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On the Hadwiger number of Kneser graphs and their random subgraphs

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For $n, k \in \mathbb{N}$, let $\mathrm{KG}(n, k)$ be the usual Kneser graph (whose vertices are k-sets of $\{1, 2, \ldots, n\}$ with $A \sim B$ if and only if $A \cap B = \varnothing$). The Hadwiger number of a graph G, denoted by $h(G)$, is $\max\{t : K_t \preccurlyeq G\}$, where $H \preccurlyeq G$ if H is a minor of G. Previously, lower bounds have been given on the Hadwiger number of a graph in terms of its average degree. In this paper we give lower bounds on $h(\mathrm{KG}(n, k))$ and $h(\mathrm{KG}(n, k)_p)$, where $\mathrm{KG}(n, k)_p$ is the binomial random subgraph of $\mathrm{KG}(n, k)$ with edge probability p. Each of these bounds is larger than previous bounds under certain conditions on k and p.

1. Introduction

Over the past few decades graph parameters of Kneser graphs have been studied extensively. The Kneser graph with parameters n and k has the k-sets of $\{1, 2, \ldots, n\}$ as its vertex set, with $A \sim B$ if and only if $A \cap B = \varnothing$. In particular, the independence number, chromatic number, diameter, and bandwidth parameters have been examined for members of this family (see [Erdős et al. 1961; Lovász 1978; Valencia-Pabon and Vera 2005; Jiang et al. 2017], respectively). In the present paper we continue the study of parameters of Kneser graphs by giving lower bounds on the *Hadwiger number* of Kneser graphs and random subgraphs of Kneser graphs. The Hadwiger number of a graph G, denoted by $h(G)$, is $\max\{t : K_t \preccurlyeq G\}$, where we say $H \preccurlyeq G$ if H is a minor of G and K_t is the complete graph, or *clique*, on t vertices.

To introduce the Hadwiger number, it's worth mentioning one of the most important open problems in graph theory — Hadwiger's conjecture. The conjecture is that if a graph has chromatic number t, then it contains K_t as a minor. It has been shown that this conjecture holds for $t \leq 6$; see [Seymour 2016] for a survey of the problem. A few decades ago, the following were proven which relate the Hadwiger number of a graph to its average degree.

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Theorem 1.1 [Kostochka 1984]. *There is a constant $c > 0$ such that if G is a graph with average degree $d \geq 2$, then*

$$h(G) \geq c\frac{d}{\sqrt{\ln(d)}}. \tag{1}$$

Theorem 1.2 [Kostochka 1982]. *There is a constant $C > 0$ such that if $\mathcal{G}$ is the set of graphs with average degree d for d sufficiently large, then*

$$\min_{G\in\mathcal{G}} h(G) \leq C\frac{d}{\sqrt{\ln(d)}}. \tag{2}$$

Notice in particular that (1) gives a lower bound on $h(G)$ for graphs with average degree d and (2) implies that up to a constant factor (1) cannot be improved when considering the collection of all graphs with average degree d as long as d is big enough. In this paper we begin by focusing on Kneser graphs $\bigl($for which $d = \binom{n-k}{k}\bigr)$ and prove the following theorem in Section 2.

Theorem 1.3. *Suppose $n = t(k^2+k)+r$ for natural numbers t and r with $0 \leq r \leq k^2+k-1$. Then*

$$h(\mathrm{KG}(n,k)) \geq \frac{1}{k+1}\binom{n-r}{k}. \tag{3}$$

In particular, when k is small compared to n (up to about $\ln(n)$) the lower bound in (3) exceeds that in (1). More precisely, suppose $k \ll \ln(n)$ (where we say $f(n) \ll g(n)$ or, equivalently, $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$). Then for $\mathrm{KG}(n,k)$ the bounds in Theorems 1.1 and 1.3 are, up to a constant factor, $\binom{n}{k}/\sqrt{k\ln(n)}$ and $\binom{n}{k}/k$, respectively. So in this case the coefficient of $\binom{n}{k}$ in Theorem 1.3 is larger, which verifies the claim.

We next consider the Hadwiger number for binomial random subgraphs of Kneser graphs. Since the introduction of the Erdős–Rényi random graph [1959], there has been interest in finding parameters of binomial random graphs. For context, the Erdős–Rényi random graph results from forming a binomial random subgraph of the complete graph; over the past couple of decades binomial random subgraphs underlying other graphs, specifically Kneser graphs, have been examined. In particular, the independence number, see [Bollobás et al. 2016; Devlin and Kahn 2016], and the chromatic number, see [Kupavskii 2016], for this type of random graph have been studied. We further this study by obtaining a lower bound on $h(\mathrm{KG}(n,k)_p)$ (where $\mathrm{KG}(n,k)_p$ is the binomial random subgraph of $\mathrm{KG}(n,k)$ with edge probability p) in the following theorem.

Theorem 1.4. *Let $k \ll \sqrt{n}$ and $N = \binom{n}{k}$. If m and p satisfy $\sqrt{\ln(N)} \ll m \ll n/k$, $2k \leq m$, and $p \gg \max\{\ln(m)/m, \ln(N)/m^2\}$, then*

$$h(\mathrm{KG}(n,k)_p) \geq \frac{1}{2m}\binom{n}{k} \text{ w.h.p.} \tag{4}$$

As is standard, for an event E depending on the (often hidden) parameter n, we say E occurs *with high probability* (*w.h.p.*) if $\Pr(E) \to 1$ as $n \to \infty$.

Additionally, we obtain the following corollary which relates the bound of Theorem 1.4 to (1).

Corollary 1.5. *For each $k \ll \sqrt{n}$, there are values of m and p (as in Theorem 1.4) such that the lower bound on $h(\mathrm{KG}(n,k)_p)$ in* (4) *exceeds that of* (1).

We will prove Theorem 1.4 and Corollary 1.5 in Section 4 after giving some preliminary notation and lemmas in Section 3. Finally, in Section 5 we state a generalization of Theorem 1.4 and mention a couple of open problems.

2. Proof of Theorem 1.3

Before presenting the proof of Theorem 1.3, we need a couple of preliminaries. First, as is standard, we will let $[n] := \{1, \dots, n\}$. We will also need the following theorem, sometimes referred to as Baranyai's theorem, which gives the existence of a particular decomposition of the collection of k-sets of $[n]$.

Theorem 2.1 [Baranyai 1975]. *If $k \mid n$, there are perfect matchings $\mathcal{A}_i$ such that*

$$\binom{[n]}{k} = \bigsqcup_{i=1}^{\binom{n-1}{k-1}} \mathcal{A}_i.$$

By its very statement, Baranyai's theorem concerns *set systems*. In this context, a *perfect matching* is a collection of k-sets of $[n]$ which are pairwise disjoint and whose union is $[n]$. The conclusion of Baranyai's theorem, then, is that we may partition the collection of all k-sets of $[n]$ into perfect matchings. Of course, we can view the collection of k-sets of $[n]$ as vertices of the Kneser graph with parameters n and k. With this perspective a perfect matching from Baranyai's theorem is a clique on n/k vertices in $\mathrm{KG}(n,k)$. As such, if we have that $k \mid n$, then we can use Theorem 2.1 to give a partition of $V(\mathrm{KG}(n,k))$ into complete graphs each on n/k vertices.

Proof of Theorem 1.3. Let n, k, and r be natural numbers such that $n = t(k^2+k)+r$, where $0 \le r \le k^2+k-1$, and take $G' = \mathrm{KG}(n-r,k)$. Notice that $G' \le G$ and so any minor of G' is a minor of G. Since $k \mid n-r$, Theorem 2.1 applies to G' yielding $K_1, K_2, \dots, K_T$, where each K_i is a complete graph on $(n-r)/k$ vertices and $T = \binom{n-r-1}{k-1}$. By the divisibility assumption, we can partition the vertices of each K_i (arbitrarily) into sets of size $k+1$ which yields $(n-r)/(k(k+1))$ "clusters" for each i (and thus $(n-r)T/(k(k+1))$ total clusters). Note that each cluster is a clique of size $k+1$ in G'. This gives that every vertex within a cluster has at least one edge to every other cluster. This is so because if v is a vertex of G' and W is cluster not containing v, then v can have nonempty intersection with at most k vertices of W (since the $k+1$ vertices in W are pairwise disjoint) and therefore v

is disjoint from at least one of the vertices in W. So if we contract each cluster to a vertex, the result is a complete graph on $(n-r)T/(k(k+1)) = (1/(k+1))\binom{n-r}{k}$ vertices, which is the desired lower bound. □

3. Preliminaries for Theorem 1.4

In this section we gather a few preliminaries from the study of random graphs which we will need to prove Theorem 1.4. The first gives the threshold value for p which ensures that the random graph $G_{n,p}$ is connected.

Theorem 3.1 [Erdős and Rényi 1959]. *If $p \gg \log(n)/n$, then $G_{n,p}$ is connected w.h.p.*

We will also need a couple of basic probability inequalities. The first is standard and the second is stated in the form of Theorem 2.1 in [Janson et al. 2000].

Theorem 3.2 (Markov's inequality). *If $\mathbb{X}$ is a nonnegative random variable and $a > 0$, then*

$$\Pr(\mathbb{X} \geq a) \leq \frac{\mathbb{E}[\mathbb{X}]}{a}.$$

Theorem 3.3 (Chernoff bound). *If $\mathbb{X}$ is the sum of n independent indicator random variables and $0 < \delta < 1$, then*

$$\Pr(\mathbb{X} \leq (1-\delta)\mathbb{E}[\mathbb{X}]) \leq \exp\left[-\tfrac{1}{2}\delta^2\mathbb{E}[\mathbb{X}]\right].$$

Throughout the paper, we use big O notation in the standard way and make repeated use of:

$$\text{if } k \ll \sqrt{n}, \text{ then } \binom{n-k}{k} \sim \binom{n}{k} \text{ as } n \to \infty.$$

For disjoint vertex sets X and Y of a graph G, let $\nabla_G(X, Y)$ be the set of edges of G with one vertex in X and the other in Y; if the underlying graph G is understood, we will neglect the subscript. In the context of taking a binomial random subgraph of G, we will let $\nabla_p(X, Y)$ be the set of edges in G_p with one vertex in X and the other in Y in order to avoid having a double subscript. For the remainder of the paper, we will take $G := \mathrm{KG}(n, k)$. Before proceeding to the proofs of Theorems 1.3 and 1.4, we will need the following lemma.

Lemma 3.4. *If X and Y are disjoint cliques of size m in G, then $|\nabla(X, Y)| \geq m^2 - km$.*

Proof. For each $x \in X$, x has nonempty intersection with at most k vertices in Y (since the vertices of Y form a clique in G). So $d_Y(x) \geq m-k$ and summing over all vertices in X proves the claim. □

4. Proofs for Theorem 1.4 and Corollary 1.5

Before proceeding, we recall the statement and discuss how the proof will unfold.

Theorem 1.4. *Let $k \ll \sqrt{n}$ and $N = \binom{n}{k}$. If m and p satisfy $\sqrt{\ln(N)} \ll m \ll n/k$, $2k \le m$, and $p \gg \max\{\ln(m)/m, \ln(N)/m^2\}$, then*

$$h(\mathrm{KG}(n,k)_p) \geq \frac{1}{2m}\binom{n}{k} \ \textit{w.h.p.} \tag{4}$$

Our strategy is to first use Baranyai's theorem to give a clique decomposition of G into somewhat "large" cliques as in the proof of Theorem 1.3. After randomizing the edges of G, the size of these cliques along with the assumed lower bound on p will ensure that *most* of them will be connected in G_p; each which remains connected will be contracted to a vertex. Next we will use Lemma 3.4 and the size of the cliques to say that if p is big enough, then every pair of cliques will have at least one edge between them in G_p. These two observations combine to say that $h(G_p)$ is at least the number of cliques which remain connected after randomizing edges. Then to prove Corollary 1.5, we simply must choose parameters so that the lower bound in (4) is greater than the lower bound in (1).

We now turn to the proof.

Proof of Theorem 1.4. For ease of reading, recall that $k \ll \sqrt{n}$, $N = \binom{n}{k}$, m satisfies $\sqrt{\ln(N)} \ll m \ll n/k$ and $k \le m/2$, and $p \gg \max\{\ln(m)/m, \ln(N)/m^2\}$. Fix $\varepsilon > 0$. Since $k \ll \sqrt{n}$ implies $N \sim \binom{n-k}{k}$, we may assume that $k \mid n$, otherwise our argument would, as in the proof of Theorem 1.3, pass to $G' \le G$ so that the divisibility assumption would be met. Now we may apply Theorem 2.1 to obtain cliques $W_1, \dots, W_T$, where $T = \binom{n-1}{k-1}$ and $|W_i| = n/k$ for each i. Each of these cliques can be (arbitrarily) partitioned into cliques of size m yielding cliques $U_1, \dots, U_S$, where $S = N/m$. We will now form G_p by sampling edges in two rounds; first we will sample edges within each U_i and thereafter will sample the remaining edges in G. By Theorem 3.1 (note that $m = \omega_n(1)$) and the first lower bound on p, we have that $G_p[U_i]$ (which is the induced subgraph of G_p on U_i) is connected with probability at least $1-\varepsilon$ provided that n is large enough. Since S is so large, it is unlikely that every $G_p[U_i]$ is connected; instead we will show that

$$\text{at least half of the } G_p[U_i]\text{'s are connected w.h.p.} \tag{5}$$

This gives away quite a bit, but the "loss" only affects our lower bound by a constant factor.

To prove (5), let $X = \sum X_i$, where each X_i is the indicator of the event $\{G_p[U_i] \text{ is connected}\}$. Thus (5) is the same as

$$\Pr\Big(X < \frac{N}{2m}\Big) \to 0. \tag{6}$$

Using linearity of expectation and the lower bound on $\Pr(X_i = 1)$ mentioned before (5), we have $\mathbb{E}[X] > (1-\varepsilon)N/m$. Note that these events are independent since they depend on disjoint sets of edges. So, using Theorem 3.3, we obtain

$$\Pr\Big(X < \frac{N}{2m}\Big) \le \Pr\big(X \le \big(1-\tfrac{1}{3}\big)\mathbb{E}[X]\big) \le \exp\big[-\big(\tfrac{1}{3}\big)^2 \tfrac{1}{2}\mathbb{E}[X]\big].$$

So as long as m is chosen so that $N/m \to \infty$ as $n \to \infty$ (and hence $\mathbb{E}[X] \to \infty$ as $n \to \infty$), the right-hand side will tend to zero as $n \to \infty$, which gives (6).

It remains to show,

$$\text{for } i \neq j, \quad |\nabla_p(U_i, U_j)| \neq 0 \text{ w.h.p.} \tag{7}$$

To do so, we let $Y = \sum Y_{i,j}$, where each $Y_{i,j}$ is the indicator of the event $\{\nabla_p(U_i, U_j) = \varnothing\}$ for $i \neq j$. Thus (7) is the same as $\Pr(Y > 0) \to 0$. For this, we have

$$\Pr(Y > 0) \le \mathbb{E}[Y] \le \binom{N/m}{2}(1-p)^{(m^2-km)} \le \frac{1}{2m^2} N^2 e^{-p(m^2-km)},$$

where the first inequality comes from Theorem 3.2, the second inequality from linearity of expectation and Lemma 3.4, and the third inequality from the fact that $1-p \le e^{-p}$. The right-hand side can be bounded by

$$N^2 e^{-p(m^2-km)} = \exp[2\ln(N) - p(m^2-km)].$$

The right-hand side tends to zero if and only if $p \gg \ln(N)/(m^2-km)$, which we have by assumption so long as $m \gg \sqrt{\ln(N)}$ and $m \ge 2k$. Indeed for such m, we can choose p appropriately so that the conditions of Theorem 1.4 are satisfied.

So to summarize, provided that $p \gg \ln(m)/m$ there are at least S pods that remain connected after the first round of randomization, where

$$S \ge \frac{1}{2m}\binom{n}{k}.$$

We will then contract all pods which are connected to a vertex and delete all vertices in pods which are disconnected. Provided that $p \gg \ln(N)/m^2$, there is an edge between every pair of remaining vertices and so $h(G_p) \ge S$ as desired. □

We conclude this section by proving Corollary 1.5 after recalling its statement.

Corollary 1.5. *For each $k \ll \sqrt{n}$, there are values of m and p (as in Theorem 1.4) such that the lower bound on $h(\mathrm{KG}(n,k)_p)$ in* (4) *exceeds that of* (1).

Proof of Corollary 1.5. In order to give parameter values so that the lower bound of (4) is greater than that of (1), we will first need that the average degree in G_p is $(1+o(1))\binom{n-k}{k}p$ (which follows from a straightforward application of Theorem 3.3

on the number of edges in G_p). So for G_p, (1) is

$$\frac{c\binom{n-k}{k}p}{\sqrt{\ln\left(\binom{n-k}{k}p\right)}}, \tag{8}$$

where c is some positive constant.

Notice that the bound of (4) is largest when m is as small as possible and the bound of (1) shrinks with p. Since $k \ll \sqrt{n}$, we have $\binom{n-k}{k} \sim N$. Before defining p, we now must consider two cases which depend on k. If there is some $0 < \alpha < \frac{1}{2}$ so that $k = (1+o(1))n^{\alpha}$, then we will take $m = n^{\beta}$, where β satisfies $\frac{1}{2}\alpha < \beta < 1-\alpha$; this is a nonempty interval since $0 < \alpha < \frac{1}{2}$ and a straightforward calculation shows that such an m satisfies the assumptions of Theorem 1.4. In this case, the larger bound on p in the conditions of Theorem 1.4 is $\ln(m)/m$. It is routine to check that for $p \gg \ln(m)/m$, $\ln\left(\binom{n-k}{k}p\right)$ is at most a constant multiple of $\ln(N)$ and so (8) is on the order of $Np/\sqrt{\ln(N)}$. This means the bound of (4) exceeds the bound of (1) for G_p provided that $p \ll \sqrt{\ln(N)}/m$. It remains to verify that $\ln(m)/m \ll \sqrt{\ln(N)}/m$ (i.e., a suitable p-value may be designated). For this observe that $k = (1+o(1))n^{\alpha}$ and so $\ln(N) \sim k\ln(n/k)$, which means $\sqrt{\ln(N)}/m$ is a constant multiple of $\ln(n)/n^{\beta-(\alpha/2)}$. On the other hand, by the choice of m, $\ln(m)/m = O(\ln(n)/n^{\beta})$, which gives the desired relation.

If $k = o(n^{\alpha})$ for every $0 < \alpha < \frac{1}{2}$, then in order to define p, let $f(n)$ be some slowly growing function which tends to infinity with n and take $m = f(n)\sqrt{\ln(N)}$. For such m, we have $\ln(m)/m \le \ln(N)/m^2$ and so the restriction on p in Theorem 1.4 is $p \gg \ln(N)/m^2$. Similar to the previous case, we have that for $p \gg \ln(N)/m^2$, $\ln\left(\binom{n-k}{k}p\right)$ is at most a constant multiple of $\ln(N)$. So (4) exceeds the bound of (1) for G_p provided that $p \ll \sqrt{\ln(N)}/m = 1/(f(n))$. Finally notice that for m-values like this, $1/(f(n)) < 1$, which means that $\ln(N)/m^2 = 1/(f(n))^2 \ll 1/(f(n))$ since $f(n) \to \infty$ as $n \to \infty$, and so such p-values may be chosen. □

5. Concluding remarks

We should note that the proof of Theorem 1.4 presented above gives rise to a slightly more general statement which is:

Theorem 5.1. *Suppose $\{G_n\}$ is an infinite family of graphs such that G_n has $g(n)$ vertices, where $g(n) \to \infty$ with n. Suppose there is an $m(n)$ such that $m := m(n) \to \infty$ with n so that*

(1) *the vertex set of G_n can be partitioned into $\{V_i\}_{i=1}^{t}$ (where $t = n/m$),*

(2) *$G_n[V_i] \cong K_m$ for each $i = 1, \ldots, t$,*

(3) *$t \to \infty$ with n, and*

(4) *$|\nabla(V_i, V_j)| \ge cm^2$ for $i \ne j$ and $c > 0$ a constant independent of n.*

Then if $\varepsilon > 0$ *and* $p \gg \max\{\ln(m)/m, \ln(n)/m^2\}$ *as* $n \to \infty$, *then w.h.p.* $K_S \preccurlyeq G_p$, *where* $S = (1-\varepsilon)t$.

Because the proof of Theorem 5.1 follows the proof of Theorem 1.4 with only the obvious modifications necessary, we will omit it. We remark that the statement applies naturally to the family of complete balanced m-partite graphs (where the size of each part is parametrized to tend to infinity) and to graph products which are fairly dense and admit a clique decomposition (e.g., $(K_n \,\square\, K_n)^C$).

We conclude by mentioning a couple of open questions. First, we will return to the Hadwiger number of Kneser graphs. As remarked above, the conclusion of Theorem 1.3 only exceeds the bound in Theorem 1.1 if $k \ll \ln(n)$. It may, therefore, be worthwhile to examine the other end of the spectrum, namely the case $n = 2k+1$. In this case, (1) gives that $h(\mathrm{KG}(2k+1, k))$ is bounded below by roughly $k/\ln(k)$. This is not, in general, best possible; if $\binom{2k+1}{k}$ is even (e.g., if k is a power of two), then $\mathrm{KG}(2k+1, k)$ has a 1-factor. A straightforward calculation shows that if we contract each edge of any 1-factor, then the resulting graph is $2k$-regular, which shows that the lower bound in (1) can be effectively doubled. This naturally gives rise to the following question.

Question 5.2. *What is the order of magnitude for* $h(\mathrm{KG}(2k+1, k))$?

Second, it is worth pointing out that the proof of Theorem 1.4 requires that $k \ll \sqrt{n}$. This is because if $k \gg \sqrt{n}$, then the cliques given from Baranyai's theorem are of size n/k, which is much smaller than k. For these cliques of size n/k, the conclusion of Lemma 3.4 becomes trivial, meaning that in this case we cannot exploit the property that every pair of cliques has many edges between them. It seems plausible that either choosing the clique decomposition carefully or using another decomposition of $\mathrm{KG}(n, k)$ (e.g., into complete bipartite graphs) may yield a suitable edge count between pieces of the decomposition like the bound Lemma 3.4, but we have not pursued this. We, therefore, put forth the following question.

Question 5.3. *Can* (1) *be improved for* $(\mathrm{KG}(n, k))_p$ *if* $k \gg \sqrt{n}$?

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