a_k x^{km} \Big)^{t_m} \in 1 + x^{N+1} \mathbb{R}[\![x]\!]$$

for every integer $N \ge 0$, we have

$$\sum_{n=0}^{N} T_n x^n \in \prod_{m=1}^{N} \Big(\sum_{k=0}^{\infty} a_k x^{km} \Big)^{t_m} + x^{N+1} \mathbb{R}[\![x]\!]$$

and thus

$$\sum_{n=0}^{N} T_n x^n \in \Big(\sum_{s=0}^{\lfloor N/2 \rfloor} T_s x^s \Big) \prod_{m=\lfloor N/2 \rfloor + 1}^{N} \Big(\sum_{k=0}^{\infty} a_k x^{km} \Big)^{t_m} + x^{N+1} \mathbb{R}[\![x]\!]. \tag{11}$$

On the other hand, by assumption t_m is a nonnegative integer for all m and thus by the binomial theorem

$$\Big(\sum_{k=0}^{\infty} a_k x^{km} \Big)^{t_m} \in 1 + a_1 t_m x^m + x^{2m} \mathbb{R}[\![x]\!]. \tag{12}$$

Since the sequences $\{a_k\}$, $\{t_m\}$ and $\{T_n\}$ are nonnegative, the lemma follows by substituting (12) into (11) and comparing coefficients. □

Lemma 4. *Let* $(\{T_n\}, \{t_n\}, \{a_n\})$ *be a Warlimont triple such that* $\log(a_n) = O(n)$. *Then for every nonnegative integer* R

$$|v_n - a_1 t_n| = \begin{cases} O(T_{n-R}) & \text{if } T_{n-1} = o(T_n), \\ O(t_{n-R}) & \text{if } t_{n-1} = o(t_n). \end{cases}$$

Proof. Assume $T_{n-1} = o(T_n)$. Since $\log(a_n) = O(n)$, there exists $r > 1$ such that $a_n \le r^n$ for all n. By induction on the definition of $\{b_n\}$, we obtain $|b_n| \le (3r)^n$ for all n. Moreover, since $T_{n-1} = o(T_n)$ and condition (iii) in the definition of Warlimont triple implies $T_n > 0$ for all but finitely many n, there exists a positive integer N such that $0 < T_n \le (3r)^{-2} T_{n+1}$ for all $n \ge N$. If

$$C = \max\Big\{ 1, \frac{(3r)^{2N} T_0}{T_N}, \frac{(3r)^{2N} T_1}{T_{N+1}}, \dots, \frac{(3r)^{2N} T_{N-1}}{T_{2N-1}} \Big\}$$

then for any $n > 0$ and for any $m \ge N$ we obtain

$$T_n \le C(3r)^{-2N} T_{n+N} \le C(3r)^{-2(N+1)} T_{n+N+1} \le \cdots \le C(3r)^{-2m} T_{n+m}.$$

Therefore, using Lemmas 2 and 3

$$|v_n - a_1 t_n| \le \sum_{d/n} \frac{d}{n} t_d |b_{n/d}| \le \sum_{d/n} T_d (3r)^{n/d} \le C T_{n-R} \sum_{d/n} (3r)^{-n+2R+2d} = O(T_{n-R}).$$

The proof for the case $t_{n-1} = o(t_n)$ is similar and left to the reader. □

Theorem 5. *Let* $(\{T_n\}, \{t_n\}, \{a_n\})$ *be a Warlimont triple such that* $\log(a_n) = O(n)$ *and let* R *be a fixed positive integer. Then the following are equivalent*:

(i) $T_{n-1} = o(T_n)$ *and*

$$\sum_{s=R}^{n-R} T_s T_{n-s} = O(T_{n-R}).$$

(ii) $T_{n-1} = o(T_n)$ *and*

$$a_1 t_n = \sum_{s=0}^{R-1} \beta_s T_{n-s} + O(T_{n-R}).$$

(iii) $t_{n-1} = o(t_n)$ *and*

$$T_n = a_1 \sum_{s=0}^{R-1} T_s t_{n-s} + O(t_{n-R}).$$

(iv) $t_{n-1} = o(t_n)$ *and*

$$\sum_{s=R}^{n-R} t_s t_{n-s} = O(t_{n-R}).$$

Proof. Assume (i) holds. Using Lemmas 3 and 4 and $T_{n-1} = o(T_n)$, we obtain

$$|v_n| \le |v_n - a_1 t_n| + a_1 t_n = O(T_n).$$

Therefore, there exist an integer $N > R$ and a constant $C > 0$ such that $|v_n| \le C T_n \le C T_{n+r}$ for all $n \ge N$ and for all $r \in \{0, \dots, R-1\}$. Combining this observation with Lemma 2 yields

$$\begin{aligned}
\left| v_n - \sum_{s=0}^{R-1} \beta_s T_{n-s} \right| &\le \sum_{r=0}^{R-1} \beta_r \sum_{s=R-r}^{n-R} \frac{s}{n} |v_s| T_{n-s} \\
&\le \sum_{r=0}^{R-1} \beta_r \left(\sum_{s=R-r}^{N-1} |v_s| T_{n-r-s} + C \sum_{s=N}^{n-R} T_s T_{n-r-s} \right) \\
&\le \sum_{r=0}^{R-1} \beta_r \left(\sum_{s=R-r}^{N-1} |v_s| T_{n-r-s} + C^2 \sum_{s=R}^{n-R} T_s T_{n-s} \right) \\
&= O(T_{n-R}),
\end{aligned}$$

and thus

$$\left|a_1 t_n - \sum_{s=0}^{R-1} \beta_s T_{n-s}\right| \le |a_1 t_n - v_n| + \left|v_n - \sum_{s=0}^{R-1} \beta_s T_{n-s}\right| = O(T_{n-R}).$$

Hence, (i) implies (ii). Assume (ii) holds. Then

$$\begin{aligned} a_1 \sum_{s=0}^{R-1} T_s t_{n-s} &= \sum_{s=0}^{R-1} T_s \sum_{r=0}^{R-1-s} \beta_r T_{n-s-r} + O(T_{n-R}) \\ &= \sum_{s=0}^{R-1} \sum_{u=s}^{R-1} T_s \beta_{u-s} T_{n-u} + O(T_{n-R}) \\ &= \sum_{u=0}^{R-1} T_{n-u} \sum_{s=0}^{u} T_s \beta_{u-s} + O(T_{n-R}) \\ &= T_n + O(T_{n-R}), \end{aligned}$$

where the last equality is obtained using the definition of the sequence $\{\beta_n\}$. In particular, setting $R = 1$ we obtain $a_1 t_n - T_n = O(T_{n-1}) = o(T_n)$, which implies $a_1 t_n \sim T_n$ and hence $o(t_{n-1}) = O(T_{n-1}) = o(T_n) = o(t_n)$. Therefore, (ii) implies (iii). Assume (iii) holds. By Lemma 3

$$a_1^2 \sum_{s=R}^{\lfloor n/2 \rfloor} t_s t_{n-s} \le a_1 \sum_{s=R}^{\lfloor n/2 \rfloor} T_s t_{n-s} \le T_n - a_1 \sum_{s=0}^{R-1} T_s t_{n-s} = O(t_{n-R}).$$

This proves (iv) since

$$\sum_{s=R}^{n-R} t_s t_{n-s} = 2 \sum_{s=R}^{\lfloor n/2 \rfloor} t_s t_{n-s} + O(t_{n-R}).$$

Finally, assume (iv) holds. Lemma 4 implies $|v_n - a_1 t_n| = O(t_{n-R}) = o(t_n)$ and thus $v_n \sim a_1 t_n$. This implies that there exist an integer $N \ge R$ and constants $c, C > 0$ such that

$$0 < c t_n \le v_n \le C t_n \tag{13}$$

for all $n \ge N$. As a consequence,

$$v_{n-1} = O(t_{n-1}) = o(t_n) = o(v_n) \tag{14}$$

and

$$\sum_{j=N}^{n-N} v_{n-j} v_j \le C^2 \sum_{j=R}^{n-R} t_{n-j} t_j = O(t_{n-R}) = O(v_{n-R}). \tag{15}$$

For each $n \ge N$, let

$$M_n = \max\left\{ \frac{T_j}{v_j} \;\middle|\; N \le j \le n \right\}.$$

By Lemma 2, we obtain

$$|T_n - v_n| \le \sum_{j=1}^{n-1} |v_{n-j}| T_j \le \sum_{j=1}^{N-1} v_{n-j} T_j + M_{n-1}\Bigg(\sum_{j=N}^{n-N} v_{n-j} v_j + \sum_{j=n-N+1}^{n-1} |v_{n-j}| v_j \Bigg)$$
$$= o(v_n)(1+M_{n-1}),$$

where (14) and (15) were used to obtain the last equality. Hence there exists $N_1 \ge N$ such that for all $n \ge N_1$

$$\frac{T_n}{v_n} \le \frac{3+M_{n-1}}{2}$$

and thus

$$M_n = \max\left\{ M_{n-1}, \frac{T_n}{v_n} \right\} \le \max\{M_{n-1}, 3\}.$$

This shows that the sequence $\{M_n\}$ is bounded; i.e., there exists a constant $K > 0$ such that $T_n \le K v_n$ for all $n \ge N$. Therefore, using (13) and Lemma 3, we obtain

$$T_{n-1} = O(v_{n-1}) = O(t_{n-1}) = o(t_n) = o(T_n).$$

Moreover (14) yields

$$\sum_{s=R}^{n-R} T_{n-s} T_s \le 2 \sum_{s=R}^{N} T_s T_{n-s} + K^2 \sum_{s=N}^{n-N} v_{n-s} v_s = O(T_{n-R}) + O(v_{n-R}) = O(T_{n-R}).$$

This concludes the proof that (iv) implies (i) and the theorem is proved. □

Remark 6. Let $(\{T_n\}, \{t_n\}, \{a_n\})$ be a Warlimont triple that satisfies the equivalent conditions of Theorem 5 for some $R > 2$. Then

$$\sum_{s=R-1}^{n-R+1} T_s T_{n-s} = \sum_{s=R}^{n-R} T_s T_{n-s} + 2T_{R-1} T_{n-R+1} = O(T_{n-R}) + O(T_{n-R+1}) = O(T_{n-R+1})$$

and thus $(\{T_n\}, \{t_n\}, \{a_n\})$ satisfies the equivalent conditions of Theorem 5 for any fixed positive integer less than or equal to R. In particular, $t_{n-1} = o(t_n)$ and $T_n \sim a_1 t_n$.

3. Warlimont functions and Wright semigroups

Definition 7. An *additive arithmetical semigroup* is a pair $(G, +, \partial)$ consisting of an abelian semigroup $(G, +)$ with identity and a semigroup homomorphism $\partial : (G, +) \to (\mathbb{Z}_{\ge 0}, +)$ such that

(i) the cardinality G_n of the preimage $\partial^{-1}(n)$ is finite for all n,

(ii) G is freely generated by $G^+ \subseteq G$.

We denote by G_n^+ the cardinality of the set $\partial^{-1}(n) \cap G^+$.

Remark 8. Let $(G, +, \partial)$ be an additive arithmetical semigroup. As pointed out in [Knopfmacher 1976; Warlimont 1993], $(\{G_n\}, \{G_n^+\}, \{1\})$ is a Warlimont triple.

Definition 9. A *Wright semigroup* is an additive arithmetical semigroup $(G, +, \partial)$ satisfying

$$\log(G_n) = \alpha n^{a+1} + \beta n \log(n) + \gamma n + O(n^b) \tag{16}$$

for some real numbers α, β, γ, a, b such that $\alpha > 0$ and $0 < b < a$.

Definition 10. Let R be a positive integer. We say that an additive arithmetical semigroup $(G, +, \partial)$ satisfies *axiom* $\mathcal{W}_R$ if $G_{n-1} = o(G_n)$ and

$$\sum_{s=R}^{n-R} G_s G_{n-s} = O(G_{n-R}).$$

Remark 11. Let $(G, +, \partial)$ be an additive arithmetical semigroup that satisfies axiom $\mathcal{W}_R$ for some positive integer R. Combining Remarks 6 and 8, we conclude that $(G, +, \partial)$ satisfies axiom $\mathcal{W}_{R'}$ for any positive integer $R' \leq R$. In particular, $G_n \sim G_n^+$ and $G_{n-1}^+ = o(G_n^+)$; i.e., the additive arithmetical semigroup $(G, +, \partial)$ satisfies both axiom $\mathcal{G}_1$ and axiom $\mathcal{G}_2$ as defined in [Knopfmacher 1976]. Notice that the combination of Axioms $\mathcal{G}_1$ and $\mathcal{G}_2$ is slightly weaker than axiom $\mathcal{W}_1$ since $\sum_{s=1}^{n-1} G_s G_{n-s} = o(G_n)$ does not necessarily imply $\sum_{s=1}^{n-1} G_s G_{n-s} = O(G_{n-1})$.

Proposition 12. *Every Wright semigroup satisfies axiom* $\mathcal{W}_R$ *for every positive integer* R.

Proof. This is a straightforward consequence of the definitions and Theorem 7 of [Wright 1970]. □

Definition 13. Let $(G, +, \partial)$ be an additive arithmetical semigroup. A function $F : G \to \mathbb{R}$ is *multiplicative* if $F(g_1 + g_2) = F(g_1)F(g_2)$ for all $g_1, g_2 \in G$ coprime. We say that F is *prime-independent* if there exists a sequence $\{F_n^+\}$ such that $F_n^+ = F(np)$ for every $p \in G^+$ and every positive integer n. For every function $F : G \to \mathbb{R}$, we denote by $\{F_n\}$ the sequence defined by setting

$$F_n = \sum_{\partial(g)=n} F(g)$$

for each nonnegative integer n. A *Warlimont function* is a nonnegative multiplicative prime-independent function such that $\log(F_n^+) = O(n)$ and $F_1^+ > 0$. The *normalization* of a Warlimont function F is the (not necessarily multiplicative) function $\widetilde{F} : G \to \mathbb{R}$ such that $\widetilde{F}(g) = F(g)/F_1^+$ for all $g \in G$.

Example 14. Let $(G, +, \partial)$ be an additive arithmetical semigroup and let $F : G \to \mathbb{R}$ be such that $F(g) = 1$ for all $g \in G$. Then F is a Warlimont function and $F_n = \widetilde{F}_n = G_n$ for all n.

Example 15. Let $(G, +, \partial)$ be an additive arithmetical semigroup and, for each $k \geq 2$, consider the *generalized divisor function* $d_k : G \to \mathbb{R}$ that to each $g \in G$ assigns the number $d_k(g)$ of k-tuples $(g_1, \ldots, g_k) \in G^k$ such that $g = g_1 + \cdots + g_k$. Then d_k is multiplicative, prime-independent and $(d_k)_n^+ = \binom{n+k-1}{k-1}$ for each integer $n \geq 1$. Therefore, d_k is Warlimont.

Example 16. Let $(G, +, \partial)$ be an additive arithmetical semigroup and consider the *unitary divisor function* $d_* : G \to \mathbb{R}$ that to each $g \in G$ assigns the number $d_*(g)$ of coprime pairs (g_1, g_2) such that $g = g_1 + g_2$. Then d_* is multiplicative, prime-independent and $(d_*)_n^+ = 2$ for each integer $n \geq 1$. Therefore d_* is Warlimont.

Example 17. Let $(G, +, \partial)$ be an additive arithmetical semigroup and consider the *prime divisor function* $B : G \to \mathbb{R}$ such that $B(k_1 p_1 + k_2 p_2 + \cdots + k_r p_r) = k_1 k_2 \cdots k_r$ for any $p_1, \ldots, p_r \in G$ primes and $k_1, \ldots, k_r$ positive integers. Then B is multiplicative, prime-independent and $B_n^+ = n$ for each integer $n \geq 1$. Therefore, B is Warlimont.

Remark 18. Let F be a Warlimont function on an additive arithmetical semigroup $(G, +, \partial)$. Then the function $F^m : G \to \mathbb{R}$ such that $F^m(g) = (F(g))^m$ for all $g \in G$ is again a Warlimont function for every integer $m \geq 1$ since

$$\log((F^m)_n^+) = m \log(F_n^+) = O(n).$$

Moreover, $\widetilde{F^m} = (\widetilde{F})^m$.

Remark 19. Let F be a Warlimont function on an additive arithmetical semigroup $(G, +, \partial)$. Then, as observed in [Warlimont 1993], $(\{F_n\}, \{G_n^+\}, \{F_n^+\})$ is a Warlimont triple.

Theorem 20. *Let $(G, +, \partial)$ be an additive arithmetical semigroup that satisfies axiom $\mathcal{W}_R$ and let F be a Warlimont function on G. Then for every positive integer M there exist constants $\xi_1, \ldots, \xi_{R-1}$ such that*

$$\sum_{\partial(g)=n} (\widetilde{F}(g) - 1)^M = \sum_{s=1}^{R-1} \xi_s G_{n-s} + O(G_{n-R}). \tag{17}$$

Proof. By Remark 19 and Example 14, $(\{G_n\}, \{G_n^+\}, \{1\})$ and $(\{F_n\}, \{G_n^+\}, \{F_n^+\})$ are both Warlimont triples. Since $\{G_n\}$ satisfies axiom $\mathcal{W}_R$, it follows from Theorem 5 applied to the Warlimont triple $(\{G_n\}, \{G_n^+\}, \{1\})$ that $G_{n-1}^+ = o(G_n^+)$,

$$\sum_{s=R}^{n-R} G_s^+ G_{n-s}^+ = O(G_{n-R}^+). \tag{18}$$

Moreover, if $\{\beta_n\}$ is the sequence defined recursively by setting $\beta_0 = 1$ and

$$\beta_n = -\sum_{s=0}^{n-1} \beta_s G_{n-s} \tag{19}$$

for every positive integer n, then

$$G^+_{n-s} = \sum_{r=0}^{R-1} \beta_r G_{n-s-r} + O(G_{n-s-R}) \tag{20}$$

for all $s \geq 0$. In particular, we can apply Theorem 5 to the Warlimont triple $(\{F_n\}, \{G^+_n\}, \{F^+_n\})$ and obtain

$$F_n = F^+_1 \sum_{s=0}^{R-1} F_s G^+_{n-s} + O(G^+_{n-R}). \tag{21}$$

Since by definition $G^+_n \leq G_n$ for all n and $G_{n-s-R} = o(G_{n-R})$ for all $s > 0$, substituting (20) into (21) yields

$$\widetilde{F}_n = \sum_{s=0}^{R-1} \Bigl(\sum_{r=0}^{s} \beta_r F_{s-r} \Bigr) G_{n-s} + O(G_{n-R}). \tag{22}$$

Using the binomial theorem and Remark 18 we obtain

$$\begin{aligned} \sum_{\partial(g)=n} (\widetilde{F}(g) - 1)^M &= (-1)^M \sum_{\partial(g)=n} \sum_{m=0}^{M} (-1)^m \binom{M}{m} \widetilde{F^m}(g) \\ &= (-1)^M \sum_{m=0}^{M} (-1)^m \binom{M}{m} (\widetilde{F^m})_n. \end{aligned} \tag{23}$$

Applying (22) to the Warlimont function F^m and substituting into the last line of (23) (after an obvious rearrangement) yields

$$\sum_{\partial(g)=n} (\widetilde{F}(g) - 1)^M = \sum_{s=0}^{R-1} \xi_s G_{n-s} + O(G_{n-R}), \tag{24}$$

with

$$\begin{aligned} \xi_s &= (-1)^M \sum_{m=0}^{M} (-1)^m \binom{M}{m} \sum_{r=0}^{s} \beta_r (F^m)_{s-r} \\ &= (-1)^M \sum_{m=1}^{M} (-1)^m \binom{M}{m} \sum_{r=0}^{s} \beta_r (F^m)_{s-r} \end{aligned} \tag{25}$$

for all $s \in \{0, \ldots, R-1\}$ where the second equality follows from (19) and Example 14. This implies (17) since, combining Remarks 18 and 19, $(F^m)_0 = 1$

for all m and thus

$$\xi_0 = (-1)^M \sum_{m=0}^{M} (-1)^m \binom{M}{m} \beta_0(F^m)_0 = 0. \qquad \square$$

Definition 21. Let F be a Warlimont function on an additive arithmetical semigroup $(G, +, \partial)$ and let M be a positive integer. We define the *normalized M-th moments of F* to be the functions $\mu_{F,M} : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ defined by

$$\mu_{F,M}(n) = \frac{1}{G_n} \sum_{\partial(g)=n} (\widetilde{F}(g) - 1)^M \tag{26}$$

for all $n \geq 0$.

Remark 22. Let F be a Warlimont function on an additive arithmetical semigroup $(G, +, \partial)$. The average value of F on $\partial^{-1}(n)$ is given by

$$\frac{F_n}{G_n} = F_1^+(1 + \mu_{F,1}(n)).$$

The higher normalized moments can be thought of as capturing the deviation of F from F_1^+. For instance, if $\mu_{F,1}(n) = o(1)$, then

$$\frac{1}{G_n} \sum_{\partial(g)=n} (F(g) - F_1^+)^2 = (F_1^+)^2 \mu_{F,2}(n)$$

can be thought of as an asymptotic measure of the variance of F on $\partial^{-1}(n)$.

Corollary 23. *Let F be a Warlimont function on an additive arithmetical semigroup $(G, +, \partial)$ that satisfies axiom W_1. Then*

$$\lim_{n\to\infty} \frac{F_n}{G_n} = F_1^+,$$

$$\lim_{n\to\infty} \frac{1}{G_n} \sum_{\partial(g)=n} (F(g) - F_1^+)^2 = 0.$$

Proof. Combining Remark 22 and Theorem 20 (with $R = M = 1$), we obtain

$$\frac{F_n}{G_n} = F_1^+(1 + \mu_{F,1}(n)) = F_1^+ + o(1).$$

Similarly,

$$\frac{1}{G_n} \sum_{\partial(g)=n} (F(g) - F_1^+)^2 = (F_1^+)^2 \left(\xi_1 \frac{G_{n-1}}{G_n} + O\left(\frac{G_{n-1}}{G_n} \right) \right) = o(1). \qquad \square$$

Remark 24. A slightly stronger (see Remark 11) version of Corollary 23 is proved in [Knopfmacher 1976] for particular choices of F. A sharper result is given in [Warlimont 1993] where it is shown that the assumption $G_{n-1} = O(G_n)$ (which is part of axiom $\mathcal{W}_1$) is unnecessary.

Theorem 25. *Let F be a Warlimont function on a Wright semigroup $(G,+,\partial)$ with α, a, b as in Definition 9 and let $q = e^{\alpha(a+1)}$:*

(i) *For every positive integer M there exists a sequence $\{\lambda_s\}$ of functions $\lambda_s : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ such that $\log(\lambda_s(n)) = O(n^{a-1}+n^b)$ and, for every fixed positive integer R, the asymptotic relation*

$$\mu_{F,M}(n) = \sum_{s=1}^{R-1} \lambda_s(n) q^{-sn^a} + O(\lambda_R(n) q^{-Rn^a}) \tag{27}$$

holds in the limit $n \to \infty$.

(ii) *Assume further that there exist constants $0 \leq d_2 \leq d_1$ and a sequence $\{\psi_s\}$ of polynomials such that $\deg(\psi_s) \leq d_1 s - d_2$ for all $s \geq 1$ and, for every fixed positive integer R, the asymptotic relation*

$$\frac{G_{n-1}}{G_n} = \sum_{s=1}^{R-1} \psi_s(n) q^{-ns} + O(n^{d_1 R - d_2} q^{-Rn}) \tag{28}$$

holds in the limit $n \to \infty$. Then there exists a sequence $\{\tau_s\}$ of polynomials such that $\deg(\tau_s) \leq d_1 s - d_2$ and, for every positive integer R, the asymptotic relation

$$\mu_{F,M}(n) = \sum_{s=1}^{R-1} \tau_s(n) q^{-sn} + O(n^{d_1 R - d_2} q^{-Rn}) \tag{29}$$

holds in the limit $n \to \infty$.

Proof. Let ξ_s be defined by (25) for all $s \geq 1$. By Proposition 12 and Theorem 20, we obtain

$$\mu_{F,M}(n) = \sum_{s=1}^{R-1} \xi_s \frac{G_{n-s}}{G_n} + O\left(\frac{G_{n-R}}{G_n}\right) \tag{30}$$

for every fixed integer $R > 0$. Since

$$\log\left(\frac{G_{n-s}}{G_n}\right) = \alpha((n-s)^{a+1} - n^{a+1}) + O(n^b) = -\alpha(a+1)sn^a + O(n^{a-1}+n^b),$$

in order to prove (i) it suffices to choose λ_s such that

$$\lambda_s(n) = \xi_s q^{sn^a} \frac{G_{n-s}}{G_n}$$

for all $n \geq s \geq 1$. Using (28) repeatedly and induction on t, for every fixed positive integer R, we obtain

$$\frac{G_{n-t}}{G_n} = \frac{G_{n-1}}{G_n} \cdots \frac{G_{n-t}}{G_{n-t+1}} = \sum_{s=t}^{R-1} \nu_{s,t}(n) q^{-sn} + O(n^{DR} q^{-Rn}), \tag{31}$$

where

$$\nu_{s,t}(n) = \sum_{i_1+\cdots+i_t=s} \psi_{i_1}(n)\psi_{i_2}(n-1)\cdots\psi_{i_t}(n-t+1)q^{i_2+2i_3+\cdots+(t-1)i_t} \tag{32}$$

is a polynomial in n of degree at most $d_1 s - d_2 t$ for all $1 \le t \le s$. Substituting (31) into (30) yields, for every fixed positive integer R,

$$\begin{aligned}\mu_{F,M}(n) &= \sum_{t=1}^{R-1} \xi_t \sum_{s=t}^{R-1} \nu_{s,t}(n) q^{-ns} + O(n^{DR} q^{-nR}) \\ &= \sum_{s=1}^{R-1} \left(\sum_{t=1}^{s} \xi_t \nu_{s,t}(n) \right) q^{-sn} + O(n^{DR} q^{-nR}),\end{aligned}$$

which proves (ii) upon setting

$$\tau_s(n) = \sum_{t=1}^{s} \xi_t \nu_{s,t}(n) \tag{33}$$

for all s, n. □

Remark 26. Comparison of (28) and (16) shows that the assumptions of (ii) in Theorem 25 require in particular that (16) holds with $a = 1$.

4. Examples

4.1. ***Graphs.*** Let $(G, +)$ be the semigroup of (simple, unlabeled) graphs with semigroup operation $+$ given by disjoint union. If ∂ is the map that to each graph g assigns the cardinality of its set of vertices, then $(G, +, \partial)$ is an additive arithmetical semigroup and $g \in G^+$ if and only if the graph g is connected. As proved in [Wright 1969], there exists a sequence $\{\varphi_s\}$ of polynomials such that φ_s has degree $2s$ for every s and, for every fixed positive integer R, the asymptotic relation

$$G_n = \frac{2^{\binom{n}{2}}}{n!} \left(\sum_{s=0}^{R-1} \varphi_s(n) 2^{-sn} + O(n^{2R} 2^{-Rn}) \right) \tag{34}$$

holds in the limit $n \to \infty$. The polynomials φ_s can be calculated explicitly, the first few being

$$\begin{aligned}\varphi_0(n) &= 1, \\ \varphi_1(n) &= 2n^2 - 2n, \\ \varphi_2(n) &= 8n^4 - \tfrac{128}{3}n^3 + 72n^2 - \tfrac{112}{3}n, \\ \varphi_3(n) &= \tfrac{256}{3}n^6 - \tfrac{3712}{3}n^5 + \tfrac{20672}{3}n^4 - \tfrac{54272}{3}n^3 + 21952n^2 - 9600n.\end{aligned}$$

In particular,

$$\log(G_n) = \log(\sqrt{2})n^2 - n\log(n) + (1 - \log(\sqrt{2}))n + O(n^b)$$

for any $b > 0$ and thus $(G, +, \partial)$ is a Wright semigroup. Moreover, using (34) and expanding the denominator as a geometric series we obtain, for every fixed $R > 0$,

$$\begin{aligned}\frac{G_{n-1}}{G_n} &= 2n2^{-n}\left(\sum_{s=0}^{R-1} 2^s \varphi_s(n-1)2^{-sn}\right)\sum_{r=0}^{R-1}\left(-\sum_{s=1}^{R-1}\varphi_s(n)2^{-sn}\right)^r + O(n^{2R+1}2^{-(R+1)s}) \\ &= \sum_{s=1}^{R-1}\psi_s(n)2^{-sn} + O(n^{2R-1}2^{-Rn}),\end{aligned}$$

where the ψ_s are polynomials of degree $\deg(\psi_s) = 2s-1$ which can be explicitly calculated in terms of the polynomials φ_s in (34). For instance

$$\begin{aligned}\psi_1(n) &= n, \\ \psi_2(n) &= 4n^3 - 20n^2 + 16n, \\ \psi_3(n) &= 40n^5 - 464n^4 + 1768n^3 - 2624n^2 + 1280n, \\ \psi_4(n) &= \tfrac{3248}{3}n^7 - 24176n^6 + \tfrac{630608}{3}n^5 - 908496n^4 + \tfrac{6137792}{3}n^3 - 2250240n^2 + 925696n.\end{aligned}$$

Substitution into (32) yields $\nu_{s,1}(n) = \psi_s(n)$ for all s and

$$\begin{aligned}\nu_{2,2}(n) &= 8n^2 - 8n, \\ \nu_{3,2}(n) &= 48n^4 - 352n^3 + 688n^2 - 384n, \\ \nu_{3,3}(n) &= 64n^3 - 192n^2 + 128n, \\ \nu_{4,2}(n) &= 864n^6 - 13472n^5 + 77216n^4 - 203488n^3 + 245376n^2 - 106496n, \\ \nu_{4,3}(n) &= 896n^5 - 9728n^4 + 35200n^3 - 50944n^2 + 24576n, \\ \nu_{4,4}(n) &= 1024n^4 - 6144n^3 + 11264n^2 - 6144n.\end{aligned}$$

Inspection of graphs with up to four vertices shows that $G_1 = 1,\ G_2 = 2,\ G_3 = 4$ and $G_4 = 11$. Substitution into (19) yields $\beta_1 = \beta_2 = \beta_3 = -1$ and $\beta_4 = -4$.

Example 27. Consider the Warlimont function d_2 from Example 15. When specialized to the semigroup of graphs, d_2 counts the number of ways of writing a given graph as the disjoint union of two graphs. The order is taken into account, so that if g_1 is not isomorphic to g_2, then $g = g_1 + g_2$ and $g = g_2 + g_1$ count as two distinct decompositions. Moreover, decompositions in which one of the components is the empty graph are allowed. Combining Remark 22 and Theorem 25 we obtain (4). In particular, setting $M = 1$ yields a full asymptotic expansion for the average of d_2 of the form

$$\frac{1}{G_n}\sum_{\partial(g)=n} d_2(g) = 2 + 2\sum_{s=1}^{R-1}\tau_s(n)2^{-sn} + O(n^{2R-1}2^{-Rn}),$$

valid for every fixed positive integer R, where the $\tau_s(n)$ are polynomials of degree $2s-1$. For instance, direct inspection of graphs with up to four vertices yields

$(d_2)_1 = 2,\ (d_2)_2 = 5,\ (d_2)_3 = 12$ and $(d_2)_4 = 34$. Substituting into (25) and then into (33) we obtain

$$\tau_1(n) = 2n,$$
$$\tau_2(n) = 4n^3 - 4n^2,$$
$$\tau_3(n) = 40n^5 - 368n^4 + 1320n^3 - 2016n^2 + 1024n,$$
$$\tau_4(n) = \tfrac{3248}{3}n^7 - 22448n^6 + \tfrac{560528}{3}n^5 - 781712n^4 + \tfrac{5136512}{3}n^3 - 1839360n^2 + 743424n.$$

4.2. *Graphs with an even number of edges.* Let $(G, +)$ be the semigroup of (simple, unlabeled) graphs with an even number of edges and semigroup operation $+$ given by disjoint union. If ∂ is the map that to each graph g assigns the cardinality of its set of vertices, then $(G, +, \partial)$ is an additive arithmetical semigroup. G^+ consists of graphs g with an even number of edges that cannot be written as the disjoint union of two nonempty graphs with an even number of edges. While G is a subsemigroup of the semigroup of all unlabeled graphs, not all graphs in G^+ are connected. For instance, while $2K_1$ is not connected, it is nevertheless prime in the semigroup of graphs with even edges. As pointed out in [Aldi 2019], for every fixed positive integer R, the asymptotic relation

$$G_n = \frac{2^{\binom{n}{2}}}{2n!}\left(\sum_{s=0}^{R-1} \varphi_s(n) 2^{-sn} + O(n^{2R} 2^{-Rn})\right)$$

holds in the $n \to \infty$ limit, where the polynomials $\varphi_s(n)$ coincide with those of Section 4.1. In particular, $(G, +, \partial)$ is a Wright semigroup and, for every fixed positive integer R,

$$\frac{G_{n-1}}{G_n} = \sum_{s=1}^{R-1} \psi_s(n) 2^{-sn} + O(n^{2R-1} 2^{-Rn}),$$

where the polynomials $\psi_s(n)$ coincide with those calculated in Section 4.1. Inspection of graphs with up to four vertices shows that $G_1 = G_2 = 1,\ G_3 = 2$ and $G_4 = 6$. Substitution into (19) yields $\beta_1 = -1,\ \beta_2 = 0,\ \beta_3 = -1$ and $\beta_4 = -3$.

Example 28. Consider the Warlimont function d_* from Example 16. Combining Remark 22 and Theorem 25 we obtain a full asymptotic expansion for the second moment of d_* about 2:

$$\frac{1}{G_n} \sum_{\partial(g)=n} (d_*(g) - 2)^2 = 4 \sum_{s=1}^{R-1} \tau_s(n) 2^{-sn} + O(n^{2R-1} 2^{-Rn})$$

for every fixed positive integer R, where the $\tau_s(n)$ are polynomials of degree $2s - 1$. To calculate these explicitly for small values of s, we first observe by direct calculation that $(d_*)_1 = 2,\ (d_*)_2 = 2,\ (d_*)_3 = 4,\ (d_*)_4 = 14$ as well as $(d_*^2)_1 = 4$, $(d_*^2)_2 = 4,\ (d_*^2)_3 = 8,\ (d_*^2)_4 = 36$. Substitution into (25) (upon setting $M = 2$) and then into (33) yields

$$\tau_1(n)=2n,$$

$$\tau_2(n)=4n^3-20n^2+16n,$$

$$\tau_3(n)=40n^5-464n^4+1832n^3-2816n^2+1408n,$$

$$\tau_4(n)=\tfrac{3248}{3}n^7-24176n^6+\tfrac{633296}{3}n^5-906960n^4+\tfrac{6040640}{3}n^3-2177280n^2+882688n.$$

4.3. ***Polynomials over a finite field.*** Consider the field $\mathbb{F}_q$ with q elements and let G be the set of nonzero polynomials in $\mathbb{F}_q[x_1,\dots,x_k]$ modulo the equivalence relation such that $f \sim g$ if and only if $f = \lambda g$ for some $\lambda \in \mathbb{F}_q$. G has a natural structure of additive semigroup with semigroup operation $+$ given by multiplication of polynomials. If ∂ is the semigroup homomorphism that to each polynomial $f \in G$ assigns its total degree, then $(G, +, \partial)$ is an additive arithmetical semigroup and G^+ is the set of equivalent classes of irreducible polynomials in $\mathbb{F}_q[x_1,\dots,x_k]$. Since

$$G_n = \frac{q^{\binom{n+k}{k}} - q^{\binom{n-1+k}{k}}}{q-1} \tag{35}$$

for every n,

$$\log(G_n) = \log(q)\frac{n^k}{k!} + O(n^{k-1})$$

for every $k \ge 2$. On the other hand if $k = 1$, then $\log(G_n) = \log(q)n$ for every n. Hence $(G, \cdot, \partial)$ is a Wright semigroup if and only if $k \ge 2$. If $k = 2$ then for every fixed positive integer R

$$\frac{G_{n-1}}{G_n} = q^{-n-1}\frac{1-q^{-n}}{1-q^{-n-1}} = \sum_{s=1}^{R-1}\psi_s(n)q^{-sn} + O(q^{-Rn}),$$

where $\psi_1(n) = q^{-1}$ and $\psi_s(n) = q^{-s}(1-q)$ for all $s \ge 2$. By Theorem 25, each $\mu_{F,M}$ admits an asymptotic expansion as a power series in q^{-n} with *constant* coefficients. For instance, substitution into (32) yields

$$\nu_{2,1}(n) = q^{-2} - q^{-1}, \quad \nu_{2,2}(n) = q^{-1},$$

$$\nu_{3,1}(n) = q^{-3} - q^{-2}, \quad \nu_{3,2}(n) = q^{-2} - 1, \quad \nu_{3,3}(n) = 1.$$

Example 29. We further specialize to the case where G is the semigroup of nonzero polynomials in two variables over the field with two elements. By Theorem 25, there exist constants τ_s such that for every fixed positive integer R the average of the Warlimont function B (as defined in Example 17) on polynomials of degree n is

$$\frac{B_n}{G_n} = 1 + \sum_{s=1}^{R-1}\tau_s 2^{-sn} + O(2^{-Rn}). \tag{36}$$

Since $B_1 = 6$, $B_2 = 62$ and $B_3 = 1002$, substituting (35) into (19) and then into (25) shows that in particular $\tau_1 = 0$, $\tau_2 = 3$ and $\tau_3 = \frac{3}{2}$.

Example 30. If $k > 2$, then by Remark 26 we are in the second part of Theorem 25. Nevertheless, the asymptotic behavior of Warlimont functions can be described using (27) as follows. Consider for instance the Warlimont function B of Example 17 on the semigroup of polynomials in three variables with coefficients in $\mathbb{F}_q$. Since

$$G_1 = -\beta_1 = (B^m)_1$$

for all m, substitution in (25) yields $\xi_1 = 0$ and thus

$$\frac{1}{G_n} \sum_{\partial(g)=n} (B(g) - 1)^M = O\left(\frac{G_{n-2}}{G_n}\right) = O(q^{-n^2-2n})$$

for all M.

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