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# Some combinatorics from Zeckendorf representations

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We explore some properties of the so-called Zeckendorf representations of integers, where we write an integer as a sum of distinct, nonconsecutive Fibonacci numbers. We examine the combinatorics arising from the arithmetic of these representations, with a particular emphasis on understanding the Zeckendorf tree that encodes them. We introduce some possibly new results related to the tree, allowing us to develop a partial analog to Kummer's classical theorem about counting the number of "carries" involved in arithmetic. Finally, we finish with some conjectures and possible future projects related to the combinatorics of these representations.

## 1. Introduction

Given an integer $b \geq 2$, we can write each natural number uniquely in its base-$b$ representation $n = \sum_{i=0}^{k} n_i b^i$, where $0 \leq n_i < b$ and $n_k \neq 0$. The classical version of Kummer's theorem yields a connection between the prime factorizations of binomial coefficients and base-$p$ arithmetic.

**Theorem 1.1** (Kummer). *Let $n, m, p \in \mathbb{N}$ with $p$ prime. Then the exponent of the largest power of $p$ dividing $\binom{n+m}{n}$ is the sum of the carries when adding the base-$p$ representations of $n$ and $m$.*

Ball et al. [2014] constructed families of generalized binomial coefficients demonstrating a similar phenomenon for base-$b$ arithmetic even when $b$ is composite, and Edgar et al. [2014] did the same for rational base arithmetic, i.e., base-$p/q$ when $p > q \geq 1$ are relatively prime integers.

In this paper, we consider the so-called Zeckendorf representation of integers, where we write an integer as the sum of distinct, nonconsecutive Fibonacci numbers, and we construct a family of generalized binomial coefficients that provide partial information about the "carries" involved in the arithmetic of Zeckendorf representations. The paper is organized as follows. In Section 2, we formally

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*Keywords:* Fibonacci, Zeckendorf, digital dominance order.

introduce Zeckendorf representations, some related combinatorial structures, and some relevant integer sequences. In particular, we include some results that are likely already known but for which proofs and citations are difficult to find. In Section 3, we describe one method for adding Zeckendorf representations and then discuss our main result; we construct a sequence whose generalized binomial coefficients give us the appropriate generalization of Kummer's theorem for Zeckendorf representations. Finally, in Section 4, we discuss how these results are related to a partial order on the set of natural numbers arising from Zeckendorf representations and describe some questions and conjectures arising from this partial order and Zeckendorf arithmetic.

## 2. Preliminaries and background

Let $F : \mathbb{N} \to \mathbb{N}$ be the Fibonacci sequence defined by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$, and let $f : \mathbb{N} \to \mathbb{N}$ be the related sequence defined by $f(n) = F(n+1)$; we will write $f_i$ in place of $f(i)$. The sequence $f$ yields the standard indexing for a combinatorial interpretation of the Fibonacci sequence, as $f_n$ counts the number of ways to tile an $n \times 1$ board with tiles of size $1 \times 1$ and $1 \times 2$.

Now, a Fibonacci representation of a natural number $n$ is a list $(n_1, n_2, \ldots, n_k)_f$, satisfying $n = \sum_{i=1}^{k} n_i f_i$, $n_k \neq 0$, and $n_i \in \{0, 1\}$. Unfortunately, Fibonacci representations are not necessarily unique for a given natural number. For example, the number 6 has exactly two Fibonacci representations:

$$6 = 1\cdot 1 + 1\cdot 2 + 1\cdot 3 = (1, 1, 1)_f \quad \text{and} \quad 6 = 1\cdot 1 + 0\cdot 2 + 0\cdot 3 + 1\cdot 5 = (1, 0, 0, 1)_f.$$

To guarantee uniqueness, we add an extra condition: we require that there are never consecutive 1's in the list. More formally, the *Zeckendorf representation* of $n$ is the list $(n_1, n_2, \ldots, n_k)_z$, where $n = \sum_{i=1}^{k} n_i f_i$, $n_k \neq 0$, $n_i \in \{0, 1\}$, and $n_i \cdot n_{i+1} = 0$ for all $i < k$. It is well known that the Zeckendorf representation is unique [Zeckendorf 1972]. We will write $n = (n_1, n_2, \ldots, n_k)_z$ to mean that $(n_1, n_2, \ldots, n_k)_z$ is the Zeckendorf representation of $n$; note that the list is written in order from the least-significant to most-significant digit. We may often also refer to $n_i$ when $i > k$, in which case we mean $n_i = 0$ since appending 0's to the list will not change the value of the sum.

Next, we can define $s_z(n) = \sum_{i=1}^{k} n_i$ when $n = (n_1, n_2, \ldots, n_k)_z$; we call $s_z$ the *Zeckendorf sum-of-digits function*. For instance, the following lemma determines the Zeckendorf sum-of-digits for numbers that are one less than a Fibonacci number (note that $s_z(f_i) = 1$ for all $i \geq 1$).

**Lemma 2.1.** *Let $\ell \in \mathbb{N}$. Then*

$$f_{2\ell} - 1 = (\underbrace{1, 0, 1, 0, \ldots, 1}_{2\ell-1})_z, \qquad f_{2\ell+1} - 1 = (\underbrace{0, 1, 0, 1, 0, \ldots, 1}_{2\ell})_z.$$

*Consequently*, $s_z(f_j - 1) = \left\lfloor \frac{1}{2} j \right\rfloor$ *for all* $j$.

*Proof.* These are standard Fibonacci identities and can be found, for instance, in [Benjamin and Quinn 2003]. □

Many sources (for instance [Marsault and Sakarovitch 2014; 2017] and the Sillke link from A005206 on [OEIS]) describe a tree structure for building the Zeckendorf representations but don't include proof. We describe two functions that allow us to build this Zeckendorf representation tree that make it clear the tree does in fact give the Zeckendorf representations. In particular, we define $b:\mathbb{N}\to\mathbb{N}$ and $p:\mathbb{N}\to\mathbb{N}$ by

$$b(n)=\lfloor (n+1)\phi\rfloor-1,\qquad p(n)=\left\lfloor \frac{n+2}{\phi}\right\rfloor-1,$$

where $\phi=\frac{1}{2}(1+\sqrt{5})$ is the golden ratio (i.e., the unique positive solution to the equation $x^2-x-1=0$).

The function $b$ seems to have been widely studied (for instance see [Kimberling 1995]) and is given by A022342 in [OEIS]. The function $p$ is missing from OEIS; however, after some searching we did discover that $p(n)+1$ is A005206 in [OEIS] and contains a link due to Sillke that mentions its connection (without proof) to Zeckendorf representations. The following (seemingly known) theorem describes the relevance of the two functions to Zeckendorf representations.

**Theorem 2.2.** *Let $n=(n_1,n_2,n_3,\ldots,n_k)_z$. Then $b(n)=(0,n_1,n_2,n_3,\ldots,n_k)_z$ and $p(n)=(n_2,n_3,\ldots,n_k)_z$.*

*Proof.* Let $n=(n_1,n_2,n_3,\ldots,n_k)_z$ so that $n=\sum_{i=1}^{k} n_i f_i$. Using the Binet formula for Fibonacci numbers, which says $f_i=(\phi^{i+1}-\phi^{-(i+1)})/\sqrt{5}$, we have

$$\begin{aligned}
b(n)=\lfloor (n+1)\phi\rfloor-1&=\left\lfloor\left(\sum_{i=1}^{k} n_i\left(\frac{\phi^{i+1}-\phi^{-(i+1)}}{\sqrt{5}}\right)\phi\right)+\phi\right\rfloor-1\\
&=\left\lfloor\left(\sum_{i=1}^{k} n_i\left(\frac{\phi^{i+2}-\phi^{-i}}{\sqrt{5}}\right)\right)+\phi\right\rfloor-1\\
&=\left\lfloor\left(\sum_{i=1}^{k} n_i\left(\frac{\phi^{i+2}-\phi^{-(i+2)}}{\sqrt{5}}+\frac{\phi^{-(i+2)}-\phi^{-i}}{\sqrt{5}}\right)\right)+\phi\right\rfloor-1\\
&=\left\lfloor\sum_{i=1}^{k} n_i\left(\frac{\phi^{i+2}-\phi^{-(i+2)}}{\sqrt{5}}\right)+\sum_{i=1}^{k} n_i\left(\frac{\phi^{-(i+2)}-\phi^{-i}}{\sqrt{5}}\right)+\phi\right\rfloor-1\\
&=\left\lfloor\sum_{i=1}^{k} n_i f_{i+1}+\sum_{i=1}^{k} n_i\left(\frac{-\phi^{-(i+1)}}{\sqrt{5}}\right)+\phi\right\rfloor-1\\
&=\sum_{i=1}^{k} n_i f_{i+1}+\left\lfloor\sum_{i=1}^{k} n_i\left(\frac{-\phi^{-(i+1)}}{\sqrt{5}}\right)+\phi\right\rfloor-1.
\end{aligned}$$

The second-to-last inequality uses the Binet formula and that $\phi^{-(i+2)}-\phi^{-i}=\phi^{-(i+1)}$.

Thus, it suffices to demonstrate that

$$1 \le \sum_{i=1}^{k} n_i \left( \frac{-\phi^{-(i+1)}}{\sqrt{5}} \right) + \phi < 2.$$

First, we see that $\phi - \sum_{i=1}^{k} n_i (\phi^{-(i+1)}/\sqrt{5}) \le \phi < 2$. Next, we note that $0 \le n_i \le 1$ for all $i$ so that

$$\phi - \sum_{i=1}^{\infty} \left( \frac{\phi^{-(i+1)}}{\sqrt{5}} \right) \le \phi - \sum_{i=1}^{k} \left( \frac{\phi^{-(i+1)}}{\sqrt{5}} \right) \le \phi - \sum_{i=1}^{k} n_i \left( \frac{\phi^{-(i+1)}}{\sqrt{5}} \right).$$

However, the series $\sum_{i=1}^{\infty} (\phi^{-(i+1)}/\sqrt{5})$ is a geometric series so that

$$\sum_{i=1}^{\infty} \left( \frac{\phi^{-(i+1)}}{\sqrt{5}} \right) = \frac{\phi^{-2}}{\sqrt{5}(1 - \phi^{-1})} = \frac{1}{\sqrt{5}},$$

which means

$$1 < \phi - \frac{1}{\sqrt{5}} \le \phi - \sum_{i=1}^{k} n_i \left( \frac{\phi^{-(i+1)}}{\sqrt{5}} \right),$$

as required. Thus, we see that $b(n) = \sum_{i=1}^{k} n_i f_{i+1}$; i.e., $b(n) = (0, n_1, n_2, \ldots, n_k)$.

For the second part, we again use Binet's formula and see that

$$\begin{aligned}
p(n) = \left\lfloor \frac{n+2}{\phi} \right\rfloor - 1 &= \left\lfloor \frac{1}{\phi} \left( \sum_{i=1}^{k} (n_i f_i) + 2 \right) \right\rfloor - 1 \\
&= \left\lfloor \frac{1}{\phi} \sum_{i=2}^{k} (n_i f_i) + \frac{2+n_1}{\phi} \right\rfloor - 1 \\
&= \left\lfloor \frac{1}{\phi} \sum_{i=2}^{k} \left( n_i \frac{\phi^{i+1} - \phi^{-(i+1)}}{\sqrt{5}} \right) + \frac{2+n_1}{\phi} \right\rfloor - 1 \\
&= \left\lfloor \sum_{i=2}^{k} n_i \left( \frac{\phi^{i} - \phi^{-(i+2)}}{\sqrt{5}} \right) + \frac{2+n_1}{\phi} \right\rfloor - 1 \\
&= \left\lfloor \sum_{i=2}^{k} n_i \left( \frac{\phi^{i} - \phi^{-i} + \phi^{-i} - \phi^{-(i+2)}}{\sqrt{5}} \right) + \frac{2+n_1}{\phi} \right\rfloor - 1 \\
&= \left\lfloor \sum_{i=2}^{k} n_i \left( \frac{\phi^{i} - \phi^{-i}}{\sqrt{5}} + \frac{\phi^{-i} - \phi^{-(i+2)}}{\sqrt{5}} \right) + \frac{2+n_1}{\phi} \right\rfloor - 1 \\
&= \left\lfloor \sum_{i=2}^{k} n_i f_{i-1} + \sum_{i=2}^{k} n_i \left( \frac{\phi^{-i} - \phi^{-(i+2)}}{\sqrt{5}} \right) + \frac{2+n_1}{\phi} \right\rfloor - 1 \\
&= \sum_{i=2}^{k} n_i f_{i-1} + \left\lfloor \sum_{i=2}^{k} n_i \left( \frac{\phi^{-(i+1)}}{\sqrt{5}} \right) + \frac{2+n_1}{\phi} \right\rfloor - 1.
\end{aligned}$$

Thus, again, it will suffice to show that

$$1 \le \sum_{i=2}^{k} n_i \left( \frac{\phi^{-(i+1)}}{\sqrt{5}} \right) + \frac{2+n_1}{\phi} < 2.$$

To do this, we consider two cases: $n_1 = 0$ and $n_1 = 1$.

Case 1: Let $n_1 = 0$. Then $n_{2\ell}(\phi^{-(2\ell+1)}/\sqrt{5}) + n_{2\ell+1}(\phi^{-(2\ell+2)}/\sqrt{5}) \le (\phi^{-(2\ell+1)}/\sqrt{5})$ for all $\ell \ge 1$ since at most one of $n_{2\ell}$ and $n_{2\ell+1}$ is nonzero. This fact along with $n_1 = 0$ implies

$$\begin{aligned} 1 < \frac{2}{\phi} &\le \sum_{i=2}^{k} n_i \left( \frac{\phi^{-(i+1)}}{\sqrt{5}} \right) + \frac{2+n_1}{\phi} \\ &\le \sum_{\ell=1}^{\infty} \left( \frac{\phi^{-(2\ell+1)}}{\sqrt{5}} \right) + \frac{2}{\phi} = \frac{\phi^{-3}}{\sqrt{5}(1-\phi^{-2})} + \frac{2}{\phi} = \frac{1+2\sqrt{5}\phi}{\sqrt{5}\phi^2} < 2, \end{aligned}$$

as required.

Case 2: Let $n_1 = 1$. Then $n_2 = 0$ and $n_{2\ell-1}(\phi^{-2\ell}/\sqrt{5}) + n_{2\ell}(\phi^{-(2\ell+1)}/\sqrt{5}) \le (\phi^{-2\ell}/\sqrt{5})$ for all $\ell \ge 2$ since at most one of $n_{2\ell-1}$ and $n_{2\ell}$ is nonzero. This fact along with $n_1 = 0$ implies

$$\begin{aligned} 1 < \frac{2}{\phi} &\le \sum_{i=2}^{k} n_i \left( \frac{\phi^{-(i+1)}}{\sqrt{5}} \right) + \frac{2+n_1}{\phi} \\ &\le \sum_{\ell=2}^{\infty} \left( \frac{\phi^{-2\ell}}{\sqrt{5}} \right) + \frac{3}{\phi} = \frac{\phi^{-4}}{\sqrt{5}(1-\phi^{-2})} + \frac{3}{\phi} = \frac{1+3\sqrt{5}\phi^2}{\sqrt{5}\phi^3} < 2, \end{aligned}$$

as required.

So, in either case, we see that $p(n) = \sum_{i=2}^{k} n_i f_{i-1}$, as we wanted to show. □

We can also investigate a few properties of these integer sequences.

**Corollary 2.3.** *Let $n = \sum_{i=1}^{k} n_i f_i$. Then $n_1 = 0$ if and only if $b(p(n)) = n$.*

*Proof.* By Theorem 2.2, we note that

$$b(p(n)) = b\left(p\left(\sum_{i=1}^{k} n_i f_i\right)\right) = b\left(\sum_{i=2}^{k} n_i f_{i-1}\right) = \sum_{i=2}^{k} n_i f_i,$$

so that $n - b(p(n)) = n_1 f_1 = n_1$. □

The previous proof provides a formula for $n_1$. We can extend this idea to provide a formula for each digit in the Zeckendorf representation of a number.

**Corollary 2.4.** *Let $n = \sum_{i=1}^{k} n_i f_i$. Then $n_j = p^{j-1}(n) - b(p^j(n))$ for any $1 \le j \le k$.*

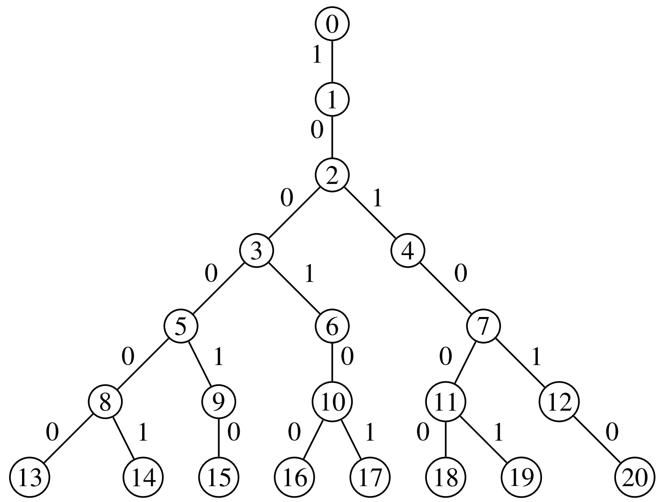

**Figure 1.** The Zeckendorf tree up to $n = 20$.

*Proof.* By repeated application of Theorem 2.2, we see

$$\begin{aligned} p^{j-1}(n) - b(p^j(n)) &= \sum_{i=1+j-1}^{k} n_i f_{i-j+1} - b\biggl(\sum_{i=1+j}^{k} n_i f_{i-j}\biggr) \\ &= \sum_{i=j}^{k} n_i f_{i-j+1} - \sum_{i=1+j}^{k} n_i f_{i-j+1} \\ &= n_j f_1 + \sum_{i=j+1}^{k} n_i f_{i-j+1} - \sum_{i=j+1}^{k} n_i f_{i-j+1} = n_j. \qquad \square \end{aligned}$$

A greedy algorithm is typically used to find the Zeckendorf representation of a positive integer, but Corollary 2.4 provides an alternate method for producing the representation, and this method can be encoded in a tree. Consider the graph $(\mathbb{N}, E)$ with vertex set $\mathbb{N}$ and edge set defined by $E = \{\{n, p(n)\} \mid n \in \mathbb{N} \setminus \{0\}\}$. This graph is a tree with root 0. Furthermore, we can define an edge-label function $e : E \to \{0, 1\}$ by $e(n) = n - b(p(n))$. We call this labeled tree the *Zeckendorf tree*, and we have drawn this tree (up to $n = 20$) in Figure 1. In the Zeckendorf tree, we refer to the vertex $p(n)$ as the *parent* of $n$, the vertex $b(n)$ as the *young child* of $n$, and the vertex $b(n) + 1$ as the *old child* of $n$. For example, 10 is the parent of 16 and 17, where 16 is the old child of 10 and 17 is the young child of 10; since 12 has only one child, 20, we refer to 20 as the old child of 12.

As noted, we are not the first to describe this tree, and it can be found various places in the literature. However, this tree is often constructed by the out-degrees of vertices (see [Marsault and Sakarovitch 2014; 2017]); by our construction and

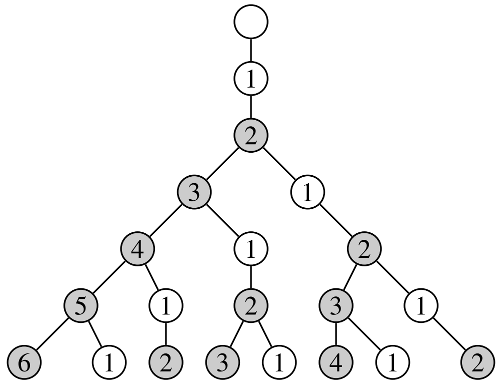


**Figure 2.** Zeckendorf tree with alternate labels. The shaded vertices represent old children so the labels are thus one more than the parent. The other nodes are young children and hence the labels are 1.

Corollaries 2.3 and 2.4, it is clear that the edge labels on the unique path from $n$ to 0 do in fact yield the Zeckendorf representation of $n$.

**Corollary 2.5.** *The list of edge labels on the path from n to* 0 *in the Zeckendorf tree gives the Zeckendorf representation of n.*

For instance we see that $15 = 0\cdot 1+1\cdot 2+0\cdot 3+0\cdot 5+0\cdot 8+1\cdot 13 = (0, 1, 0, 0, 0, 1)_z$ and these are precisely the edge labels on the path from 15 to 0 in the Zeckendorf tree.

Next, for $n = (n_1, n_2, \ldots, n_k)_z$, we let $w(n) = \min\{i \mid n_i = 1\}$; i.e., $f_{w(n)}$ is the least Fibonacci number used in the Zeckendorf representation of $n$. Thus, $w(n) - 1$ counts the number of 0's at the beginning of the Zeckendorf representation. The sequence $w$ is given by A035612 in [OEIS] and has been extensively studied since it is connected to the Wythoff array (A035513). The first few values of $w$ are listed in the following table:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w(n)$ | 1 | 2 | 3 | 1 | 4 | 1 | 2 | 5 | 1 | 2 | 3 | 1 | 6 | 1 | 2 | 3 | 1 | 4 | 1 | 2 |

Now, consider the following alternative labeling of the vertices of the Zeckendorf tree (see Figure 2):

(1) Omit a label on the vertex 0 and label vertex 1 with 1.

(2) For $n > 1$, if $n$ is a young child, label it 1.

(3) If $n$ is an old child, label it with one more than its parent's label.

In light of Corollary 2.3, we see that this labeling produces the function $w$.

**Theorem 2.6.** *For* $n \geq 1$,

$$w(n) = \begin{cases} w(p(n)) + 1 & \text{if } b(p(n)) = n, \\ 1 & \text{otherwise.} \end{cases}$$

We are not aware if this recursive formula for A035612 was previously known. We can also use our results to establish some further results about this and related integer sequences that will be utilized in the following section.

**Theorem 2.7.** *For* $n \geq 1$,

$$\left\lfloor \tfrac{1}{2} w(n) \right\rfloor = 1 + s_z(n-1) - s_z(n),$$
$$\left\lfloor \tfrac{1}{2}(w(n) - 1) \right\rfloor = 1 - n_1 + s_z(p(n-1)) - s_z(p(n)).$$

*Proof.* First, we note that the Zeckendorf representation of $n$ is $n = \sum_{i=w(n)}^{k} n_i f_i$; in particular, we note that $n = (0, 0, \ldots, 0, 1, n_{w(n)+1}, \ldots, n_k)_z$. Thus, there exists an integer $m$ with $n = f_{w(n)} + m$, so that $m = (0, 0, \ldots, 0, 0, n_{w(n)+1}, \ldots, n_k)_z$, and $s_z(n) = s_z(f_{w(n)}) + s_z(m)$. Also, by Lemma 2.1 we can see that $n - 1 = (f_{w(n)} - 1) + m$ so that $s_z(n-1) = s_z(f_{w(n)} - 1) + s_z(m)$. We thus have

$$1 + s_z(n-1) - s_z(n) = 1 + s_z(f_{w(n)} - 1) + s_z(m) - (s_z(f_{w(n)}) + s_z(m)) = s_z(f_{w(n)} - 1),$$

since $s_z(f_{w(n)}) = 1$. Thus, again by Lemma 2.1, we have

$$1 + s_z(n-1) - s_z(n) = s_z(f_{w(n)} - 1) = \left\lfloor \tfrac{1}{2} w(n) \right\rfloor,$$

as required for the first part.

The second part can be shown using two cases. First, we suppose that $w(n) \neq 1$. Then $p(n) = (0, \ldots, 0, 1, n_{w(n)+1}, \ldots, n_k)_z$ and so $s_z(p(n)) = 1 + s_z(m)$. Now, again we know that $n - 1 = (f_{w(n)} - 1) + m$, which implies $s_z(p(n-1)) = s_z(f_{w(n)-1} - 1) + s_z(m)$. Putting these together, we see

$$s_z(p(n-1)) - s_z(p(n)) = s_z(f_{w(n)} - 1) + s_z(m) - (1 + s_z(m)) = \left\lfloor \tfrac{1}{2}(w(n) - 1) \right\rfloor - 1.$$

Next, we suppose that $w(n) = 1$. Then Theorem 2.2 implies $p(n) = p(n-1)$ so that

$$s_z(p(n-1)) - s_z(p(n)) = 0 = \left\lfloor \tfrac{1}{2}(w(n) - 1) \right\rfloor.$$

The result now follows. $\square$

The previous result allows us to give a closed form for $w$ in terms of the sum-of-digits function.

**Corollary 2.8.** *For* $n \geq 1$, $w(n) = 3 - n_1 + s_z(n-1) - s_z(n) + s_z(p(n-1)) - s_z(p(n))$.

*Proof.* First, we know that for any natural number $a$,

$$\left\lfloor \tfrac{1}{2} a \right\rfloor + \left\lfloor \tfrac{1}{2}(a-1) \right\rfloor = a - 1$$

since 2 divides either $a$ or $a-1$ but not both. Thus, by Theorem 2.7

$$\begin{aligned} w(n) &= \left\lfloor \tfrac{1}{2} w(n) \right\rfloor + \left\lfloor \tfrac{1}{2}(w(n)-1) \right\rfloor + 1 \\ &= 1 + s_z(n-1) - s_z(n) + 1 - n_1 + s_z(p(n-1)) - s_z(p(n)) + 1 \\ &= 3 - n_1 + s_z(n-1) - s_z(n) + s_z(p(n-1)) - s_z(p(n)). \qquad \square \end{aligned}$$

We will make use of these functions in the next section as we describe our analog of Kummer's theorem.

## 3. Zeckendorf arithmetic, generalized binomial coefficients and Kummer's theorem

In order to generalize Kummer's theorem to Zeckendorf representations, we will describe one algorithm for producing the Zeckendorf representations of the sum of two numbers using only their Zeckendorf representations; moreover, we will also construct a suitable replacement for binomial coefficients.

Fenwick [2003] demonstrated a method for determining the Zeckendorf representation for $a+b$ in terms of the Zeckendorf representations for $a$ and $b$. Ahlbach et al. [2013] described a more efficient method of performing the same task based on a result of [Frougny 1991]. Here, we provide a slight modification of the arithmetic described in [Fenwick 2003] instead of the more efficient algorithm due to the fact that this requires us to remember fewer rules. The rules that we employ depend on the fact that the defining relation of the Fibonacci numbers, $f_{n+1} = f_n + f_{n-1}$, implies $2f_n = f_{n+1} + f_{n-2}$ when $n > 2$ (along with the facts that $2f_2 = f_1 + f_3$ and $2f_1 = f_2$). Using these facts, we have the following four rules that we use to transform a list into another list:

Rule 1: $(\dots, x, y, \mathbf{2}, z, \dots) \mapsto (\dots, x+1, y, \mathbf{0}, z+1, \dots)$.

Rule 2: $(x, \mathbf{2}, y, \dots) \mapsto (x+1, \mathbf{0}, y+1, \dots)$.

Rule 3: $(\mathbf{2}, x, \dots) \mapsto (\mathbf{0}, x+1, \dots)$.

Rule 4: $(\dots, 1, \mathbf{1}, x, \dots) \mapsto (\dots, 0, \mathbf{0}, x+1, \dots)$.

Now, given two Zeckendorf representations, $n = (n_1, n_2, \dots, n_k)_z$ and $m = (m_1, m_2, \dots, m_k)_z$ (where we can append 0's to ensure both lists are the same length), we can obtain the Zeckendorf representation for $n+m$ by using the following procedure, which has three stages.

Stage 1: Add the two lists $(n_1, n_2, \dots, n_k)_z$ and $(m_1, m_2, \dots, m_k)_z$ digit by digit to produce the new list $(n_1+m_1, n_2+m_2, \dots, n_k+m_k)$.

Stage 2: From left to right (least significant to most significant), apply Rules 1, 2, and 3 until the list contains no 2's.

$$
\begin{array}{rl}
 & (1,0,1,0,1,0)_z \\
 & +\,(1,0,1,0,1,0)_z \\ \hline
\text{add digits} \longrightarrow & (2,0,2,0,2,0) \\
\text{Rule 3} \longrightarrow & (0,1,2,0,2,0) \\
\text{Rule 1} \longrightarrow & (1,1,0,1,2,0) \\
\text{Rule 1} \longrightarrow & (1,1,1,1,0,1) \\
\text{Rule 4} \longrightarrow & (1,1,0,0,1,1) \\
\text{Rule 4} \longrightarrow & (1,1,0,0,0,0,1) \\
\text{Rule 4} \longrightarrow & (0,0,1,0,0,0,1)_z
\end{array}
$$

**Figure 3.** Addition of Zeckendorf representations: adding $12+12=24$.

Stage 3: From right to left (most significant to least significant), apply Rule 4 until the list contains no consecutive 1's.

Note that after applying a rule while in Stage 2 or Stage 3, we must start again from the left/right of the list since we might have created an earlier 2 (in Stage 2) or a later instance of (1, 1) (in Stage 3). For example, Figure 3 demonstrates this algorithm when finding the Zeckendorf representation of $24 = (0, 0, 1, 0, 0, 0, 1)_z$ given that $12 = (1, 0, 1, 0, 1)_z$ and $12 + 12 = 24$.

As noted, this algorithm is far from efficient but will terminate according to [Fenwick 2003].

We refer to each of the Rules 1–4 as a *carry rule*. Moreover, we note that Rules 1 and 2 maintain the sum of digits in the list, while both Rules 3 and 4 reduce the sum of digits in the list by 1, so we refer to Rules 3 and 4 as *drop carries*. In traditional base-$b$ arithmetic, the number of carries when adding the base-$b$ representations of $n$ and $m$ using the standard algorithm is given by $s_b(n) + s_b(m) - s_b(n+m)$, where $s_b$ is the base-$b$ sum-of-digits function. Our definition of drop carries and the Zeckendorf addition algorithm hence provide the following analogous result for the Zeckendorf sum-of-digits function.

**Theorem 3.1.** *For two natural numbers $n$ and $m$, the quantity $s_z(n) + s_z(m) - s_z(n+m)$ is the number of drop carries utilized when adding the Zeckendorf representations of $n$ and $m$.*

We can visualize the two-dimensional sequence of drop carries in a triangular form: entry $\ell$ in row $n$ of the triangle in Figure 4 shows the number of drop carries when adding the Zeckendorf representations of $\ell$ and $n - \ell$, which is given by $s(\ell) + s(n-\ell) - s(n)$ according to Theorem 3.1.

With this theorem in place, we turn our attention to defining a new family of "binomial coefficients" that will have the desired property. Given any sequence of

```
                        0
                       0 0
                      0 1 0
                     0 1 1 0
                    0 0 0 0 0
                   0 2 1 1 2 0
                  0 0 1 0 1 0 0
                 0 1 0 1 1 0 1 0
                0 2 2 1 3 1 2 2 0
               0 0 1 1 1 1 1 1 0 0
              0 1 0 1 2 0 2 1 0 1 0
             0 1 1 0 2 1 1 2 0 1 1 0
            0 0 0 0 0 0 1 0 0 0 0 0 0
           0 3 2 2 3 1 3 3 1 3 2 2 3 0
          0 0 2 1 2 1 1 2 1 1 2 1 2 0 0
         0 1 0 2 2 1 2 1 1 2 1 2 2 0 1 0
        0 1 1 0 3 1 2 2 0 2 2 1 3 0 1 1 0
       0 0 0 0 0 1 1 1 0 0 1 1 1 0 0 0 0 0
      0 2 1 1 2 0 3 2 1 2 1 2 3 0 2 1 1 2 0
     0 0 1 0 1 0 0 2 0 1 1 0 2 0 0 1 0 1 0 0
    0 1 0 1 1 0 1 0 1 1 1 1 1 0 1 0 1 1 0 1 0
   0 3 3 2 4 2 3 3 1 4 3 3 4 1 3 3 2 4 2 3 3 0
  0 0 2 2 2 2 2 2 1 1 3 2 3 1 1 2 2 2 2 2 2 0 0
 0 1 0 2 3 1 3 2 1 2 1 3 3 1 2 1 2 3 1 3 2 0 1 0
0 1 1 0 3 2 2 3 1 2 2 1 4 1 2 2 1 3 2 2 3 0 1 1 0
```

**Figure 4.** The "drop-carry triangle".

positive integers, $g : \mathbb{N} \to \mathbb{N}_{>0}$, we can define the $g$-factorial function, $g_!$, to be the sequence of partial products of $g$:

$$g_!(n) = \prod_{i=1}^{n} g(i).$$

Then, we can use this generalized factorial function to define the generalized binomial coefficients for $g$, commonly called the $g$-binomial coefficients:

$$\binom{n}{\ell}_g = \begin{cases} g_!(n)/(g_!(\ell) \cdot g_!(n-\ell)), & \text{if } 0 \le \ell \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Of course, for an arbitrary integer sequence, we cannot expect $\binom{n}{\ell}_g$ to be an integer. We use this construction with the sequence $c : \mathbb{N} \to \mathbb{N}$ defined by

$$c(n) = 2^{\lfloor w(n)/2 \rfloor},$$

whose first few terms are listed in the table below:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lfloor \frac{1}{2} w(n) \rfloor$ | 0 | 1 | 1 | 0 | 2 | 0 | 1 | 2 | 0 | 1 | 1 | 0 | 3 | 0 | 1 | 1 | 0 | 2 | 0 | 1 |
| $c(n)$ | 1 | 2 | 2 | 1 | 4 | 1 | 2 | 4 | 1 | 2 | 2 | 1 | 8 | 1 | 2 | 2 | 1 | 4 | 1 | 2 |

1

1 1

1 2 1

1 2 2 1

1 1 1 1 1

1 4 2 2 4 1

1 1 2 1 2 1 1

1 2 1 2 2 1 2 1

1 4 4 2 8 2 4 4 1

1 1 2 2 2 2 2 2 1 1

1 2 1 2 4 1 4 2 1 2 1

1 2 2 1 4 2 2 4 1 2 2 1

1 1 1 1 1 1 2 1 1 1 1 1 1

1 8 4 4 8 2 8 8 2 8 4 4 8 1

1 1 4 2 4 2 2 4 2 2 4 2 4 1 1

1 2 1 4 4 2 4 2 2 4 2 4 4 1 2 1

1 2 2 1 8 2 4 4 1 4 4 2 8 1 2 2 1

1 1 1 1 1 2 2 2 1 1 2 2 2 1 1 1 1 1

1 4 2 2 4 1 8 4 2 4 2 4 8 1 4 2 2 4 1

1 1 2 1 2 1 1 4 1 2 2 1 4 1 1 2 1 2 1 1

1 2 1 2 2 1 2 1 2 2 2 2 2 1 2 1 2 2 1 2 1

1 8 8 4 16 4 8 8 2 16 8 8 16 2 8 8 4 16 4 8 8 1

1 1 4 4 4 4 4 4 2 2 8 4 8 2 2 4 4 4 4 4 4 1 1

1 2 1 4 8 2 8 4 2 4 2 8 8 2 4 2 4 8 2 8 4 1 2 1

1 2 2 1 8 4 4 8 2 4 4 2 16 2 4 4 2 8 4 4 8 1 2 2 1

**Figure 5.** The triangle of $c$-binomial coefficients.

Knuth and Wilf [1989] showed that if a sequence $g$ is strongly divisible then $\binom{n}{\ell}_g$ will always be an integer, and Edgar and Spivey [2016] showed that if a sequence $g$ is both divisible and multiplicative, then $\binom{n}{\ell}_g$ will always be an integer. Unfortunately, the sequence $c$ is not divisible (note that $c(2) = 2$ and $c(4) = 1$) and thus not strongly divisible. However, it turns out that every $c$-binomial coefficient $\binom{n}{\ell}_c$ is still an integer.

**Theorem 3.2.** *Let $n$ and $m$ be natural numbers. Then*

$$\binom{n+m}{n}_c = 2^{s_z(n)+s_z(m)-s_z(n+m)}.$$

*Proof.* Now, by Theorem 2.7 we know that, for any $j \geq 0$, we have

$$\begin{aligned}
\sum_{i=1}^{j} \left\lfloor \tfrac{1}{2} w(i) \right\rfloor &= \sum_{i=1}^{j} (1 + s_z(i-1) - s_z(i)) \\
&= \sum_{i=1}^{j} 1 + \sum_{i=1}^{j} s_z(i-1) - \sum_{i=1}^{j} s_z(i) \\
&= j - s_z(j) + \sum_{i=1}^{j} s_z(i-1) - \sum_{i=1}^{j-1} s_z(i) \\
&= j - s_z(j) + \sum_{i=2}^{j} s_z(i-1) - \sum_{i=2}^{j} s_z(i-1) = j - s_z(j),
\end{aligned}$$

where the fourth equality follows by reindexing and by noticing that $s_z(0) = 0$. Now, using this fact, we see that

$$\binom{n+m}{n}_c = \frac{2^{\sum_{i=1}^{n+m}\lfloor w(i)/2\rfloor}}{2^{\sum_{i=1}^{n}\lfloor w(i)/2\rfloor}\cdot 2^{\sum_{i=1}^{m}\lfloor w(i)/2\rfloor}} = 2^{\sum_{i=1}^{n+m}\lfloor w(i)/2\rfloor-\sum_{i=1}^{n}\lfloor w(i)/2\rfloor-\sum_{i=1}^{m}\lfloor w(i)/2\rfloor}$$
$$= 2^{(n+m)-s_z(n+m)-(n-s_z(n))-(m-s_z(m))}$$
$$= 2^{s_z(n)+s_z(m)-s_z(n+m)}. \qquad \square$$

Figure 5 shows the triangle of $c$-binomial coefficients, where the entry $\ell$ in row $n$ is given by $\binom{n}{\ell}_c$; the previous theorem implies that this triangle contains the same information as the "drop-carry triangle" pictured in Figure 4. Thus, when we put Theorem 3.1 together with Theorem 3.2, we obtain our generalization of Kummer's theorem for Zeckendorf representations.

**Corollary 3.3.** *Let $n$ and $m$ be natural numbers. Then the exponent of 2 in $\binom{n+m}{n}_c$ is the number of drop carries when adding the Zeckendorf representations of $n$ and $m$.*

## 4. Digital dominance, carries and conjectures

Let $n = (n_1, n_2, \ldots, n_k)_z$ and $m = (m_1, m_2, \ldots, m_k)_z$ (where again we append zeroes to each list to ensure they all have the same length.). We say *$m$ Zeckendorf digitally dominates $n$*, denoted by $n \preceq_z m$, if $n_i \le m_i$ for all $i$. This relation $\preceq_z$ is a (lower-finite) partial order on the set of natural numbers (with minimum element 0). Figure 6 provides a visualization of this partial order as a triangular array: entry $\ell$

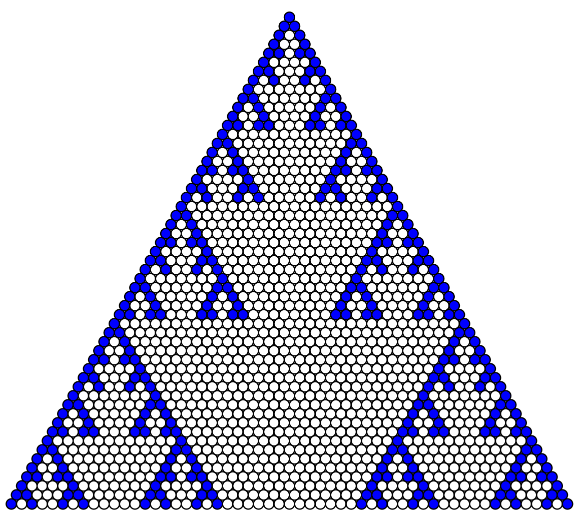

**Figure 6.** The first 55 rows (starting at 0) of the triangular representation of the Zeckendorf digital dominance order.

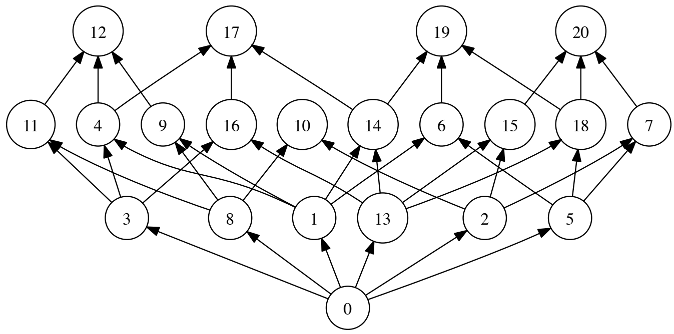


**Figure 7.** The Hasse diagram of the Zeckendorf digital dominance order up to $n = 20$.

in row $n$ is shaded if and only if $\ell \preceq_z n$. Figure 7 shows the Hasse diagram of the poset $(\mathbb{N}, \preceq_z)$ (up to $n = 20$). This poset is graded with rank function given by the Zeckendorf sum-of-digits.

De Castro et al. [2018] described some connections between the base-$b$ digital dominance order, base-$b$ arithmetic, and binomial coefficients extending some observations by [Ball et al. 2014] related to Fine's theorem [1947] describing how to use Lucas' theorem to count the number of binomial coefficients modulo a prime $p$. In this section, we introduce some of the same connections and discuss some questions that could be pursued in the future. To begin, we discuss two results about digital dominance.

**Proposition 4.1.** *Let $n$ and $m$ both be natural numbers with $n = (n_1, n_2, \ldots, n_k)_z$, $m = (m_1, m_2, \ldots, m_k)_z$ and $n+m = ((n+m)_1, (n+m)_2, \ldots, (n+m)_k)_z$, where we append zeroes to each list to ensure they all have the same length*:

(1) *If $n \preceq_z n+m$, then $m \preceq_z n+m$.*

(2) *We have $n \preceq_z n+m$ if and only if $(n+m)_i = n_i + m_i$ for all $i$.*

*Proof.* For part (1), let $h_i = (n+m)_i - n_i$. We note that since $n_i \le (n+m)_i \le 1$ for all $i$, we have $0 \le h_i \le 1$. Then

$$\begin{aligned} h_i \cdot h_{i+1} &= (n+m)_{i+1} \cdot (n+m)_i - (n+m)_{i+1} \cdot n_i - n_{i+1} \cdot (n+m)_i + n_{i+1} \cdot n_i \\ &= -(n+m)_{i+1} \cdot n_i - n_{i+1} \cdot (n+m)_i. \end{aligned}$$

Now, if $(n+m)_i = 0$, then $n_i = 0$ by assumption so that $h_i h_{i+1} = 0$. On the other hand, if $(n+m)_i = 1$, then $(n+m)_{i+1} = 0$ (since we have the Zeckendorf representation), and thus by assumption $n_{i+1} = 0$ since $n_{i+1} \le (n+m)_{i+1}$; hence $h_i \cdot h_{i+1} = 0$. Therefore, $(h_1, h_2, \ldots, h_k)_z$ is a Zeckendorf representation and moreover

$$\sum_{i=1}^{k} h_i f_i = \sum_{i=1}^{k}((n+m)_i - n_i) f_i = \sum_{i=1}^{k}(n+m)_i f_i - \sum_{i=1}^{k} n_i f_i = n+m-n = m.$$

Since Zeckendorf representations are unique, we conclude that $h_i = m_i$ for all $i$. Finally, note that since $n_i \geq 0$ for each $i$, we have $m_i = h_i = (n+m)_i - n_i \leq (n+m)_i$ for all $i$, which means that $m \preceq_z n+m$.

For part (2), we first assume that $n \preceq_z n+m$. The result follows by the proof of part (1) since we proved in this situation that $m_i = (n+m)_i - n_i$.

Conversely, if we assume that $(n+m)_i = n_i + m_i$ for all $i$, then, for each $i$, we see $n_i \leq n_i + m_i = (n+m)_i$ so that $n \preceq_z n+m$. □

If $n_i + m_i = (n+m)_i$ for all $i$, then $s_z(n+m) = s_z(n) + s_z(m)$. If this is the case, we say the addition of $n$ and $m$ is *carry-free* since we must only perform the first step of the Zeckendorf addition algorithm to obtain the Zeckendorf representation of $n+m$ from $n$ and $m$.

Part (1) of Proposition 4.1 explains the symmetry apparent in the digital dominance triangle pictured in Figure 6. Part (2) of the proposition demonstrates a connection between Figure 6 and the drop carry triangle in Figure 4, which is a consequence of the following corollary.

**Corollary 4.2.** *Let $n$ and $m$ be natural numbers. Then $s_z(n+m) = s_z(n) + s_z(m)$ if and only if $n \preceq_z n+m$.*

In particular, we have that if entry $\ell$ in row $n$ is shaded in the digital dominance triangle, then entry $\ell$ in row $n$ of the drop carry triangle is 0 (note that the converse is not true since the digital dominance triangle can detect carries other than drop carries).

The notion of carry-free addition leads us to define a new operation using Zeckendorf representations. For $n = (n_1, n_2, \ldots, n_k)_z$ and $m = (m_1, m_2, \ldots, m_k)_z$ (again with zeroes appended as necessary), we let $n \boxplus m = [n_1+m_1, n_2+m_2, \ldots, n_k+m_k]$. Note that $n \boxplus m$ is a list but typically not a Zeckendorf representation since $n \boxplus m$ represents the first list obtained in the Zeckendorf addition algorithm (before utilizing any carry rules). With this notation, we see by Corollary 4.2 that $n \boxplus m$ is the Zeckendorf representation of $n+m$ if and only if $n, m \preceq n+m$. Turning the previous idea around, we fix an integer $n$ and consider the set

$$H_z(n) = \{\ell \boxplus (n-\ell) \mid 0 \leq \ell \leq n\}.$$

We call $H_z(n)$ the set of hyper-Zeckendorf partitions of $n$. Figure 8 shows the first few values of $h_z(n) := |H_z(n)|$ and then lists them in an irregular table where row $y$ has $f_y$ elements (we start with row 0 and column 1).

The table in Figure 8 has some interesting patterns. In particular, we have the following conjectures.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $h_z(n)$ | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 3 | 5 | 4 | 4 | 5 | 5 | 5 | 6 | 5 | 7 | 6 | 5 |

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | | | | | | | | | |
| 2 | 2 | | | | | | | | | | | | | | | | | | | |
| 3 | 3 | 3 | | | | | | | | | | | | | | | | | | |
| 4 | 3 | 5 | 4 | 4 | | | | | | | | | | | | | | | | |
| 5 | 5 | 5 | 6 | 5 | 7 | 6 | 5 | | | | | | | | | | | | | |
| 7 | 6 | 8 | 7 | 6 | 8 | 8 | 8 | 9 | 7 | 10 | 8 | 7 | | | | | | | | |
| 9 | 8 | 10 | 10 | 9 | 11 | 10 | 8 | 11 | 9 | 13 | 11 | 10 | 12 | 12 | 11 | 13 | 10 | 14 | 11 | 9 |

**Figure 8.** The sequence $h_z(n)$ counting the number of hyper-Zeckendorf partitions of $n$ in list form and in irregular table form (where row $y$ has $f_y$ elements).

**Conjecture 4.3.** *Let $T(n, \ell)$ represent the entry in row $n$ and column $\ell$ of the irregular table given in Figure 8* (*where the first row is* 0 *and the first column is* 1):

(1) *For all $n \geq 1$, we have $T(n, 1) = T(n, f_n)$.*

(2) *For all $n$ and $\ell$ with $T(n, \ell)$ defined, we have $T(n, \ell)+T(n+1, \ell)=T(n+3, \ell)$.*

Any formula for $h_z(n)$ would be interesting; part (2) in the conjecture would give a recursion for $h_z(n)$ provided we can find the first three defined values in any column. Part (2) of the conjecture also implies that the column 1 is the Padovan sequence (A000931 in [OEIS]).

Finally, let $n$ be a natural number. For any hyper-Zeckendorf representation $L \in H_z(n)$, we define the set $S(L) \subseteq \{1, 2, 3, \dots, n\}$ by

$$S(L) = \{\ell \mid \ell \boxplus (n - \ell) = L\}.$$

Now, for any two natural numbers $a$ and $b$, we let

$$[a, b]_z = \{x \in \mathbb{N} \mid a \preceq_z x \preceq_z b\}$$

and we call $[a, b]_z$ a *dominance interval*. We believe that the set $S(L)$ can be decomposed into dominance intervals.

**Conjecture 4.4.** *Let $n$ be a natural number and $L \in H_z(n)$. Then there exist integers $a_1, \dots, a_j$ and $b_1, \dots, b_j$ such that*

$$S(L) = \bigcup_{i=1}^{j} [a_i, b_i]_z$$

*and the union is disjoint.*

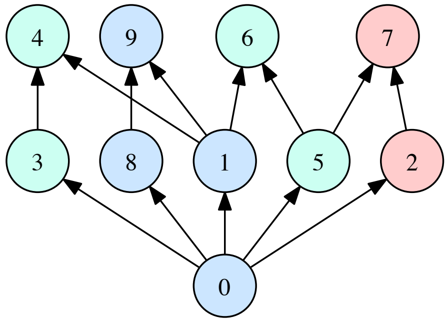


**Figure 9.** Zeckendorf dominance intervals for $n = 9$.

To demonstrate this conjecture visually, we let $n = 9$ and perform all additions of the form $\ell + (9 - \ell)$:

$$\begin{array}{rl} 0: & (0, 0, 0, 0, 0)_z \\ 9: & \boxplus(1, 0, 0, 0, 1)_z \\ \hline & [1, 0, 0, 0, 1] \end{array} \qquad \begin{array}{rl} 1: & (1, 0, 0, 0, 0)_z \\ 8: & \boxplus(0, 0, 0, 0, 1)_z \\ \hline & [1, 0, 0, 0, 1] \end{array} \qquad \begin{array}{rl} 2: & (0, 1, 0, 0, 0)_z \\ 7: & \boxplus(0, 1, 0, 1, 0)_z \\ \hline & [0, 2, 0, 1, 0] \end{array}$$

$$\begin{array}{rl} 4: & (1, 0, 1, 0, 0)_z \\ 5: & \boxplus(0, 0, 0, 1, 0)_z \\ \hline & [1, 0, 1, 1, 0] \end{array} \qquad \begin{array}{rl} 3: & (0, 0, 1, 0, 0)_z \\ 6: & \boxplus(1, 0, 0, 1, 0)_z \\ \hline & [1, 0, 1, 1, 0] \end{array}$$

We see that $h_z(9) = 3$ since $H_z(9) = \{[1, 0, 0, 0, 1], [1, 0, 1, 1, 0], [0, 2, 0, 1, 0]\}$. Furthermore, $S([1, 0, 0, 0, 1]) = \{0, 1, 8, 9\}$, $S([1, 0, 1, 1, 0]) = \{3, 4, 5, 6\}$ and $S([0, 2, 0, 1, 0]) = \{2, 7\}$. The Hasse diagram for $\preceq_z$ up to $n = 9$ is pictured in Figure 9.

We see from Figure 9 that $S([1, 0, 0, 0, 1]) = [0, 9]_z$ (as implied by Corollary 4.2), $S([1, 0, 1, 1, 0]) = [3, 4]_z \cup [5, 6]_z$ and $S([0, 2, 0, 1, 0]) = [2, 7]_z$. We note that the analogous idea using base-$b$ representations always yields $S(L)$ as a single dominance interval with a constructible minimal element [de Castro et al. 2018]; as such, it would be interesting to know (if Conjecture 4.4 is true) how many intervals are in the union and to find the set of minimal/maximal elements of each dominance interval of which $S(L)$ is the union.

Finally, as we have noted, the algorithm for adding Zeckendorf representations we utilize is not efficient. Additionally, we have forced a particular order in which to perform our rules, but [Fenwick 2003] says that any rule can be used at any time. Theorem 3.1 tells us that we will always have to use the same number of drop carries (regardless of the order we choose to perform rules). Is there some way to determine the minimal number of rules we must use in our Zeckendorf addition? For example,

$$
\begin{array}{rl}
 & (1, 0, 1, 0, 1, 0)_z \\
 & \underline{+\,(1, 0, 1, 0, 1, 0)_z} \\
\text{add digits} \longrightarrow & (2, 0, 2, 0, 2, 0) \\
\text{Rule 3} \longrightarrow & (0, 1, 2, 0, 2, 0) \\
\text{Rule 4} \longrightarrow & (0, 0, 1, 1, 2, 0) \\
\text{Rule 4} \longrightarrow & (0, 0, 1, 0, 1, 1) \\
\text{Rule 4} \longrightarrow & (0, 0, 1, 0, 0, 0, 1)_z
\end{array}
$$

**Figure 10.** Addition of Zeckendorf representations: adding $12+12=24$.

Figure 3 required us to use four drop carries and two Rule 1 carries. In Figure 10, we show that if we modify Rule 4 to allow $(\dots, 1, 2, 0, \dots) \mapsto (\dots, 0, 1, 1, \dots)$, then we can perform the same addition $(12 + 12 = 24)$ using only four drop carries and no others.

If $x = n + m$ and $x = (x_1, x_2, \dots, x_k)_z$, then we can model the carry rules by vectors of length $k$. The $k - 3$ vectors

$$
\begin{aligned}
r_1^1 &:= (1, 0, -2, 1, \dots, 0), \\
r_1^2 &:= (0, 1, 0, -2, 1, \dots, 0), \\
&\vdots \\
r_1^{k-3} &:= (0, \dots, 0, 1, 0, -2, 1)
\end{aligned}
$$

model Rule 1, the vector $r_2 := (1, -2, 1, 0, \dots 0)$ models Rule 2, the vector $r_3 := (-2, 1, 0, \dots, 0)$ models Rule 3, and the $k - 2$ vectors

$$
\begin{aligned}
r_4^1 &:= (1, 1, -1, 0, \dots, 0), \\
r_4^2 &:= (0, 1, 1, -1, 0, \dots, 0), \\
&\vdots \\
r_4^{k-2} &:= (0, \dots, 0, 1, 1, -1)
\end{aligned}
$$

model Rule 4. If we let $\epsilon$ be the vector with entries given by $\epsilon_i = x_i - n_i - m_i$, then any positive integer solution $(u, y_1, \dots, y_{k-2}, w, z_1, \dots, z_{k-3})$ to the linear system

$$
ur_3 + \sum_{i=1}^{k-2} y_i r_4^i + wr_2 + \sum_{i=1}^{k-3} z_i r_1^i = \epsilon
$$

will give us instructions about which rules to apply (though not which order). Therefore, we would be interested in positive integer solutions to the system such that the sum of the entries is minimized. The $k \times (2k - 3)$ matrix for this system of

$$M = \begin{bmatrix} -2 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{rref}(M) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Figure 11.** One particular form of the matrix of carry-rule (column) vectors and its reduced row echelon form.

linear equations may thus also be of interest. For instance, Figure 11 shows this matrix when $k = 7$, that is, when the most significant digit of $x$ is $x_7$. We also include the reduced row echelon form of this matrix and note that if we input the columns systematically, the pattern shown there will continue for all $k$.

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