involve
a journal of mathematics

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2019 vol. 12, no. 8

INVOLVE 12:8 (2019)
dx.doi.org/10.2140/involve.2019.12.1279

The variable exponent Bernoulli differential equation

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(Communicated by Toka Diagana)

We investigate the realization of a Bernoulli-type first-order differential equation with a variable exponent. Using substitution methods, we show the existence of an implicit solution to the Bernoulli problem. Numerical simulations applied to several examples are also provided.

1. Introduction

The aim of this paper is to investigate a Bernoulli-type first-order ordinary differential equation with a variable exponent, formally written as

$$\frac{dy}{dx} + a(x)y = b(x)y^{p(x)}. \tag{1-1}$$

Here $a(x)$, $b(x)$ are continuous functions and $p(x)$ is a function of class C^1 in a bounded interval $[\alpha, \beta]$, with $p(x) \neq 1$ for all x.

Equation (1-1) is well known and standard in the case when $p(x) = p$, a constant; e.g., see [Boyce and DiPrima 2012; Edwards and Penney 2008; Zill and Cullen 2012]. However, when the exponent is variable, to the best of our knowledge, this problem has not been investigated up to the present time. The focus of this work is to provide a first attempt to solve the generalized Bernoulli-type problem (1-1) for particular functions $p(x)$. Unfortunately, even for simple types of functions $p(x)$, the solution of problem (1-1) cannot be given explicitly, and its formulation is in most cases quite complicated. At the end, for the main examples, we will provide numerical simulations for the solutions of ODEs of the type presented in this paper, and we will analyze and compare them with the analytical solutions.

Problem (1-1) for p a constant, known as the Bernoulli ODE, was proposed by James Bernoulli in 1695. A year later, Leibniz solved the equation by making

MSC2010: 34A34, 34A09, 34B15, 65L06.
Keywords: variable exponent differential equations, Bernoulli differential equation, implicit solutions, numerical simulations.

substitutions and simplifying to a linear equation, similar to the method employed in this work. This type of ODE can be viewed as a generalization of the frictional forces equation. Furthermore, modern physics uses Bernoulli differential equations for modeling the dynamics behind certain circuit elements, known as Bernoulli memristors (for more details, we refer to [O'Neil 2012], among others). The Bernoulli differential equation also shows up in some economic utility maximization problems; see, e.g., [Merton 1969]. As mentioned above, all these models consider p to be constant, and there is no literature known for the case when $p = p(x)$ is nonconstant.

Over the recent years, various mathematical problems with variable exponent have attracted the attention of many authors. Interest in variational problems and differential equations with nonstandard growth conditions has grown, highly motivated by various applications, such as elastic mechanics, electrorheological fluids, fluid dynamics, and image restoration; see [Acerbi and Mingione 2002; Bollt et al. 2009; Chen et al. 2006; Cruz-Uribe and Fiorenza 2013; Diening et al. 2011; Diening and Růžička 2003], among others. However, to the best of our knowledge, there is no work done on variable exponent ordinary differential equations.

The paper is organized as follows. In Section 2 we work with (1-1) in all its generality. By making proper substitutions, we transform (1-1) into an exponential-type first-order ODE with variable coefficients, which depends on the variable exponent function $p(x)$. We show that under appropriate conditions on $p(x)$, the corresponding initial value problem of type (1-1) is well-posed. Section 3 is devoted entirely to the solvability of problem (1-1) in the case when the coefficients a, b are constant. However, up to the present time, there are no known appropriate tools that could allow us to solve the problem (1-1) in a general form. Consequently, in this section we focus on the realization of problem (1-1) under particular choices of the function $p(x)$. Even under such restrictions, the solution of problem (1-1) turns out to be of a very complicated structure, and in almost all cases only implicit solutions are achieved. Under some additional restrictions, we are able to provide a concrete formula for the solution of problem (1-1) (under the assumptions of Section 3), which is given as an elaborated convergent infinite series which involves complicated expressions, such as exponential integral functions. In Section 4, we consider a particular case when the coefficients are variable with a specific structure directly related with the exponent $p(x)$. Several examples will be illustrated, whose structure will coincide with the structure outlined at Section 2. Consequently, solutions can only be given implicitly, as argued in the previous section. Finally, in Section 5, some numerical methods are performed over the solutions of particular examples of ODEs of types given by problem (1-1). When possible, we will discuss the relationship between the behavior shown by the solution deduced through numerical methods, in comparison with the analytic solution.

2. Reformulation of the problem

In this section, simple calculations to transform the original Bernoulli equation (1-1) into a simple differential equation will be employed.

Let us start by performing the substitution

$$v = y^{1-p(x)}. \tag{2-1}$$

Then one has

$$\begin{aligned} y &= v^{1/(1-p(x))}, \\ y' &= \frac{d}{dx}(v^{1/(1-p(x))}) = v^{1/(1-p(x))}\left(\frac{p'(x)}{(1-p(x))^2}\ln v + \frac{1}{(1-p(x))}\frac{v'}{v}\right). \end{aligned} \tag{2-2}$$

Substituting (2-2) and (2-1) into (1-1), and multiplying both sides by v we obtain

$$v^{1/(1-p(x))}\left(\frac{p'(x)}{(1-p(x))^2}v\ln v + \frac{1}{(1-p(x))}v' + a(x)v\right) = b(x)v^{1/(1-p(x))},$$

where we recall that $p = p(x)$. Dividing both sides of the equality above by $v^{1/(1-p(x))}$, we arrive at

$$\frac{p'(x)}{(1-p(x))^2}v\ln v + \frac{1}{(1-p(x))}v' + a(x)v = b(x). \tag{2-3}$$

Performing the substitution $w = \ln v$ in (2-3), we have the nonlinear ODE

$$\frac{p'(x)}{(1-p(x))^2}e^w w + \frac{1}{(1-p(x))}e^w w' + a(x)e^w = b(x),$$

which, in turn, can be further simplified into the ODE

$$w' = b(x)e^{-w}(1-p) - a(x)(1-p) - \frac{p'}{1-p}w. \tag{2-4}$$

Note that (2-4) is fully nonlinear, and cannot be linearized, and consequently its solvability is quite nontrivial (as we will see in the subsequent section, even for particular cases). However, the following result asserts that the ODE (2-4) can be solved under certain conditions.

Theorem 2.1 (see [Edwards and Penney 2008]). *Assume that both $f(x, y)$ and its partial derivative $\partial_y f(x, y)$ are continuous over a rectangular region R in the xy-plane that contains the point (a, b) in its interior. Then, there exists some open interval I containing the point a such that the initial value problem*

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b,$$

is uniquely solvable over I.

For our case of interest, namely, the solution of (2-4), under suitable conditions on the functions $a(x)$, $b(x)$ and $p(x)$, we have

$$f(x,w) = b(x)e^{-w}(1-p) - a(x)(1-p) - \frac{p'}{1-p}w,$$

$$\partial_w f(x,w) = -e^{-w}b(x)(1-p) - \frac{p'}{1-p}.$$

Hence we can easily find a rectangle in $\mathbb{R}^2$ in which both $f(x,w)$ and $\partial_w f(x,w)$ are continuous. Consequently, we can apply Theorem 2.1 to obtain that (2-4) is solvable over some interval $I = (\alpha, \beta)$.

3. Solvability of problem (1-1): constant coefficients case

The following section will be devoted in finding tools to solve the problem (1-1). Because of the generality and difficulty of the original problem (1-1), we will investigate the solvability for particular constant coefficient cases. It is shown that even in very simple cases the problem will be highly nontrivial, as will its solution, and basically impossible to be solved explicitly.

3A. ***The case: $a = 0$ and $b = 1$.*** Consider the situation when $a(x) = 0$ and $b(x) = 1$. Then (1-1), using the substitution argument in (2-4), becomes the simplified differential equation

$$w' = e^{-w}(1-p) - \frac{p'}{1-p}w. \tag{3-1}$$

We seek an even more simplified version of the problem (3-1). In fact, below we present some particular cases when the problem (3-1) can be solved implicitly (under suitable conditions that will be explained in more detail).

3A1. *A separable case.* We consider the case when the exponent $p = p(x)$ satisfies the ordinary differential equation

$$\frac{p'}{(1-p)} = \lambda(1-p), \tag{3-2}$$

where $\lambda \in \mathbb{R}\setminus\{0\}$ is a fixed constant. Then (3-2) becomes

$$\frac{1}{(1-p)^2}\,dp = \lambda\,dx, \tag{3-3}$$

which is clearly separable. The function $p(x) = 1 - 1/(\lambda x)$ is a particular solution to the problem (3-3). For this particular case, substituting the function $p(x)$ in (3-1) yields

$$\frac{dw}{dx} = \frac{1}{\lambda x}(e^{-w} - \lambda w), \tag{3-4}$$

which is also a separable first-order ODE. Hence solving (3-4), we get the implicit equation

$$\int \frac{1}{e^{-w} - \lambda w}\, dw = \frac{\ln|x|}{\lambda} + C, \tag{3-5}$$

where the integral on the left-hand side cannot be computed explicitly. We then examine the case when

$$\left|\frac{e^{-w}}{\lambda w}\right| < 1. \tag{3-6}$$

This condition guarantees the uniform convergence of the series in the right-hand side of (3-5) in the function $h(w)$, defined by

$$h(w) = \frac{1}{e^{-w} - \lambda w} = -\frac{1}{\lambda w}\left(\frac{1}{1 - e^{-w}/(\lambda w)}\right) = -\sum_{k=0}^{\infty} \frac{1}{(\lambda w)^{k+1} e^{kw}}. \tag{3-7}$$

The uniform convergence of the series in $h(w)$ allows us to perform term by term integration, arriving at

$$\int h(w)\, dw = -\sum_{k=0}^{\infty} \int \frac{1}{(\lambda w)^{k+1} e^{kw}}\, dw = \frac{1}{\lambda}\sum_{k=0}^{\infty} (\lambda w)^{-k} E_{k+1}(kw), \tag{3-8}$$

where $E_n(x)$ is the so called *n-th exponential integral function*, defined by

$$E_n(x) = \int_1^{\infty} \frac{e^{-xt}}{t^n}\, dt \quad (n \in \mathbb{N}). \tag{3-9}$$

Thus in view of (3-5) and (3-8) (under the special assumption (3-6)), the implicit solution to (3-4) with $a(x) = 0,\ b(x) = 1$ and $p(x) = 1 - 1/(\lambda x)$ is given by

$$\sum_{k=0}^{\infty} (\lambda w)^{-k} E_{k+1}(kw) - \ln|x| = C. \tag{3-10}$$

Performing backward substitutions on w and using the explicit formula for the variable exponent $p(x)$, the solution for (3-1) becomes

$$G(x, y) = C$$

for

$$G(x, y) := \sum_{k=0}^{\infty} \left(\frac{x}{\ln y}\right)^k E_{k+1}(k(\lambda x)^{-1} \ln y) - \ln|x|, \tag{3-11}$$

whenever

$$0 < y < \left(\frac{|\ln y|}{\lambda x}\right)^{\lambda x}.$$

A further analysis on the solution (3-11) together with the condition above shows that the solution $y = y(x)$ fulfills $y(x) \geq e^x$, with $y \approx e^x$ as x is large enough. In particular, the solution blows up as x tend to infinity.

3A2. *The exact method.* The previous case can be worked with an exact ODE by taking (3-4) and rewriting it such that

$$(e^{-w} - \lambda w)\,dx - \lambda x\,dw = 0, \tag{3-12}$$

where $M(x, w) = e^{-w} - \lambda w$ and $N(x, w) = -\lambda x$, and looking over the partial derivatives M_w and N_x it is clear that (3-12) is not exact; see, e.g., [Boyce and DiPrima 2012]. Thus, suppose that an integration factor $\mu(x, w) = \mu(w)$ exists such that

$$\mu(w)(e^{-w} - \lambda w)\,dx - \mu(w)\lambda x\,dw = 0 \tag{3-13}$$

is an exact differential equation. Then $\widetilde{M}_w = \widetilde{N}_x$, where $\widetilde{M}(x, w) = \mu(w)(e^{-w} - \lambda w)$ and $\widetilde{N}(x, w) = -\mu(w)\lambda x$, from which we obtain

$$\mu(w) = \exp\left(\int \frac{1}{1 - \lambda w e^w}\,dw\right). \tag{3-14}$$

Now let

$$\Phi(w) = \frac{1}{1 - \lambda w e^w} = \sum_{k=0}^{\infty} \lambda^k w^k e^{kw}$$

for $|\lambda w e^w| < 1$, and where the integration of this series is

$$\int \Phi(w)\,dw = 1 + \sum_{k=1}^{\infty} \int \lambda^k w^k e^{kw}\,dw = 1 + \sum_{k=1}^{\infty} -w(\lambda w)^k E_{-k}(-kw). \tag{3-15}$$

Substitution of (3-15) into (3-14) yields

$$\mu(w) = \exp\left(1 + \sum_{k=1}^{\infty} -w(\lambda w)^k E_{-k}(-kw)\right). \tag{3-16}$$

With this function we are now able to perform partial integration over (3-13) and obtain an expression for the implicit solution of problem (3-1) for $a = 0$, $b = 1$ and $p(x) = 1 - 1/(\lambda x)$:

$$F(x, w) = \mu(w)(e^{-w} - \lambda w)x = C, \tag{3-17}$$

where $\mu(w)$ is as shown in (3-14).

3B. *The case $a = b = 1$.* In this subsection we take a quick look into the case when $a \neq 0$; for simplicity we take $a = 1$. In fact, setting $a(x) = b(x) = 1$, (1-1) and (2-4) become

$$\frac{dy}{dx} + y = y^{p(x)}, \tag{3-18}$$

$$w' = (e^{-w} - 1)(1 - p) - \frac{p'}{1-p} w, \tag{3-19}$$

respectively. Then as in the previous subsection, we concentrate on the separable case for (3-19).

In fact, to have separability, as before we require that the function $p(x)$ fulfill (3-2) (here for simplicity we take $\lambda = 1$). Proceeding as in Section 3A1, we have that $p(x) = 1 - 1/x$ is the required function. Inserting this function into (3-19), we get

$$\frac{dw}{dx} = \frac{1}{x}(e^{-w} - w - 1), \tag{3-20}$$

a clearly separable ODE whose integral equation is given by

$$\int \frac{dw}{e^{-w} - w - 1} = \ln|x| + C. \tag{3-21}$$

Now let

$$h(w) = \frac{1}{e^{-w} - w - 1} = -\frac{1}{1+w}\left(\frac{1}{1 - e^{-w}/(1+w)}\right) = -\sum_{k=0}^{\infty} \frac{e^{-kw}}{(1+w)^{k+1}}, \tag{3-22}$$

where we are requiring that

$$\left|\frac{e^{-w}}{1+w}\right| < 1. \tag{3-23}$$

Then under such restriction, the series appearing in (3-22) converges absolutely, and consequently, we have

$$\int h(w)\,dw = -\sum_{k=0}^{\infty} \int \frac{e^{-kw}}{(1+w)^{k+1}}\,dw = \sum_{k=0}^{\infty} e^{-k}(w+1)^{-k} E_{k+1}(k(w+1)). \tag{3-24}$$

In view of (3-23) and (3-24), the solution of the ODE (3-18) is given implicitly by $H(x, y) = C$ for

$$H(x, y) := \sum_{k=0}^{\infty} e^{-k}\left(\frac{\ln y}{x} + 1\right)^{-k} E_{k+1}\left(\frac{k \ln y}{x} + k\right) - \ln|x|, \tag{3-25}$$

whenever (3-23) holds for $w = \ln v = (\ln y)/x$. A careful examination of this condition shows that (3-23) is valid if and only if $w > 0$, or equivalently, if and only if $(\ln y)/x > 0$. Going over the solution (3-25) over the given interval of convergence shows that the solution y satisfies $y = y(x) \geq 1$ when $x > 0$, with $y \approx 1$ as x tends to infinity.

3C. ***Method of differences.*** In this subsection we consider another approach to solve (3-1) (and consequently (1-1)) in the case when $a(x) = a$ and $b(x) = b$ are constant coefficients. For simplicity, we take $a = b = 1$.

We begin by considering the ODE

$$\gamma' + \frac{p'}{1-p}\gamma = (1-p)[e^{-w} - 1], \tag{3-26}$$

where $\gamma := \gamma(x)$ and $w = w(x)$ is the solution of problem (3-1). One sees that (3-26) is a linear first-order ODE, and thus the solution $\gamma(x)$ of problem (3-26) is given by

$$\gamma(x) = (1-p(x))\int(e^{-w} - 1)\,dx = (1-p(x))\Big[-x + \int e^{-w}\,dx\Big]. \tag{3-27}$$

On the other hand, since w solves the ODE (3-19), we have

$$w' + \frac{p'}{1-p}w = (1-p)[e^{-w} - 1]. \tag{3-28}$$

Substituting the solution (3-27) into (3-26), and then taking the difference of (3-26) and (3-28), we obtain

$$\frac{d\phi}{dx} = \frac{p'}{1-p}\phi, \tag{3-29}$$

where $\phi(x) := \gamma(x) - w(x)$. Equation (3-29) is separable, and its solution is given by

$$\phi(x) = \frac{E}{1-p(x)} \tag{3-30}$$

for $E = e^D$ some arbitrary constant. Using the definition of ϕ, we arrive at the integral equation

$$w(x) = (1-p(x))\Big[-x + \int e^{-w}\,dx\Big] - \frac{E}{1-p(x)}. \tag{3-31}$$

Substituting back into the original variable $y = y(x)$, solution (3-31) becomes the exponential integral equation

$$y(x) = \exp\Big(-x + \int [y(x)]^{p(x)-1}\,dx - \frac{E}{(1-p(x))^2}\Big). \tag{3-32}$$

4. Solvability of problem (1-1): variable coefficients case

In this section, we will look into problem (1-1) for particular cases of the (variable) coefficients $a(x)$, $b(x)$. A careful examination of (2-4) shows that the cases where such problem can be solved (with the standard tools) are very few. In particular, one can deduce that a requirement is that $a(x) = b(x)$, and this may equal a particular function depending on the exponent function $p(x)$, as we show below.

In view of the above paragraph, we let

$$a(x) = b(x) = \frac{p'(x)}{(1 - p(x))^2}. \tag{4-1}$$

Then, substituting these choices into (2-4) gives the ODE

$$w' = \frac{p'}{1 - p}(e^{-w} - 1 - w). \tag{4-2}$$

Equation (4-2) is clearly a separable differential equation, and consequently, its solution is given by the integral equation

$$\int \frac{dw}{e^{-w} - 1 - w} = -\ln|p(x) - 1| + C, \tag{4-3}$$

where the solution to the left-hand side of (4-3) is given by solution (3-24) (under the assumption (3-23)). The following examples illustrate in a more concrete way the above formulations.

Example 4.1. Consider the differential equation

$$y' - e^x y = -e^x y^{1+e^{-x}}. \tag{4-4}$$

Here $a(x) = b(x) = -e^x$ and $p(x) = 1 + e^{-x}$. Observe that $a,\ b,\ p \in C^\infty(\mathbb{R})$ with $p(x) > 1$ for all $x \in \mathbb{R}$. Furthermore, one clearly sees that (4-1) holds, and consequently applying (4-3), recalling (3-24) and proceeding as in the derivation of (3-25), the solution of the differential equation (4-4) is given by

$$\sum_{k=0}^{\infty} e^{-k}\left(\frac{\ln y}{x} + 1\right)^{-k} E_{k+1}\left(\frac{k \ln y}{x} + k\right) - x = C, \tag{4-5}$$

provided that the condition $(\ln y)/x > 0$ is valid.

Example 4.2. Consider the differential equation

$$y' + 2\tan x \sec^2 x y = 2\tan x \sec^2 x y^{\sin^2 x} \tag{4-6}$$

for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $a(x) = b(x) = 2\tan x \sec^2 x \in C^\infty\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $p(x) = \sin^2 x \in C^\infty\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$ with $|p| < 1$ over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Again, (4-1) holds, and thus proceeding as in the previous example, one gets that the solution of ODE (4-6) is given implicitly by

$$\sum_{k=0}^{\infty} e^{-k}\left(\frac{\ln y}{x} + 1\right)^{-k} E_{k+1}\left(\frac{k \ln y}{x} + k\right) - 2\ln(\cos x) = D, \tag{4-7}$$

whenever the inequality $(\ln y)/x > 0$ holds.

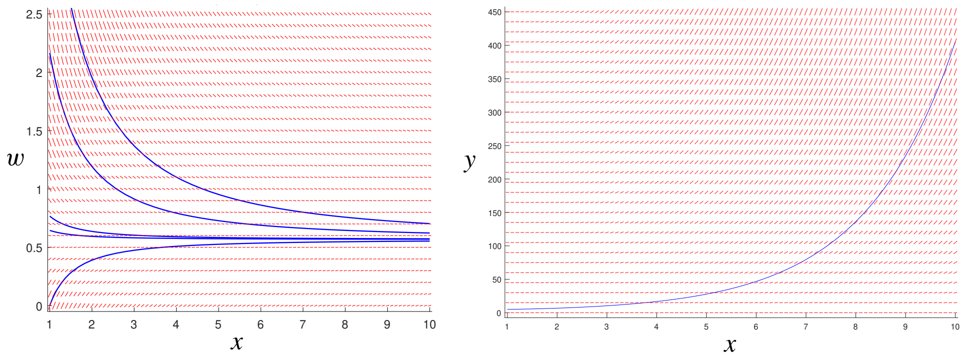

Figure 1. Solutions of $dw/dx = (1/x)(e^{-w} - w)$ with different initial conditions (left) and the solution of $dy/dx = y^{1-1/x}$ with initial condition $y(1) = 5$ (right).

5. Numerical simulations: some examples

In this section, we look at numerical solutions the problems (1-1) and (3-4) (for a, b constant). Several examples are examined for particular choices of the function $p = p(x)$. With MATLAB software we tested numerical convergence to a solution using the Runge–Kutta solution method through the built in function ode45. In each of the examples provided, the plot on the left represents the solution of the substitution problem given by (2-4) (under the specific assumptions for each example) with five initial conditions taken randomly at $x_0 = 1$. The plot on the right shows a solution with initial condition $y(1) = 5$ for the original Bernoulli-type equation (1-1), under the same assumptions as the plot on the left. All solutions are plotted over their respective vector fields.

5A. *The separable case.* As shown in Section 3, the separable case of the problem (2-4) (for a, b constants, $b \neq 0$) is given when $p(x) = 1 - 1/(\lambda x)$ (for $\lambda \neq 0$ an arbitrary fixed constant).

Example 5.1. We take the constants $a = 0$ and $b = \lambda = 1$ and $p(x) = 1 - 1/x$. The solutions produced are given in Figure 1. Notice that, as described in Section 3A1 solutions blow up as x tends to infinity.

Observe that Figure 1(left) illustrates that as x tends to infinity, the numerical solution converges, whereas the graph of the solution of the original equation, Figure 1 (right) seems to blow-up as x goes to infinity. These facts agree with the analysis performed over (3-11).

Example 5.2. We consider now the case where the constants satisfy $a = b = \lambda = 1$, and $p(x) = 1 - 1/x$. The simulations produced are given in Figure 2.

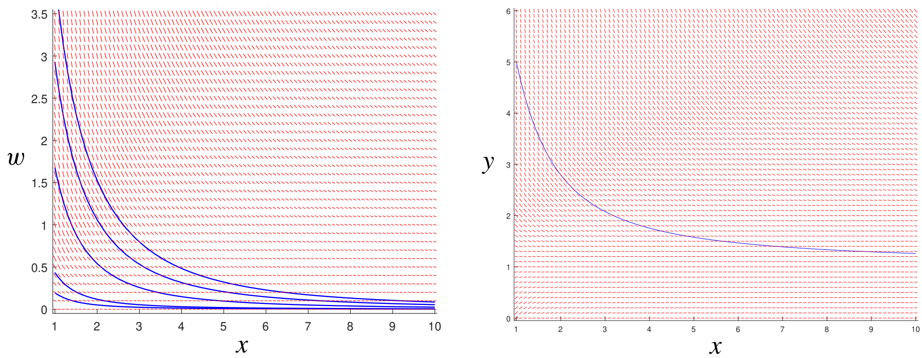

Figure 2. Solutions of $dw/dx = (1/x)(e^{-w} - w - 1)$ with different initial conditions (left) and the solution of $dy/dx = y^{1-1/x} - y$ with initial condition $y(1) = 5$ (right).

Notice, again, we have numerical convergence in Figure 2 (left) and Figure 2 (right) shows convergence to $y = 1$; this agrees with the analysis done over (3-25).

5B. ***Other examples.*** In this section, we examine numerical solutions to the (nonseparable) problem (2-4) and the original equation, by exploring other choices for the function $p(x)$, but over the same domain and conditions used in the previous simulations. Unlike the previous cases, we will be unable to provide a more rigorous examination of the solution for these examples, since in these cases both (1-1) and (2-4) become unsolvable with any of the known methods for ODEs. For simplicity, we will assume that $a(x) = 0$ and $b(x) = 1$ in the following examples.

Example 5.3. Let $p(x) = 1 - e^x$. The resulting simulations are given in Figure 3.

Example 5.4. Let $p(x) = 1 - e^{-x}$. The resulting simulations are given in Figure 4.

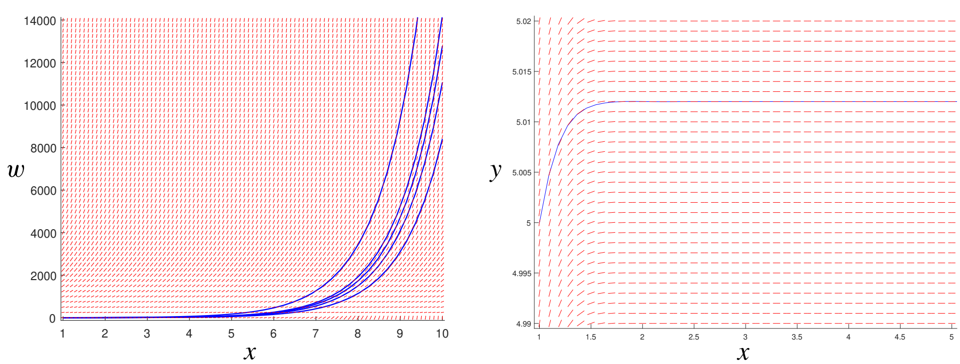

Figure 3. Solutions of $dw/dx = e^{x-w} + w$ with different initial conditions (left) and the solution of $dy/dx = y^{1-e^x}$ with initial condition $y(1) = 5$ (right).

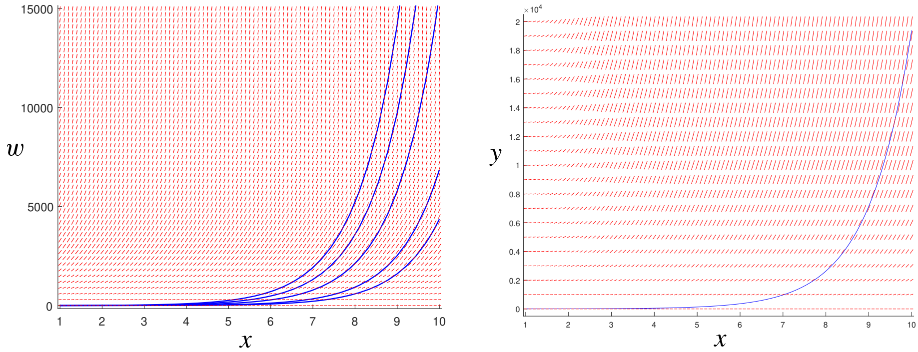

Figure 4. Solutions of $dw/dx = e^{-(x+w)} + w$ with different initial conditions (left) and the solution of $dy/dx = y^{1-e^{-x}}$ with initial condition $y(1) = 5$ (right).

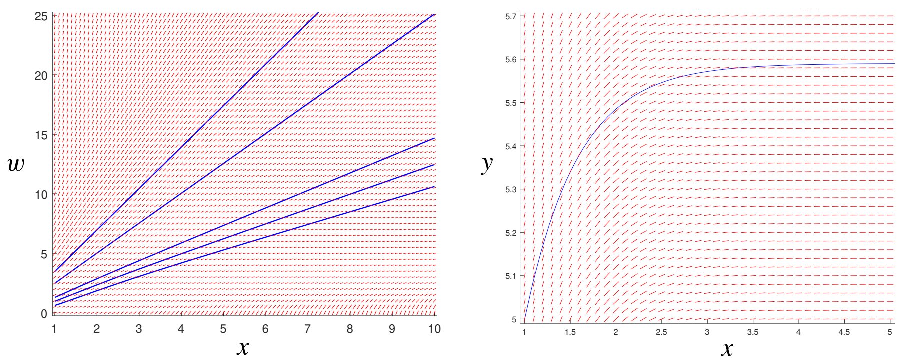

Figure 5. Solutions of $dw/dx = xe^{-w} + w/x$ with different initial conditions (left) and the solution of $dy/dx = y^{1-x}$ with initial condition $y(1) = 5$ (right).

Example 5.5. Let $p(x) = 1 - x$. The resulting simulations are given in Figure 5.

In the examples above of nonseparable ODEs, one can notice that the corresponding solutions to problem (2-4) are unbounded as x becomes large enough. Nevertheless, their corresponding solutions to the original equation (1-1) can be bounded, as Examples 5.3 and 5.5 show. Since for these particular examples, there is no method available to allow a more rigorous and deep analysis on the solutions of problems (1-1) and (2-4), further details concerning these last examples cannot be provided.

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Received: 2018-06-06 Revised: 2019-06-18 Accepted: 2019-08-31

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2019 vol. 12 no. 8

1944-4176(2019)12:8;1-Y