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On the zero-sum group-magicness of cartesian products

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Let $G = (V(G), E(G))$ be a graph and let $\mathbb{A} = (A, +)$ be an abelian group with identity 0. Then an $\mathbb{A}$-*magic labeling of* G is a function ϕ from $E(G)$ into $A \setminus \{0\}$ such that for some $a \in A$, $\sum_{e \in E(v)} \phi(e) = a$ for every $v \in V(G)$, where $E(v)$ is the set of edges incident to v. If ϕ exists such that $a = 0$, then G is *zero-sum* $\mathbb{A}$-*magic*. Let G be the cartesian product of two or more graphs. We establish that G is zero-sum $\mathbb{Z}$-magic and we introduce a graph invariant $j^*(G)$ to explore the zero-sum integer-magic spectrum (or null space) of G. For certain G, we establish $\mathcal{A}(G)$, the set of nontrivial abelian groups for which G is zero-sum group-magic. Particular attention is given to $\mathcal{A}(G)$ for regular G, odd/even G, and G isomorphic to a product of paths.

1. Introduction

Let $G = (V(G), E(G))$ be a graph. Let $\mathcal{A}$ be the set all nontrivial abelian groups and let $\mathbb{A} = (A, +) \in \mathcal{A}$, where 0 denotes the identity of $\mathbb{A}$. Then an $\mathbb{A}$-*labeling of* G is a function ϕ from $E(G)$ into $A \setminus \{0\}$. For fixed $e \in E(G)$, $\phi(e)$ is called the *label of e under* ϕ, and for fixed $v \in V(G)$, the sum of the labels of the edges incident to v is called the *weight of v under* ϕ. The graph G is $\mathbb{A}$-*magic* if and only if there exists an $\mathbb{A}$-labeling ϕ of G such that for some $a \in A$, the weight of every vertex in $V(G)$ under ϕ is a. In such a case, ϕ is called an $\mathbb{A}$-*magic labeling of* G. Additionally, G is *zero-sum* $\mathbb{A}$-*magic* if and only if there is an $\mathbb{A}$-labeling ϕ of G such that the weight of every vertex in $V(G)$ under ϕ is 0. In this case, ϕ is called a *zero-sum* $\mathbb{A}$-*magic labeling of* G. Letting $\mathcal{A}(G)$ denote the set of all $\mathbb{A} \in \mathcal{A}$ such that G is zero-sum $\mathbb{A}$-magic, we observe that if $\mathbb{H}_0 \in \mathcal{A}(G)$ and $\mathbb{H}_0$ is isomorphic to a subgroup of an abelian group $\mathbb{H}$, then $\mathbb{H} \in \mathcal{A}(G)$. Particularly, if $\mathbb{A} \in \mathcal{A}(G)$, then for a positive integer k, we have $\mathbb{A}^k \in \mathcal{A}(G)$. In Figure 1, we

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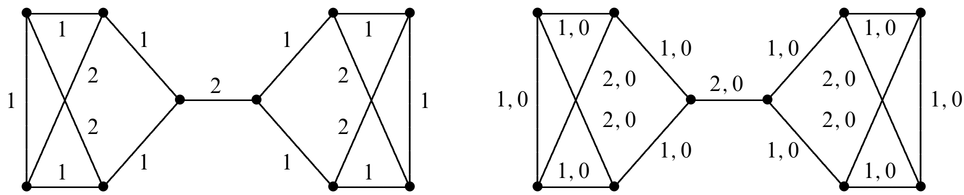

Figure 1. Zero-sum $\mathbb{Z}_4$ and $\mathbb{Z}_4^2$-magic labelings of G.

illustrate a zero-sum $\mathbb{Z}_4$-magic labeling and a zero-sum $\mathbb{Z}_4^2$-magic labeling of a graph G. (The group $\mathbb{Z}_4$ is the cyclic group of integers under addition mod 4. In the specification of elements of Z_4^2, parentheses are omitted.) Note that an alternative zero-sum $\mathbb{Z}_4^2$-magic labeling can be formed by changing each label $(a, 0)$ to (a, a). Other obvious alternatives exist.

The *zero-sum integer-magic spectrum* of a graph G, denoted by $\text{zim}(G)$, is the set of positive integers k such that G is zero-sum $\mathbb{Z}_k$-magic, where $\mathbb{Z}_1$ is the group $\mathbb{Z}$ of integers under addition and, for $k \geq 2$, $\mathbb{Z}_k$ is the cyclic group of integers under addition mod k. Let $\mathcal{N}$ denote the set of positive integers. By the fundamental theorem of finite abelian groups, $\mathcal{N} \setminus \{1\}$ is a subset of $\text{zim}(G)$ if and only if G is zero-sum $\mathbb{A}$-magic for all finite $\mathbb{A}$ in $\mathcal{A}$. Moreover, for any infinite abelian group $\mathbb{A} = (A, +)$ with nonzero $a \in A$, a generates a subgroup of $\mathbb{A}$ that is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_k$ for some $k \geq 2$. Thus $\text{zim}(G) = \mathcal{N}$ if and only if $\mathcal{A}(G) = \mathcal{A}$. We note especially that if $\text{zim}(G) = \mathcal{N} \setminus \{2\}$, then $\mathcal{A} \setminus \{\mathbb{Z}_2^k \mid k \in \mathcal{N}\} \subseteq \mathcal{A}(G)$.

Sedláček [1964] first introduced magic labelings, motivated by magic squares in number theory. Stanley [1973] later showed that the study of magic labelings is related to the study of linear homogeneous diophantine equations. It was shown independently in [Low and Lee 2006] and [Shiu et al. 2004] that if G and H are $\mathbb{A}$-magic graphs, then the cartesian product of G and H is also $\mathbb{A}$-magic. More recently, Akbari et al. [2014] proved that every r-regular graph G with $r \geq 3$, $r \neq 5$ has $\text{zim}(G) = \mathcal{N}$ if r is even; otherwise $\text{zim}(G) \supseteq \mathcal{N} \setminus \{2, 4\}$. And, Shiu and Low [2018] have determined the zero-sum integer-magic spectrum of the cartesian product of two trees. For a dynamic survey of results on magic labelings, see [Gallian 2018].

In this paper, we consider the zero-sum $\mathbb{A}$-magicness of cartesian products. In Section 2, we give definitions and preliminary results. In Section 3, we develop our main results, with particular attention given to graph parity and regularity. And in Section 4, we consider the cartesian products of paths, also known as grid graphs.

2. Definitions and preliminary results

Throughout this paper, graphs will be finite, nontrivial, simple, loopless, and connected unless specified otherwise. An *even graph* (resp. *odd graph*) shall refer

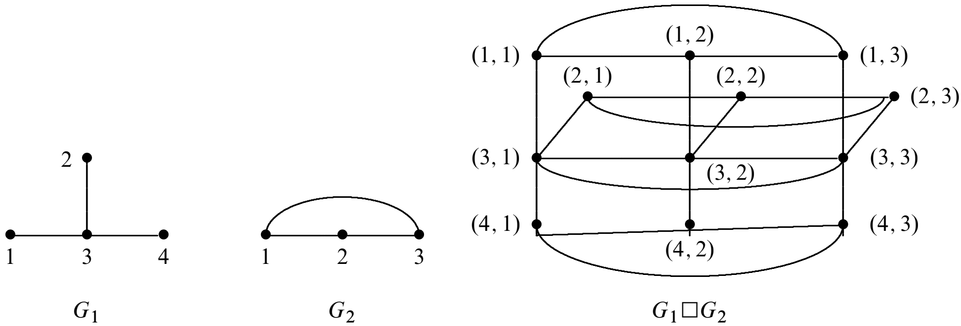

Figure 2. The cartesian product $G_1 \square G_2$.

to a graph G of which each vertex v has even (resp. odd) degree, denoted by $d_G(v)$. Abelian groups shall have identity element 0 and binary operator $+$.

For $i \in \{1, \dots, n\}$, let $G_i = (V(G_i), E(G_i))$ be a graph. The *cartesian product* of $G_1, G_2, \dots, G_n$, denoted by $\square_{i=1}^{n} G_i$ or $G_1 \square G_2 \square \cdots \square G_n$, is the graph G such that

(1) the vertex set of G is $\prod_{i=1}^{n} V(G_i)$, and

(2) vertices $(u_1, u_2, \dots, u_n)$ and $(w_1, w_2, \dots, w_n)$ of G are adjacent if and only if $(u_1, u_2, \dots, u_n)$ and $(w_1, w_2, \dots, w_n)$ differ in precisely one component i_0, and u_{i_0} is adjacent to w_{i_0} in G_{i_0}.

The following are well known. Note that the results in (b), (c), and (d) extend to cartesian products of arbitrary finite length by part (a):

(a) As a binary operator on the set of graphs, $\square$ is associative and commutative with respect to isomorphism.

(b) If (u, w) is a vertex of $G_1 \square G_2$, then $d_{G_1 \square G_2}((u, w)) = d_{G_1}(u) + d_{G_2}(w)$.

(c) $G_1 \square G_2$ is regular if and only if each of G_1 and G_2 is regular.

(d) If G_1 and G_2 are graphs with no isolated vertices, then $G_1 \square G_2$ is bridgeless.

In Figure 2, we demonstrate the cartesian product $G_1 \square G_2$ where G_1 is isomorphic to the claw on four vertices and G_2 is isomorphic to the cycle C_3.

Let G be a graph and let h be a positive integer. Then a *factor of* G is a spanning subgraph of G, and an h-factor of G is an h-regular factor of G. (The term *factor* may also refer to a factor of a cartesian product. Its usage will clarify its intended meaning.)

For $n \geq 2$, let G be a cartesian product $\square_{i=1}^{n} G_i$. For fixed i_0, $1 \leq i_0 \leq n$, suppose that H_{i_0} is an h_{i_0}-factor of G_{i_0}. Then there exists a natural h_{i_0}-factor of G, comprising the union of C disjoint subgraphs of G, each isomorphic to H_{i_0}, where C is the product of the orders of the graphs G_i, $1 \leq i \leq n$, except for $i = i_0$. If, for

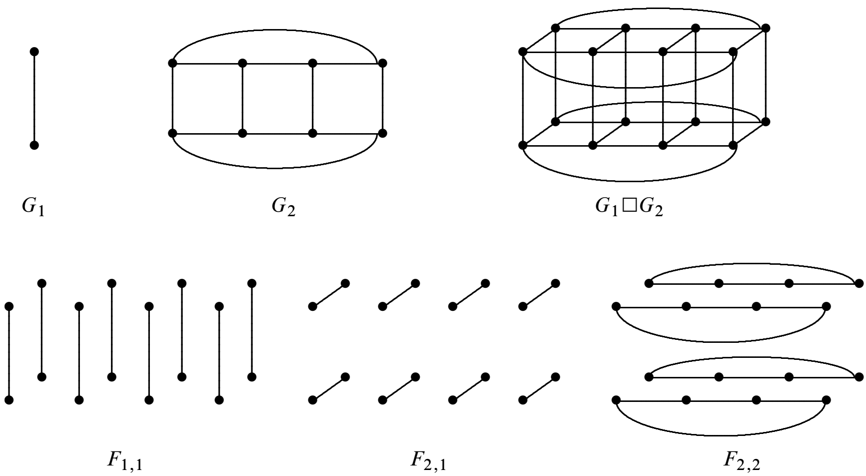

Figure 3. An illustration of regular factors.

each i, $E(G_i)$ is partitioned by the edge sets of regular factors $H_{i,1}, H_{i,2}, \ldots, H_{i,m_i}$ of G_i, then these n partitions of $E(G_1)$, $E(G_2)$, $\ldots$, and $E(G_n)$ induce a natural partition of $E(G)$ by the edge sets of regular factors $F_{i,j}$, $1 \le i \le n$, $1 \le j \le m_i$, where the regularity of $F_{i,j}$ equals the regularity of $H_{i,j}$. Note that each vertex of G is a vertex of each $F_{i,j}$ as illustrated in Figure 3. Therein, graph $G_1 \cong P_2$ has a 1-factor $H_{1,1} = G_1$ whose edge set trivially partitions $E(G_1)$. And, G_2 has a 1-factor $H_{2,1}$ isomorphic to four vertex-disjoint copies of P_2 and a 2-factor $H_{2,2}$ isomorphic to two vertex-disjoint copies of C_4, where $E(H_{2,1})$ and $E(H_{2,2})$ together partition $E(G_2)$. These regular factors respectively induce $F_{1,1}$, $F_{2,1}$, and $F_{2,2}$, which together partition $E(G_1 \square G_2)$. (Note that G_2 is itself a cartesian product isomorphic to $P_2 \square C_4$.)

Let G be a graph that has a zero-sum $\mathbb{Z}$-magic labeling and let $\mathcal{Z}(G)$ represent the (nonempty) set of all zero-sum $\mathbb{Z}$-magic labelings of G. For each $\phi \in \mathcal{Z}(G)$, let $j(\phi)$ equal $\max_{e \in E(G)} |\phi(e)|$. Then $j^*(G)$ shall denote $\min_{\phi \in \mathcal{Z}(G)} j(\phi)$. To illustrate, we note that there exists a zero-sum $\mathbb{Z}$-magic labeling ϕ of K_4 such that $\max_{e \in E(K_4)} |\phi(e)| = 2$. Yet, since K_4 is an odd graph, there is no such labeling ϕ such that $\max_{e \in E(K_4)} |\phi(e)| = 1$. Thus $j^*(K_4) = 2$. On the other hand, some graphs, including C_{2n-1} and P_n for $n \ge 2$, are not zero-sum $\mathbb{Z}$-magic. For such graphs G, $j^*(G)$ does not exist.

Theorem 1. *If G is zero-sum $\mathbb{Z}$-magic, then G is zero-sum $\mathbb{Z}_k$-magic for all $k > j^*(G)$. Additionally, if $j^*(G) = 1$, then $\mathcal{A}(G) = \mathcal{A}$.*

Proof. Let ϕ be any zero-sum $\mathbb{Z}$-magic labeling of G such that $j(\phi) = j^*(G)$, and fix $k > j^*(G)$. We form a zero-sum $\mathbb{Z}_k$-magic labeling of G by assigning $\phi(e) \pmod{k}$

to $e \in E(G)$. Consequently, if $j^*(G) = 1$, then $\operatorname{zim}(G) = \mathcal{N}$, implying that G is zero-sum $\mathbb{A}$-magic for all $\mathbb{A} \in \mathcal{A}$. □

We observe that Georges, Mauro, and Wash [Georges et al. 2017] established necessary and sufficient conditions under which a graph G is zero-sum Z_{2j}^k-magic. In particular, they showed that G has a bridge whose removal results in an isolate or bipartite component if and only if G is not zero-sum Z_{2j}^k-magic for any positive integers j, k. Theorem 1 immediately implies that such G is not zero-sum Z-magic for otherwise G would be zero-sum Z_{2j}-magic for $2j > j^*(G)$.

Theorems 2 through 8, useful in the sequel, can be found in the existing literature.

Theorem 2 [Petersen 1891]. *Let G be a $2t$-regular graph. Then there exist t 2-factors of G that partition $E(G)$.*

Theorem 3 [Ore 1957]. *Let G be a bridgeless regular graph of odd degree k and let h be an even integer, $2 \le h \le \frac{2}{3}k$. Then there exists an h-factor of G.*

Theorem 4 [Georges et al. 2010]. *Let G be a 3-regular graph. Then G is zero-sum $\mathbb{Z}_2^2$-magic if and only if the chromatic index of G is 3.*

Theorem 5 [Georges et al. 2010]. *Let G be a graph with a bridge. Then for each positive integer k, G is not zero-sum $\mathbb{Z}_2^k$-magic.*

Theorem 6 [Choi et al. 2012]. *Let G be a bridgeless graph. Then for each positive integer $k \ge 3$, G is zero-sum $\mathbb{Z}_2^k$-magic.*

Theorem 7 [Low and Lee 2006; Shiu et al. 2004]. *If $G_1, G_2, \ldots, G_n$ are zero-sum $\mathbb{A}$-magic graphs, then $\square_{i=1}^n G_i$ is zero-sum $\mathbb{A}$-magic.*

Theorem 8 [Akbari et al. 2014]. *Let G be an r-regular graph, $r \ge 3$, $r \ne 5$. If r is even, then $\operatorname{zim}(G) = \mathcal{N}$. Otherwise, $\mathcal{N} \setminus \{2, 4\} \subseteq \operatorname{zim}(G)$.*

Corollary 9. *Suppose that G is a graph with $j^*(G) \le 2$. Then the following hold:*

(a) *If G is bridgeless, then $\mathcal{A} \setminus \{\mathbb{Z}_2, \mathbb{Z}_2^2\} \subseteq \mathcal{A}(G)$.*

(b) *If G is bridgeless and even, then $\mathcal{A}(G) = \mathcal{A}$.*

(c) *If G is bridgeless with a vertex of odd degree, then*

$$\mathcal{A} \setminus \{\mathbb{Z}_2, \mathbb{Z}_2^2\} \subseteq \mathcal{A}(G) \subseteq \mathcal{A} \setminus \{\mathbb{Z}_2\}.$$

(d) *If G has a bridge, then $\mathcal{A}(G) = \mathcal{A} \setminus \{\mathbb{Z}_2^k \mid k \in \mathcal{N}\}$.*

Proof. (a) We note that $1 \in \operatorname{zim}(G)$ since $j^*(G)$ is assumed to exist. The result follows by Theorems 1 and 6.

(b) Since G is even, G is zero-sum $\mathbb{Z}_2$-magic (assign 1 to each edge of G), and hence zero-sum $\mathbb{Z}_2^2$-magic. The result follows by part (a).

(c) Part (c) follows from part (a) and the fact that any $\mathbb{Z}_2$-labeling of G must assign 1 to each edge; hence, no graph with a vertex of odd degree can be zero-sum $\mathbb{Z}_2$-magic.

(d) We again observe that $1 \in \text{zim}(G)$ and, by Theorem 1, $k \in \text{zim}(G)$ for $k \geq 3$. Thus $\mathcal{N} \setminus \{2\} \subseteq \text{zim}(G)$. The result now follows from Theorem 5. □

We note that $j^*(K_4) = 2$ and, by the upcoming Corollary 10 and Theorem 11, $j^*(K_7) = 2$ as well. Yet $\text{zim}(K_4) = \mathcal{N} \setminus \{2\}$ and, by Theorem 8, $\text{zim}(K_7) = \mathcal{N}$. Moreover, the three graphs K_4, the Petersen graph PG, and the (unique) 3-regular graph G on 10 vertices with one bridge (see Figure 1) are each easily verified to have zero-sum integer-magic spectrum $\mathcal{N} \setminus \{2\}$. Yet, by Theorem 4 and Corollary 9, $\mathcal{A}(K_4) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$, $\mathcal{A}(PG) = \mathcal{A} \setminus \{\mathbb{Z}_2, \mathbb{Z}_2^2\}$, and $\mathcal{A}(G) = \mathcal{A} \setminus \{\mathbb{Z}_2^k \mid k \in \mathcal{N}\}$.

In the following corollary, we utilize $j^*(G)$ to give an alternative proof of Theorem 8 in the case r is even, $r \geq 4$.

Corollary 10. *Let G be an even-regular graph with degree $r = 2t$ such that $t \geq 2$. Then the following hold*:

(a) *If t is even, then $j^*(G) = 1$ and $\mathcal{A}(G) = \mathcal{A}$.*

(b) *If t is odd, then $j^*(G) \leq 2$ and $\mathcal{A}(G) = \mathcal{A}$.*

Proof. By Theorem 2, $E(G)$ partitions into t 2-factors. If t is even, we construct a zero-sum $\mathbb{Z}$-magic labeling ϕ of G by assigning the label 1 to each edge of precisely $\frac{t}{2}$ 2-factors, and -1 to each edge of the remaining $\frac{t}{2}$ 2-factors. Since $j(\phi) = 1$, we have $j^*(G) = 1$, implying $\mathcal{A}(G) = \mathcal{A}$ by Theorem 1 or (since even graphs are bridgeless) Corollary 9(b). On the other hand, if t is odd, we construct a zero-sum $\mathbb{Z}$-magic labeling ϕ of G by assigning the label 1 to each edge of precisely $\frac{1}{2}(t+1)$ 2-factors, -1 to each edge of precisely $\frac{1}{2}(t-3)$ 2-factors, and -2 to each edge of the one remaining 2-factor. Since $j(\phi) = 2$, we have $j^*(G) \leq 2$, implying $\mathcal{A}(G) = \mathcal{A}$ by Corollary 9(b). □

For illustration, Figure 4 displays vertex α of a 14-regular graph G whose edge set is partitioned by the 2-factors $H_1, H_2, H_3, H_4, H_5, H_6, H_7$. Suppose that for each i, the edge set of H_i includes the edges αu_i and αw_i. Since $t = 7$, we assign the label 1 to the edges of each of four 2-factors H_1, H_2, H_3, and H_4, we assign the label -1 to the edges of each of two 2-factors H_5 and H_6, and we assign the label -2 to the edges of the remaining 2-factor H_7. Since these labels bear upon every vertex, the weight of every vertex is 0.

We observe that the graphs $K_6 \,\square\, P_2$ and K_7 illustrate the case t odd, yet $j^*(K_6 \,\square\, P_2) = 1$ (see Theorem 11), while $j^*(K_7) = 2$.

Theorem 11. *Let G be a* (*connected*) *graph. Then $j^*(G) = 1$ if and only if G is an even graph with even size.*

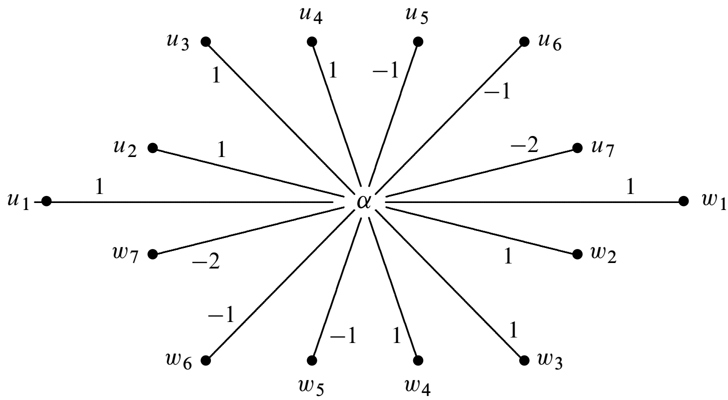

Figure 4. An illustration of Corollary 10.

Proof. Suppose $j^*(G) = 1$. Then it is clear that G is even. To show that G has even size, let ϕ be a zero-sum $\mathbb{Z}$-magic labeling of G with $j(\phi) = 1$. Also let x denote the number of edges of G which receive the label 1 under ϕ, and let y denote the number of edges of G which receive the label -1 under ϕ. Then

$$0 = \sum_{v \in V(G)} w_v(\phi) = 2 \sum_{e \in E(G)} \phi(e) = \sum_{e \in E(G)} \phi(e),$$

implying $x = y$. Thus G has even size.

Now suppose that G is even with even size. Since G is even (and connected), there exists an Eulerian circuit in G with even length. We produce a zero-sum $\mathbb{Z}$-magic labeling ϕ of G with $j(\phi) = 1$ by assigning alternating labels of 1 and -1 to the edges of the circuit. □

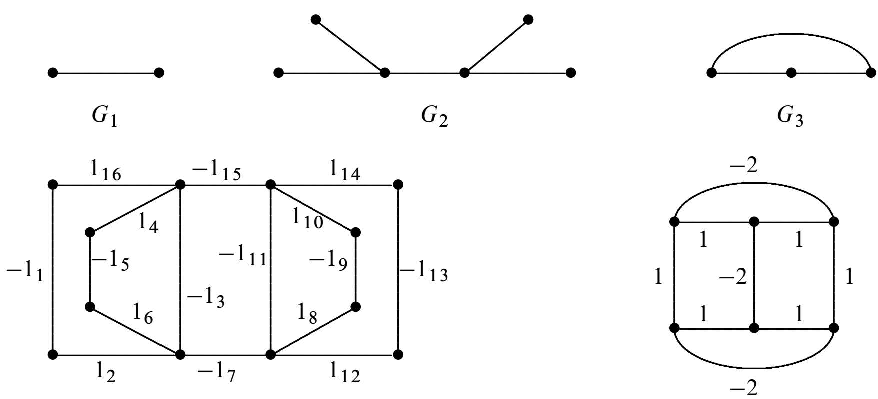

Figure 5. Zero-sum $\mathbb{Z}$-magic labelings of $G_1 \square G_2$ and $G_1 \square G_3$.

Because this paper emphasizes cartesian products, Theorem 11 is illustrated in Figure 5 with an even graph $G_1 \square G_2$ with even size, and a noneven graph $G_1 \square G_3$. We exhibit a zero-sum $\mathbb{Z}$-magic labeling ϕ of $G_1 \square G_2$ with $j(\phi) = 1$ along with indication (via subscripts) of a corresponding Eulerian circuit. And, we show $j^*(G_1 \square G_3) = 2$ by exhibiting a zero-sum $\mathbb{Z}$-magic labeling of $G_1 \square G_3$ with maximum absolute label 2.

If M and N are connected odd graphs, then each has even order and $M \square N$ is an even graph with even size $|V(M)||E(N)| + |V(N)||E(M)|$. As well, if M or N is not connected, then each component of $M \square N$ is even with even size. Therefore we have the following.

Corollary 12. *Let M and N be odd graphs (not necessarily connected). Then $j^*(M \square N) = 1$.*

3. Main results

As noted in the preceding section, not all graphs are zero-sum $\mathbb{Z}$-magic. On the other hand, if G is the cartesian product of nontrivial graphs, we have the following result.

Theorem 13. *Let M and N be graphs with $\delta(M), \delta(N) \geq 1$. Then $M \square N$ is zero-sum $\mathbb{Z}$-magic with $j^*(M \square N) \leq \max\{\Delta(M), \Delta(N)\}$.*

Proof. Let $V(M) = \{u_1, \dots, u_m\}$ and $V(N) = \{w_1, \dots, w_n\}$. For fixed i, $1 \leq i \leq n$, let $M(i)$ be the subgraph of $M \square N$ induced by the vertices in $\{(u_j, w_i) \mid 1 \leq j \leq m\}$. Similarly, for fixed i, $1 \leq i \leq m$, let $N(i)$ be the subgraph of $M \square N$ induced by the vertices in $\{(u_i, w_j) \mid 1 \leq j \leq n\}$. Form a $\mathbb{Z}$-labeling of $M \square N$ as follows: to each edge of $M(\beta)$, assign the label $d_N(w_\beta)$ and to each edge of $N(\alpha)$, assign the label $-d_M(u_\alpha)$. Then the weight of any vertex $(u_\alpha, w_\beta) \in V(M \square N)$ is $d_M(u_\alpha)d_N(w_\beta) - d_M(u_\alpha)d_N(w_\beta) = 0$. Since the maximum absolute label is $\max\{\Delta(M), \Delta(N)\}$, the result follows. □

It is easy to show the following.

Theorem 14. *Let M and N be zero-sum $\mathbb{Z}$-magic graphs. Then $M \square N$ is zero-sum $\mathbb{Z}$-magic with $j^*(M \square N) \leq \max\{j^*(M), j^*(N)\}$.*

Theorem 15. *Let $G = \square_{i=1}^{n} G_i$ where $n \geq 2$ and $\delta(G_i) \geq 1$ for each i. Then G is zero-sum $\mathbb{Z}$-magic. Moreover, for $\zeta = \max\{\Delta(G_i)\}$ over i, we have $j^*(G) \leq \zeta$ if n is even, and $j^*(G) \leq 2\zeta$ if n is odd.*

Proof. By Theorem 13, G is zero-sum $\mathbb{Z}$-magic.

If $n = 2t$, let $H_i = G_{2i-1} \square G_{2i}$ for $1 \leq i \leq t$. Then by Theorem 13, each H_i is zero-sum $\mathbb{Z}$-magic with $j^*(H_i) \leq \max\{\Delta(G_{2i-1}), \Delta(G_{2i})\} \leq \zeta$. Since G is isomorphic to $\square_{i=1}^{t} H_i$, it follows from Theorem 14 that $j^*(G) \leq \max\{j^*(H_i)\} \leq \zeta$.

If $n = 2t+1$, let $H_1 = G_1 \square (G_2 \square G_3)$ and let $H_i = G_{2i} \square G_{2i+1}$ for $2 \le i \le t$. By Theorem 13, each H_i is zero-sum $\mathbb{Z}$-magic. Observing that $\Delta(G_2 \square G_3) = \Delta(G_2) + \Delta(G_3)$, we have from Theorem 13 that $j^*(H_1) \le \max\{\zeta, 2\zeta\} = 2\zeta$. Since $j^*(H_i) \le \zeta$ for $i \ge 2$, we have $j^*(G) \le 2\zeta$ by Theorem 14. □

Lemma 16. *Let G_1, G_2 and G_3 be odd graphs, and let $M = \square_{i=1}^{3} G_i$. Then $j^*(M) = 2$ and $\mathcal{A}(M) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$.*

Proof. Since M is odd, we show $j^*(M) = 2$ by developing a zero-sum $\mathbb{Z}$-magic labeling λ of M with $j(\lambda) = 2$.

Let G_1, G_2, and G_3 have respective vertex sets $\{u_i \mid 1 \le i \le n_1\}$, $\{v_i \mid 1 \le i \le n_2\}$, and $\{w_i \mid 1 \le i \le n_3\}$. Letting $V(P_2) = \{1, 2\}$, we have $j^*(G_i \square P_2) = 1$ by Corollary 12. We thus let ϕ_i'' denote a zero-sum $\mathbb{Z}$-magic labeling of $G_i \square P_2$ such that $j(\phi_i'') = 1$. Let ϕ_i' be the $\mathbb{Z}$-labeling of G_i such that for each edge $\alpha\beta$ of G_i, $\phi_i'(\alpha\beta) = \phi_i''((\alpha, 1)(\beta, 1))$. Note that the weight of each vertex under ϕ_i' is 1 or -1.

Now consider $G_1 \square G_2$. We construct a $\mathbb{Z}$-labeling ψ of $G_1 \square G_2$ as follows. To each edge $(u_x, v)(u_y, v)$ of $G_1 \square G_2$, assign the label $\phi_1'(u_x u_y) w_{\phi_2'}(v)$. And, to each edge $(u, v_x)(u, v_y)$ of $G_1 \square G_2$, assign the label $\phi_2'(v_x v_y) w_{\phi_1'}(u)$. We observe that under ψ, each edge of $G_1 \square G_2$ has label 1 or -1 and each vertex of $G_1 \square G_2$ has weight 2 or -2.

Each edge e in $E(M)$ has one of the following two forms:

- Type I: $e = (z_x, w)(z_y, w)$, where $z_x z_y \in E(G_1 \square G_2)$ and $w \in V(G_3)$.
- Type II: $e = (z, w_x)(z, w_y)$, where $z \in V(G_1 \square G_2)$ and $w_x w_y \in E(G_3)$.

Let λ be a $\mathbb{Z}$-labeling of M such that

$$\lambda(e) = \begin{cases} \psi(z_x z_y) w_{\phi_3'}(w) & \text{if } e \text{ is of type I,} \\ -w_\psi(z) \phi_3'(w_x w_y) & \text{if } e \text{ is of type II.} \end{cases}$$

Since $\psi(z_x z_y) \in \{-1, 1\}$, $w_{\phi_3'}(w) \in \{-1, 1\}$, $w_\psi(z) \in \{-2, 2\}$, and $\phi_3'(w_x w_y) \in \{-1, 1\}$, the edges of type 1 receive ± 1 under λ and edges of type II receive labels of ± 2 under λ. It is easily checked that λ is a zero-sum $\mathbb{Z}$-magic labeling of M and that $j(\lambda) = 2$. Hence $j^*(M) = 2$. (In Figure 6, the evolution of λ is illustrated via edges incident to $(u, v, w) \in V(G_1 \square G_2 \square G_3)$, where each of u, v, and w is assumed to have degree 3 in G_1, G_2, and G_3 respectively. Labels assigned to edges under ϕ_1', ϕ_2', and ϕ_3' are notional, from which labels under ψ and λ follow.)

By Corollary 9(c), it suffices to produce a zero-sum $\mathbb{Z}_2^2$-magic labeling of M. For each edge $e \in E(M)$ of the form $(u_i, v, w)(u_j, v, w)$, let $\rho(e) = (0, 1)$. For each edge e of the form $(u, v_i, w)(u, v_j, w)$, let $\rho(e) = (1, 0)$. And for each edge e of the form $(u, v, w_i)(u, v, w_j)$, let $\rho(e) = (1, 1)$. It follows from the oddness of each G_i that the weight of each vertex under ρ is 0. □

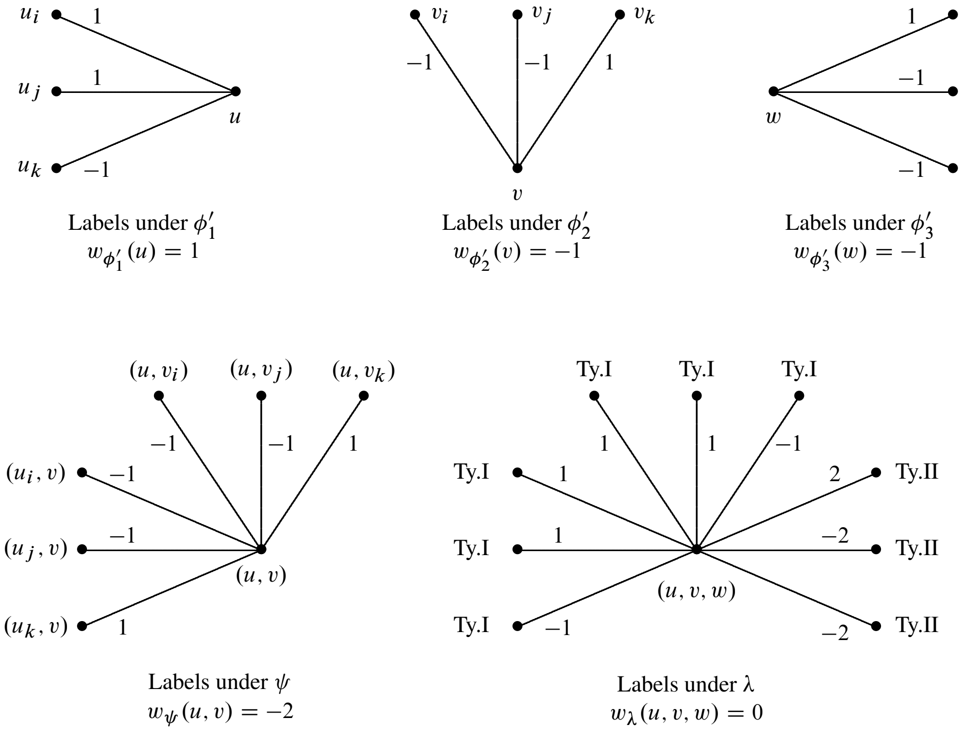

Figure 6. An illustration of the labeling λ of Lemma 16.

Theorem 17. *For $1 \le i \le n$, let G_i be an odd graph, and let $G = \square_{i=1}^{n} G_i$. Then*

(a) *If n is even, $j^*(G) = 1$ and $\mathcal{A}(G) = \mathcal{A}$.*

(b) *If n is odd, $j^*(G) = 2$ and $\mathcal{A}(G) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$.*

Proof. (a) Let $n = 2t$. Then G is isomorphic to $\square_{i=1}^{t} H_i$, where $H_i = G_{2i-1} \square G_{2i}$. By Corollary 12, $j^*(H_i) = 1$. The result now follows from Theorems 14 and 1.

(b) Let $n = 2t + 1$, where $t \ge 1$. Let $H_1 = G_1 \square G_2 \square G_3$ and for $2 \le i \le t$, let $H_i = G_{2i} \square G_{2i+1}$. By Lemma 16, $j^*(H_1) = 2$ and $\mathcal{A}(H_1) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$. And, by Corollary 12, $j^*(H_i) = 1$ for $2 \le i \le t$, implying $\mathcal{A}(H_i) = \mathcal{A}$. Thus, by Theorem 7, Theorem 14 and the fact that G is odd, $j^*(G) = 2$ and $\mathcal{A}(G) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$. □

We now consider graphs $G = \square_{i=1}^{n} G_i$ such that $n \ge 2$, G_i is nontrivial, and G is r-regular. Since the regularity of G coimplies the regularity of each G_i, we assume G_i is r_i-regular, $r_i \ge 1$. Then the degree of G is $r = \sum_{i=1}^{n} r_i$.

The next theorem follows from Theorem 8 or alternatively Corollary 10 with the 4-cycle handled as a trivial special case.

Theorem 18. *Let $n \ge 2$ and let $G = \square_{i=1}^{n} G_i$ be an even-regular graph. Then $\mathcal{A}(G) = \mathcal{A}$.*

We turn to the case r is odd. For odd-regular graphs M except 5-regular, Theorem 8 indicates that M is zero-sum $\mathbb{A}$-magic for all $\mathbb{A}$ except possibly $\mathbb{Z}_2^k$, $\mathbb{Z}_4^k$, $k \geq 1$. (Certainly M is not zero-sum $\mathbb{Z}_2$-magic.) However, if M is a cartesian product of at least two factors, then more definitive results can be given for any odd regularity.

Lemma 19. *Let $G = G_1 \,\square\, G_2$ be an odd-regular graph, where $\delta(G_i) \geq 1$. Then $\mathcal{A} \setminus \{\mathbb{Z}_2, \mathbb{Z}_2^2\} \subseteq \mathcal{A}(G) \subseteq \mathcal{A} \setminus \{\mathbb{Z}_2\}$. Moreover, if either G_1 or G_2 has degree 1, then $\mathcal{A}(G) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$.*

Proof. Since G is regular if and only if each G_i is regular, we assume without loss of generality G_i is r_i-regular for odd r_1 and even r_2. Let $r_2 = 2t$.

If $r_1 \geq 3$, then by Theorem 3, $E(G_1)$ partitions into a $\frac{1}{2}(r_1+1)$-factor and a $\frac{1}{2}(r_1-1)$-factor. By Petersen's theorem, $E(G_2)$ partitions into t 2-factors. Thus $E(G)$ naturally partitions into t 2-factors $F_1, F_2, \ldots, F_t$, a $\frac{1}{2}(r_1+1)$-factor F', and a $\frac{1}{2}(r_1-1)$-factor F''. We form a zero-sum $\mathbb{Z}$-magic labeling ϕ of G with $j(\phi) = 2$ based on the parity of t.

If $t = 2k+1$,

$$\phi(e) = \begin{cases} 1 & \text{if } e \in F_i,\ 1 \leq i \leq k, \\ -1 & \text{if } e \in F_i,\ k+1 \leq i \leq 2k+1, \\ 2 & \text{if } e \in F', \\ -2 & \text{if } e \in F''. \end{cases}$$

If $t = 2k$,

$$\phi(e) = \begin{cases} 1 & \text{if } e \in F_i,\ 1 \leq i \leq k, \\ -1 & \text{if } e \in F_i,\ k+1 \leq i \leq 2k-1, \\ -2 & \text{if } e \in F_{2k}, \\ 2 & \text{if } e \in F', \\ -2 & \text{if } e \in F''. \end{cases}$$

Since $j(\phi) = 2$ in each case, we have $j^*(G) \leq 2$. The result follows by Corollary 9(c).

Suppose $r_1 = 1$. Then the edge set $E(G)$ partitions naturally into a 1-factor F' (of which there are $|V(G_2)|$ components each isomorphic to P_2) and, by Petersen's theorem, t 2-factors $F_1, F_2, \ldots, F_t$. We form a zero-sum $\mathbb{Z}$-magic labeling ϕ of G with $j(\phi) = 2$ as above, accounting for the vacuous $\frac{1}{2}(r_1-1)$-factor.

If $t = 2k+1$ is odd,

$$\phi(e) = \begin{cases} 1 & \text{if } e \in F_i,\ 1 \leq i \leq k, \\ -1 & \text{if } e \in F_i,\ k+1 \leq 1 \leq 2k+1, \\ 2 & \text{if } e \in F'. \end{cases}$$

If $t = 2k$ is even,

$$\phi(e) = \begin{cases} 1 & \text{if } e \in F_i,\ 1 \le i \le k, \\ -1 & \text{if } e \in F_i,\ k+1 \le i \le 2k-1, \\ -2 & \text{if } e \in F_{2k}, \\ 2 & \text{if } e \in F'. \end{cases}$$

Thus $j^*(G) \le 2$, implying $\mathcal{A} \setminus \{\mathbb{Z}_2, \mathbb{Z}_2^2\} \subseteq \mathcal{A}(G) \subseteq \mathcal{A} \setminus \{\mathbb{Z}_2\}$ by Corollary 9(c). It remains to show that G is zero-sum $\mathbb{Z}_2^2$-magic.

Let H denote a 2-factor of G_2. Then $P_2 \,\square\, H$ is a 3-factor of G (not necessarily connected) in which each component is a prism. Since each prism is hamiltonian and hence 3-edge colorable, $P_2 \,\square\, H$ is zero-sum $\mathbb{Z}_2^2$-magic by Theorem 4. Letting ϕ' be a zero-sum $\mathbb{Z}_2^2$-magic labeling of $P_2 \,\square\, H$, we form a zero-sum $\mathbb{Z}_2^2$-magic labeling ϕ of G as follows:

$$\phi(e) = \begin{cases} \phi'(e) & \text{if } e \in E(P_2 \,\square\, H), \\ (1,1) & \text{if } e \in E(G - (P_2 \,\square\, H)). \end{cases}$$ □

Lemma 20. *Suppose $G = G_1 \,\square\, G_2 \,\square\, G_3$ is odd-regular, where G_i is r_i-regular, $r_i \ge 1$. Suppose also that either $r_i = 1$ for some i or r_i is odd for all i. Then $\mathcal{A}(G) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$.*

Proof. If r_i is odd for all i, then the result follows from Theorem 17. So, with no loss of generality, suppose $r_1 = 1$ and r_2, r_3 are even. Since G is isomorphic to $G_1 \,\square\, H$, where $H = G_2 \,\square\, G_3$, the result follows from Lemma 19. □

Theorem 21. *Suppose $G = \square_{i=1}^n G_i$ is odd-regular, where $n \ge 3$ and G_i is r_i-regular, $r_i \ge 1$. Let ω denote the number of odd-regular G_i. Then the following hold*:

(a) *If $\omega \ge 3$, then $\mathcal{A}(G) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$.*

(b) *If $\omega = 1$, then $\mathcal{A} \setminus \{\mathbb{Z}_2, \mathbb{Z}_2^2\} \subseteq \mathcal{A}(G) \subseteq \mathcal{A} \setminus \{\mathbb{Z}_2\}$. Moreover, if the sole odd-regular G_i has degree 1, then $\mathcal{A}(G) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$.*

Proof. We observe that ω must be odd since G is odd-regular with degree equal to $\sum_{i=1}^n r_i$.

(a) With no loss of generality, let G_1 and G_2 be odd-regular. Then $\square_{i=3}^n G_i$ is odd regular as well, which implies that G is isomorphic to the cartesian product of three odd-regular graphs G_1, G_2, and $\square_{i=3}^n G_i$. The result follows by Lemma 20.

(b) With no loss of generality, let G_1 be odd-regular. Then $\square_{i=2}^n G_i$ is even-regular, which implies that G is isomorphic to the cartesian product of one odd-regular graph G_1 and one even-regular graph $\square_{i=2}^n G_i$. The result follows by Lemma 19. □

4. Grid graphs

In this section we consider the cartesian product of paths $G = \square_{i=1}^{n} P_{a_i}$, where $n \geq 2$ and $a_i \geq 2$. We denote the vertex set of P_m by $\{1, 2, 3, 4, \ldots, m\}$, where x is adjacent to y if and only if $|x - y| = 1$. We note that if α is the number of a_i's equal to 2, then $\Delta(G) = 2n - \alpha$, $\delta(G) = n$, and for all integers j, $n \leq j \leq 2n - \alpha$, there exists a vertex of G with degree j. We also note that G is regular (in particular, n-regular) if and only if $a_i = 2$ for all i.

Lemma 22. *Let M be a graph such that $j^*(M) \leq 2$ and $j^*(M \square P_2) \leq 2$. Then $j^*(M \square P_n) \leq 2$ for $n \geq 3$.*

Proof. Let H' denote the subgraph of $M \square P_n$ induced by the vertices in $\{(v, 1) \mid v \in V(M)\}$ and let H'' denote the subgraph of $M \square P_n$ induced by the vertices in $\{(v, i) \mid v \in V(M), i = 1, 2\}$. Since H' and H'' are respectively isomorphic to M and $M \square P_2$, we can find zero-sum $\mathbb{Z}$-magic labelings ϕ' of H' and ϕ'' of H'' such that $j(\phi')$ and $j(\phi'')$ are each at most 2. As follows, we construct a zero-sum $\mathbb{Z}$-magic labeling ϕ of $M \square P_n$ that draws its labels from the images of ϕ' and ϕ'' (this labeling is illustrated in Figure 7 with $n = 5$ and a graph M that is seen by inspection to satisfy the hypotheses of the lemma):

$$\phi(e) = \begin{cases} \phi''(e) & \text{if } e = (u, 1)(w, 1), \\ \phi'((u, 1)(w, 1)) & \text{if } e = (u, i)(w, i),\ 2 \leq i \leq n-1, \\ \phi''((u, 1)(u, 2)) & \text{if } e = (u, i)(u, i+1),\ 1 \leq i \leq n-1,\ i \text{ odd}, \\ -\phi''((u, 1)(u, 2)) & \text{if } e = (u, i)(u, i+1),\ 1 \leq i \leq n-1, i \text{ even}, \\ -\phi''((u, 1)(w, 1)) & \text{if } e = (u, n)(w, n), n \text{ odd}, \\ \phi''((u, 1)(w, 1)) & \text{if } e = (u, n)(w, n), n \text{ even}. \end{cases}$$

Thus $j(\phi) \leq 2$, giving the result. □

Lemma 23. *Let M be a graph and let $\mathbb{A} \in \mathcal{A}$ such that both M and $M \square P_2$ are zero-sum $\mathbb{A}$-magic. Then $M \square P_n$ is zero-sum $\mathbb{A}$-magic for $n \geq 3$.*

Proof. Let H' and H'' denote the subgraphs of $M \square P_n$ given in the proof of Lemma 22, and let ϕ' and ϕ'' be zero-sum $\mathbb{A}$-magic labelings of H' and H'', respectively. Then the labeling ϕ of that proof is a zero-sum $\mathbb{A}$-magic labeling of $M \square P_n$. □

Theorem 24. *Suppose $G = \square_{i=1}^{n} P_{a_i}$, where $n \geq 2$. If n is even and $a_i = 2$ for all i, then $\mathcal{A}(G) = \mathcal{A}$. Otherwise, $\mathcal{A}(G) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$.*

Proof. If n is even and $a_i = 2$ for all i, the result follows from Theorem 18. If n is odd and $a_i = 2$ for all i, the result follows from Lemma 19 and the observation that G is isomorphic to $H_1 \square H_2$, where $H_1 = P_2$ and $H_2 = \square_{i=2}^{n} P_2$. We thus assume $a_i \geq 3$ for some i, which implies that G has a vertex of odd degree.

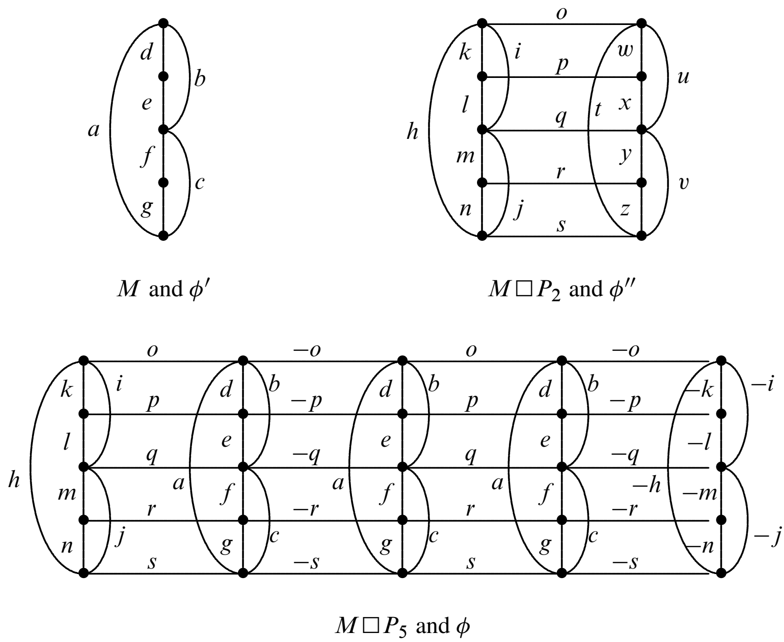

Figure 7. An illustration of the labeling ϕ of Lemma 22.

Express G as follows:

- If $n = 2k$, then $G = \square_{i=1}^{k} H_i$, where $H_i = P_{a_{2i-1}} \,\square\, P_{a_{2i}}$.
- If $n = 2k+1$, then $G = \square_{i=1}^{k} H_i$, where $H_1 = P_{a_1} \,\square\, P_{a_2} \,\square\, P_{a_3}$ and $H_i = P_{a_{2i}} \,\square\, P_{a_{2i+1}}$ for $2 \le i \le k$.

Each H_i must be isomorphic to one of the following:

- Class 1: $P_2 \,\square\, P_2$.
- Class 2: $P_r \,\square\, P_s$, $r \ge 2$, $s > 2$.
- Class 3: $P_2 \,\square\, P_2 \,\square\, P_2$.
- Class 4: $P_2 \,\square\, P_2 \,\square\, P_s$, $s \ge 3$.
- Class 5: $P_2 \,\square\, P_s \,\square\, P_t$, $s, t \ge 3$.
- Class 6: $P_s \,\square\, P_t \,\square\, P_r$, $s, t, r \ge 3$.

We first show that for each H_i, we have $j^*(H_i) \le 2$.

By Theorem 15, the graphs H_i of classes 1, 2, and 3 have $j^*(H_i) \le 2$.

Consider graphs of class 4. Since $j^*(P_2 \,\square\, P_2) \le 2$ and $j^*((P_2 \,\square\, P_2) \,\square\, P_2) \le 2$, graphs H_i of class 4 have $j^*(H_i) \le 2$ by Lemma 22.

Consider graphs of class 5. Since $j^*(P_2 \,\square\, P_s) \le 2$ and $j^*((P_2 \,\square\, P_s) \,\square\, P_2) \le 2$, graphs H_i of class 5 have $j^*(H_i) \le 2$ by Lemma 22.

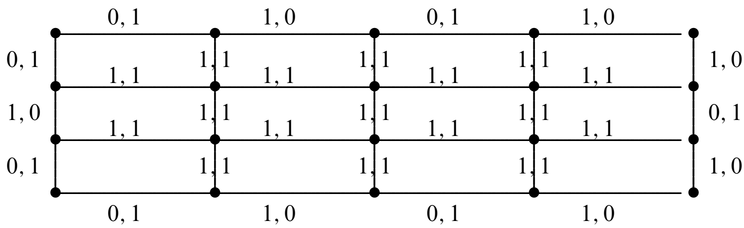

Figure 8. A zero-sum $\mathbb{Z}_2^2$-magic labeling of $P_4 \square P_5$.

Consider graphs of class 6. Since $j^*(P_s \square P_t) \leq 2$ and $j^*((P_s \square P_t) \square P_2) \leq 2$, graphs H_i of class 6 have $j^*(H_i) \leq 2$ by Lemma 22.

Thus, by Theorem 14, $j^*(G) \leq 2$, which implies by Corollary 9(c) that

$$\mathcal{A} \setminus \{\mathbb{Z}_2, \mathbb{Z}_2^2\} \subseteq \mathcal{A}(G) \subseteq \mathcal{A} \setminus \{\mathbb{Z}_2\}.$$

It now suffices to show that G is zero-sum $\mathbb{Z}_2^2$-magic. To that end, we next show that each H_i is zero-sum $\mathbb{Z}_2^2$-magic.

By inspection, the graph H_i of class 1 has $\mathcal{A}(H_i) = \mathcal{A}$.

By Corollary 9(c), the graphs H_i of class 2 are zero-sum $\mathbb{A}$-magic for all $\mathbb{A}$ except $\mathbb{Z}_2$ and possibly $\mathbb{Z}_2^2$. But it is an easy matter to construct a zero-sum $\mathbb{Z}_2^2$-magic labeling of H_i, thereby establishing that $\mathcal{A}(H_i) = \mathcal{A} \setminus \{Z_2\}$. Particularly, consider the following sequence of vertices that specifies a cycle:

$$(1, 1), \dots, (1, s), (2, s), \dots, (r, s), (r, s-1), \dots, (r, 1), (r-1, 1), \dots, (1, 1).$$

Moving around the cycle, assign each edge a label from $\{(0, 1), (1, 0)\}$ according to the following algorithm and as illustrated in Figure 8 (where parentheses are omitted): assign $(0, 1)$ to the edge $(1, 1)(1, 2)$. For every two distinct edges e' and e'' of the cycle, assign distinct labels if e' and e'' are incident to a common vertex of degree 3 in G; otherwise, assign equal labels if e' and e'' are incident to a common vertex of degree 2 in G. To each other edge of G, assign $(1, 1)$.

By Theorem 21(a), the graph H_i of class 3 has $\mathcal{A}(H_i) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$.

Consider graphs of class 4. Since $P_2 \square P_2$ is of class 1, it is zero-sum $\mathbb{A}$-magic for all $\mathbb{A}$. And since $(P_2 \square P_2) \square P_2$ is of class 3, it is zero-sum $\mathbb{A}$-magic for all $\mathbb{A}$ except Z_2. Thus, by Lemma 23 and the fact that graphs H_i of class 4 have vertices of odd degree, $\mathcal{A}(H_i) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$.

Consider graphs of class 5. Since $P_2 \square P_s$ is of class 2, it is zero-sum $\mathbb{A}$-magic for all $\mathbb{A}$ except $\mathbb{Z}_2$. And since $(P_2 \square P_s) \square P_2$ is of class 4, it is zero-sum $\mathbb{A}$-magic for all $\mathbb{A}$ except Z_2. Thus, by Lemma 23 and the fact that graphs H_i of class 5 have vertices of odd degree, $\mathcal{A}(H_i) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$.

Consider graphs of class 6. Since $P_s \square P_t$ is of class 2, it is zero-sum $\mathbb{A}$-magic for all $\mathbb{A}$ except $\mathbb{Z}_2$. And since $(P_s \square P_t) \square P_2$ is of class 5, it is zero-sum $\mathbb{A}$-magic for all $\mathbb{A}$ except Z_2. Thus, by Lemma 23 and the fact that graphs H_i of class 6 have vertices of odd degree, $\mathcal{A}(H_i) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$.

We therefore see by Theorem 7 that $\mathcal{A} \setminus \{\mathbb{Z}_2\} \subseteq \mathcal{A}(G)$. But since G has a vertex of odd degree, G is not zero-sum $\mathbb{Z}_2$-magic. Thus $\mathcal{A}(G) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$. □

We close this section with a theorem that utilizes Corollary 12.

Theorem 25. *Let $n \geq 3$ and let G be an odd graph. Then $\mathcal{A}(G \,\square\, P_n) = \mathcal{A} \setminus \{\mathbb{Z}_2\}$.*

Proof. Let H' be the subgraph of $G \,\square\, P_n$ induced by $\{(v, i) \mid v \in V(G),\ i = 1, 2\}$. Since H' is isomorphic to $G \,\square\, P_2$, Corollary 12 implies the existence of a zero-sum $\mathbb{Z}$-magic labeling ϕ' of H' such that $j(\phi') = 1$. We establish a zero-sum $\mathbb{Z}$-magic labeling ϕ of $G \,\square\, P_n$ with $j(\phi) = 2$, thereby establishing $j^*(G \,\square\, P_n) \leq 2$:

$$\phi(e) = \begin{cases} \phi'((u,1)(w,1)) & \text{if } e = (u,1)(w,1) \text{ or } (u,n)(w,n), \\ 2\phi'((u,1)(w,1)) & \text{if } e = (u,i)(w,i),\ 2 \leq i \leq n-1, \\ \phi'((u,1)(u,2)) & \text{if } e = (u,i)(u,i+1),\ 1 \leq i \leq n-1. \end{cases}$$

By Corollary 9(c), it now suffices to establish a zero-sum $\mathbb{Z}_2^2$-magic labeling ϕ of $G \,\square\, P_n$:

$$\phi(e) = \begin{cases} (0,1) & \text{if } e = (u,1)(w,1), \\ (1,1) & \text{if } e = (u,i)(w,i),\ 2 \leq i \leq n-1, \\ (0,1) & \text{if } e = (u,n)(w,n),\ n \text{ even}, \\ (1,0) & \text{if } e = (u,n)(w,n),\ n \text{ odd}, \\ (0,1) & \text{if } e = (u,i)(u,i+1),\ i \text{ odd}, \\ (1,0) & \text{if } e = (u,i)(u,i+1),\ i \text{ even}. \end{cases}$$ □

5. Closing remarks

We close this paper with suggestions for further study.

If G is the cartesian product of two odd graphs, then $j^*(G) = 1$. What can be said of $j^*(G)$ if one or each factor is even?

If $T_1, T_2, \ldots, T_n$ is a collection of nontrivial trees and $G = \square_{i=1}^{n} T_i$, we are able to show that $j^*(G) \leq 4$. Is this a sharp upper bound?

What can be said of $j^*(G)$ if G is an odd cartesian product?

We have seen that the graphs G under study have $j^*(G) \leq 2$, with $j^*(G) = 1$ if and only if G is an even graph with even size. We have also seen that $j^*(G) = 1$ is a sufficient but not necessary condition for $\mathcal{A}(G) = \mathcal{A}$. Are there cartesian products G such that $j^*(G) \geq 3$? Are there necessary and sufficient conditions for $j^*(G) = 2$?

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adam.y.w.fong@gmail.com — *Department of Mathematics, Trinity College, Hartford, CT, United States*

john.georges@trincoll.edu — *Department of Mathematics, Trinity College, Hartford, CT, United States*

david.mauro@trincoll.edu — *Department of Mathematics, Trinity College, Hartford, CT, United States*

dylanspag@gmail.com — *Department of Mathematics, Trinity College, Hartford, CT, United States*

jackwallacemv@gmail.com — *Department of Mathematics, Trinity College, Hartford, CT, United States* shufan.wang.2018@trincoll.edu *Department of Mathematics, Trinity College, Hartford, CT, United States*

kwashmath@gmail.com *Department of Mathematics, Trinity College, Hartford, CT, United States*

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2019 vol. 12 no. 8

On the zero-sum group-magicness of cartesian products 1261
Adam Fong, John Georges, David Mauro, Dylan Spagnuolo, John Wallace, Shufan Wang and Kirsti Wash

The variable exponent Bernoulli differential equation 1279
Karen R. Ríos-Soto, Carlos E. Seda-Damiani and Alejandro Vélez-Santiago

The supersingularity of Hurwitz curves 1293
Erin Dawson, Henry Frauenhoff, Michael Lynch, Amethyst Price, Seamus Somerstep, Eric Work, Dean Bisogno and Rachel Pries

Multicast triangular semilattice network 1307
Angelina Grosso, Felice Manganiello, Shiwani Varal and Emily Zhu

Edge-transitive graphs and combinatorial designs 1329
Heather A. Newman, Hector Miranda, Adam Gregory and Darren A. Narayan

A logistic two-sex model with mate-finding Allee effect 1343
Elizabeth Anderson, Daniel Maxin, Jared Ott and Gwyneth Terrett

Unoriented links and the Jones polynomial 1357
Sandy Ganzell, Janet Huffman, Leslie Mavrakis, Kaitlin Tademy and Griffin Walker

Nonsplit module extensions over the one-sided inverse of $k[x]$ 1369
Zheping Lu, Linhong Wang and Xingting Wang

Split Grothendieck rings of rooted trees and skew shapes via monoid representations 1379
David Beers and Matt Szczesny

On the classification of Specht modules with one-dimensional summands 1399
Aubrey Piper Collins and Craig J. Dodge

The monochromatic column problem with a prime number of colors 1415
Loran Crowell and Steve Szabo

Total Roman domination edge-critical graphs 1423
Chloe Lampman, Kieka (C. M.) Mynhardt and Shannon Ogden