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# Edge-transitive graphs and combinatorial designs

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A graph is said to be edge-transitive if its automorphism group acts transitively on its edges. It is known that edge-transitive graphs are either vertex-transitive or bipartite. We present a complete classification of all connected edge-transitive graphs on less than or equal to 20 vertices. We investigate biregular bipartite edge-transitive graphs and present connections to combinatorial designs, and we show that the Cartesian products of complements of complete graphs give an additional family of edge-transitive graphs.

## 1. Introduction

A graph is vertex-transitive (edge-transitive) if its automorphism group acts transitively on its vertex (edge) set. We note the alternative definition given in [Andersen et al. 1992].

**Theorem 1** (Andersen, Ding, Sabidussi, and Vestergaard). *A finite simple graph $G$ is edge-transitive if and only if $G - e_1 \cong G - e_2$ for all pairs of edges $e_1$ and $e_2$.*

We also mention the following well-known result, which appears as Proposition 15.1 in [Biggs 1974].

**Proposition 2.** *If $G$ is an edge-transitive graph, then $G$ is either vertex-transitive or bipartite; in the latter case, vertices in a given part belong to the same orbit of the automorphism group of $G$ on vertices.*

Given a graph $G$ we will denote its vertex set by $V(G)$ and edge set by $E(G)$. We will use $K_n$ to denote the complete graph with $n$ vertices, and $K_{m,n}$ to denote the complete bipartite graph with $m$ vertices in one part and $n$ in the other. The path on $n$ vertices will be denoted by $P_n$ and the cycle on $n$ vertices by $C_n$. The disjoint union of $t$ copies of a graph $H$ will be denoted by $tH$. The cube on $n$ vertices will be denoted by $Q_n$. The complement of a graph $G$ will be denoted by $\overline{G}$. For any undefined notation, please see [West 2001].

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**Definition 3.** A graph is regular if all of its vertices have the same degree. A bipartite graph is said to be biregular if all vertices on the same side of the bipartition have the same degree. Particularly, we refer to a bipartite graph with parts of size $m$ and $n$ as an $(r,s)$-biregular subgraph of $K_{m,n}$ if the $m$ vertices in the same part each have degree $r$ and the $n$ vertices in the same part each have degree $s$.

It follows from Proposition 2 that bipartite edge-transitive graphs are biregular.

**Definition 4.** Given a group $G$ and generating set $S$, the Cayley graph $\Gamma(G,S)$ is a graph with vertex set $V(\Gamma)$ and edge set

$$E(\Gamma)=\{\{x,y\}\mid x,y\in V(\Gamma),\text{there exists an integer } s \text{ in } S \text{ such that } y=xs\}.$$

It is known that all Cayley graphs are vertex-transitive. Next we recall a specialized class of Cayley graphs known as circulant graphs.

**Definition 5.** A circulant graph $C_n(L)$ is a graph on vertices $v_1, v_2, \ldots, v_n$ where each $v_i$ is adjacent to $v_{(i+j) \pmod n}$ and $v_{(i-j) \pmod n}$ for each $j$ in a list $L$. Algebraically, circulant graphs are Cayley graphs of finite cyclic groups. For a list $L$ containing $m$ items, we refer to $C_n(L)$ as an $m$-circulant. We say an edge $e$ is a chord of length $k$ when $e=v_iv_j,\ |i-j|\equiv k \pmod n$.

In our next definition we present another family of vertex-transitive graphs.

**Definition 6.** A wreath graph, denoted by $W(n,k)$, has $n$ sets of $k$ vertices each, arranged in a circle where every vertex in set $i$ is adjacent to every vertex in bunches $i+1$ and $i-1$. More precisely, its vertex set is $\mathbb{Z}_n \times \mathbb{Z}_k$ and its edge set consists of all pairs of the form $\{(i,r),(i+1,s)\}$.

It was proved in [Onkey 1995] that all wreath graphs are edge-transitive. We next recall the definition of the line graph which we use later to show that certain graph families are edge-transitive.

**Definition 7.** Given a graph $G$, the line graph $L(G)$ is a graph where $V(L(G)) = E(G)$ and two vertices in $V(L(G))$ are adjacent in $L(G)$ if and only if their corresponding edges are incident in $G$.

Finally we recall the operation of the Cartesian product of graphs.

**Definition 8.** Given two graphs $H$ and $K$, with vertex sets $V(H)$ and $V(K)$, the Cartesian product $G = H \times K$ is a graph where

$$V(G)=\{(u_i,v_j)\mid u_i\in V(H) \text{ and } v_j\in V(K)\}$$

and $\{(u_i,v_j),(u_k,v_l)\}\in E(G)$ if and only if $i=k$ and $v_j$ and $v_l$ are adjacent in $K$ or $j=l$ and $u_i$ and $u_k$ are adjacent in $H$.

The properties vertex-transitive and edge-transitive are distinct. This is clear with the following examples:

- $K_n,\ n \geq 2$, is both vertex-transitive and edge-transitive.
- $C_n(1,2),\ n \geq 6$, is vertex-transitive, but not edge-transitive.
- $K_{1,n-1}$ is not vertex-transitive, but is edge-transitive.
- $P_n,\ n \geq 4$, is neither vertex-transitive nor edge-transitive.

However the two properties are linked, as is evident from the following proposition, which is a consequence of results of [Whitney 1932; Sabidussi 1961].

**Proposition 9.** *A connected graph is edge-transitive if and only if its line graph is vertex-transitive.*

Note, however, that a graph may not be the line graph of some original graph. For example, $K_{1,3} \times C_4$ is vertex-transitive, but it follows by a theorem of [Beineke 1968] that this graph is not the line graph of some graph.

We used the databases from Brendan McKay[1] to obtain all connected edge-transitive graphs on 20 vertices or less. We then reported the number of edge-transitive graphs up to 20 vertices to the Online Encyclopedia of Integer Sequences, and they are listed under sequence #A095424. The full classification of these graphs is given in the online supplement. We can extrapolate much from this data and these results are presented in this paper. It was recently brought to our attention that Marston Conder and Gabriel Verret independently determined the edge-sets of the connected edge-transitive bipartite graphs on up to 63 vertices[2] using the Magma system, and a complete list of all connected edge-transitive graphs on up to 47 vertices[3] with their edge sets.[4] In our paper we provide additional details about these graphs, allowing us to generalize some cases to infinite families of graphs.

We note the following graph families are edge-transitive: $K_n,\ n \geq 2$; $C_n,\ n \geq 3$; $K_{n,n}$ minus a perfect matching; $K_{2n}$ minus a perfect matching; and all complete bipartite graphs $K_{t,n-t},\ 1 \leq t \leq \lfloor \frac{n}{2} \rfloor$. Wreaths [Onkey 1995] and Kneser graphs [Godsil and Royle 2001, pp. 135–161] are also edge-transitive. Besides these predictable and apparent cases, we can identify other infinite families of edge-transitive graphs, using the data up through 20 vertices.

We say that $H$ is an $(r,s)$-biregular subgraph of $K_{m,n}$ if $H$ is bipartite graph with degrees $r$ and $s$. In Section 2 of this paper we begin by exploring the problem of

[1] http://users.cecs.anu.edu.au/∼bdm/data/graphs.html

[2] https://www.math.auckland.ac.nz/∼conder/AllSmallETBgraphs-upto63-summary.txt

[3] https://www.math.auckland.ac.nz/∼conder/AllSmallETgraphs-upto47-summary.txt

[4] https://www.math.auckland.ac.nz/∼conder/AllSmallETgraphs-upto47-full.txt

determining which values of $m, n, r, s$, where $mr = ns$, result in a (connected) $(r, s)$-biregular subgraph of $K_{m,n}$ that is edge-transitive. In Section 2.1, we investigate bipartite edge-transitive graphs where one of the two vertex degrees in $G$ is 2.

Connections between balanced incomplete block designs and graphs are well-known. For some recent papers, see [Abueida and Pike 2013; Mamut et al. 2004; McKay and Pike 2007]. In Section 2.2, we investigate connections between edge-transitive graphs and balanced incomplete block designs.

## 2. Connected bipartite graphs

Given positive integers $m$ and $n$, we first describe which values of $r$ and $s$ are possible for an $(r, s)$-biregular subgraph of $K_{m,n}$. Note that if $\gcd(m, n) = 1$, the only biregular subgraph of $K_{m,n}$ is $K_{m,n}$.

**Proposition 10.** *An $(r, s)$-biregular subgraph of $K_{m,n}$ satisfies*

$$mr = ns,$$
$$r = \frac{n}{\gcd(m,n)}k, \quad k = 1, 2, \ldots, \gcd(m, n).$$

*Proof.* We know

$$s = \frac{mr}{n} = \frac{m/\gcd(m,n)}{n/\gcd(m,n)}r,$$

and since

$$\gcd\left(\frac{m}{\gcd(m,n)}, \frac{n}{\gcd(m,n)}\right) = 1,$$

$r$ is a multiple of $n/\gcd(m, n)$ (and is less than or equal to $n$). □

**Corollary 11.** *If $\gcd(m, n) = 2$, there are only two possible pairs $(r, s)$, namely, $(r, s) = \left(\frac{n}{2}, \frac{m}{2}\right)$ and $(r, s) = (n, m)$. The latter case is the complete bipartite graph $K_{m,n}$.*

We now introduce a construction for generating nontrivial edge-transitive (connected) bipartite subgraphs of $K_{m,n}$ for $\gcd(m, n) > 2$. This construction involves a process of extending a nontrivial edge-transitive (connected) bipartite graph to a larger one, which we describe in the following lemma.

**Lemma 12.** *Let $G$ be an edge-transitive (connected) $(r, s)$-biregular subgraph of $K_{m,n}$. Then, for any positive integers $a, b$, and $r$, the subgraph $G$ can be extended to an edge-transitive (connected) $(ra, sb)$-biregular subgraph of $K_{mb,na}$.*

*Proof.* It suffices to show that, by letting $G$ be a (connected) edge-transitive $(r, s)$-biregular subgraph of $K_{m,n}$, we can build a (connected) edge-transitive graph $H$ that is an $(r, 2s)$-biregular subgraph of $K_{2m,n}$. Let $G$ consist of partite sets $A$, $B$,

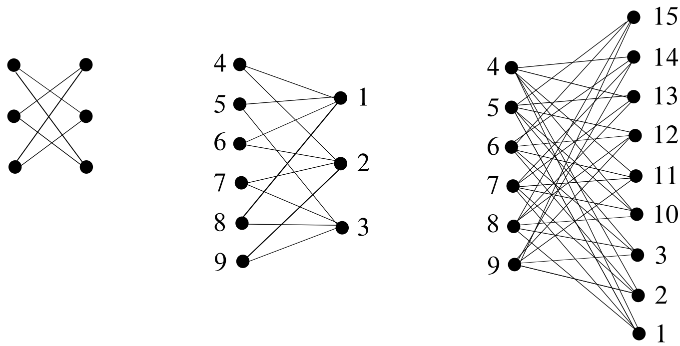


**Figure 1.** An example of the construction in the theorem, with vertices drawn in the same color being vertices that are connected to the graph in the same way.

where $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Now create the set $A' = \{a_1, a_2, \ldots, a_m, a'_1, a'_2, \ldots, a'_m\}$ and create a graph $H$ with partite sets $A'$ and $B$ as follows. For each $a_i$, let $N_H(a_i) = N_G(a_i)$. For each $a'_i$, let $N_H(a'_i) = N_G(a_i)$. Then by construction, $H$ is a (connected) $(r, 2s)$-biregular subgraph of $K_{2m,n}$. Since $G$ is edge-transitive, $H$ is edge-transitive by construction. $\square$

It turns out we can use the results above to state the following general theorem.

**Theorem 13.** *Let* $\gcd(m, n) > 2$. *Then there exists a noncomplete edge-transitive* (*connected*) *subgraph of* $K_{m,n}$.

*Proof.* We appeal to the construction in the preceding lemma, and consider the following two cases. It may be helpful to refer to Figure 1.

Case 1: $m \mid n$. Then $n = mk$ for some positive integer $k$. Let $G$ be the graph that results from removing a perfect matching from $K_{m,m}$. Then $G$ is connected, biregular, and edge-transitive but not complete. Repeating the construction in the lemma $k - 1$ times, we obtain a subgraph of $K_{m,mk} = K_{m,n}$ that is connected, biregular, edge-transitive, and not complete.

Case 2: $m \nmid n$. Let $l = \gcd(m, n)$ and $m = k_1 l$, $n = k_2 l$. Let $G$ be the graph that results from removing a perfect matching from $K_{l,l}$. Then $G$ is connected, biregular, and edge-transitive but not complete. Following the construction in the lemma, increase the left partite set by $l$ vertices $k_1 - 1$ times and the right partite set by $l$ vertices $k_2 - 1$ times. The resulting graph will be a connected, biregular, and edge-transitive subgraph of $K_{k_1 l, k_2 l} = K_{m,n}$ but not complete. $\square$

**Remark 14.** Theorem 13 gives rise to the following observations/questions:

- When $\gcd(m, n) = 1$, the only possible (connected) biregular subgraph is the complete graph $K_{m,n}$.

- When $\gcd(m,n) = 2$, the method fails because the only connected, biregular subgraph of $K_{2,2}$ is $K_{2,2}$, and we seek a noncomplete bipartite graph.
- When $\gcd(m,n) = 2$, under what additional conditions does the theorem still hold?

**2.1.** ***Edge-transitive (connected) $(r,2)$-biregular subgraphs of $K_{m,n}$.*** We now investigate bipartite edge-transitive graphs where one of the two vertex degrees in $G$ is 2. We will provide a construction for some graphs in this family. As pointed out by a referee, such a graph $G$ can be obtained by subdividing every edge of another multigraph $F$. Here $F$ is formed by taking a complete graph on $m$ vertices and "cloning" each of its edges a fixed number of times. Let $F$ be the graph with $m$ vertices and $t$ edges between each pair of distinct vertices. This forms a multigraph with $m$ vertices and $s = t\binom{m}{2}$ edges. Subdividing each edge yields a bipartite subgraph of $K_{m,s}$ with degrees $(t(m-1), 2)$. We could also create $F$ by taking other arc-transitive graphs and cloning each of the edges a fixed number of times.

Using this construction, in general $G$ is edge-transitive if and only if $F$ is arc-transitive. In these arc-transitive multigraphs, every edge must have the same multiplicity, hence reducing this case to the study of arc-transitive graphs. We formalize these ideas in the following theorem.

**Theorem 15.** *$G$ is an edge-transitive connected $(r,2)$-biregular subgraph of $K_{m,n}$ if and only if there exists an arc-transitive graph $F$ such that $F$ is obtained by contracting every edge of $G$.*

*Proof.* Let $G$ is an edge-transitive connected $(r,2)$-biregular subgraph of $K_{m,n}$. Then any two edges $e_1$ and $e_2$ incident to the same vertex in the part of size $n$ are indistinguishable. Then contracting the $P_3$ with edges $e_1$ and $e_2$ results in an edge between vertices in $F$ that is indistinguishable in either direction. Hence $F$ is arc-transitive. For the other direction, using reasoning similar to the above, note that subdividing edges of an arc-transitive graph results in a graph that is edge-transitive. □

We use this theorem for small cases of $|V(G)|$. We first consider the case where $m = 4$. Assume that $G$ is an $(r,2)$-biregular subgraph of $K_{4,n}$. Then $F$ is an arc-transitive multigraph of order 4 with degrees equal to $r$. Since the only arc-transitive graphs of order 4 are $K_4$, $C_4$, and $2P_2$, we know $F$ must be one of these three graphs with each edge cloned a fixed number of times. This will give a complete classification for $G$. This method can be generalized for cases where all of the arc-transitive graphs of a given order are known.

We next use the same procedure on graphs of up to nine vertices. A list of all of the arc-transitive graphs for small orders (with a minor correction) is found on MathWorld [Weisstein]:

- $|V(G)| = 2$:
  - $P_2$ with edges cloned $t \geq 2$ times gives a $(t, 2)$-biregular subgraph of $K_{2,t}$.
- $|V(G)| = 3$:
  - $C_3$ with edges cloned $t \geq 2$ times gives a $(2t, 2)$-biregular subgraph of $K_{3,3t}$.
- $|V(G)| = 4$:
  - $K_4$ with edges cloned $t \geq 2$ times gives a $(3t, 2)$-biregular subgraph of $K_{4,6t}$.
  - $C_4$ with edges cloned $t \geq 2$ times gives a $(2t, 2)$-biregular subgraph of $K_{4,4t}$.
- $|V(G)| = 5$:
  - $K_5$ with edges cloned $t \geq 2$ times gives a $(4t, 2)$-biregular subgraph of $K_{5,10t}$.
  - $C_5$ with edges cloned $t \geq 2$ times gives a $(2t, 2)$-biregular subgraph of $K_{5,5t}$.
- $|V(G)| = 6$:
  - $K_6$ with edges cloned $t \geq 2$ times gives a $(5t, 2)$-biregular subgraph of $K_{6,15t}$.
  - $C_6$ with edges cloned $t \geq 2$ times gives a $(2t, 2)$-biregular subgraph of $K_{6,6t}$.
  - $C_6(1,2)$ with edges cloned $t \geq 2$ times gives a $(4t, 2)$-biregular subgraph of $K_{6,12t}$.
  - $K_{3,3}$ with edges cloned $t \geq 2$ times gives a $(3t, 2)$-biregular subgraph of $K_{9,9t}$.
- $|V(G)| = 7$:
  - $K_7$ with edges cloned $t \geq 2$ times gives a $(6t, 2)$-biregular subgraph of $K_{7,21t}$.
  - $C_7$ with edges cloned $t \geq 2$ times gives a $(4, 2)$-biregular subgraph of $K_{7,7t}$.
- $|V(G)| = 8$:
  - $K_8$ with edges cloned $t \geq 2$ times gives a $(7t, 2)$-biregular subgraph of $K_{8,28t}$.
  - $C_8$ with edges cloned $t \geq 2$ times gives a $(4, 2)$-biregular subgraph of $K_{8,8t}$.
  - $C_8(2,4)$ with edges cloned $t \geq 2$ times gives a $(4t, 2)$-biregular subgraph of $K_{8,16t}$.
  - $C_8(1,2,3)$ with edges cloned $t \geq 2$ times gives a $(6t, 2)$-biregular subgraph of $K_{8,24t}$.
  - $Q_8$ doubled gives a $(3t, 2)$-biregular subgraph of $K_{8,12t}$.
  - $K_{4,4}$ doubled gives a $(4t, 2)$-biregular subgraph of $K_{8,16t}$.
- $|V(G)| = 9$:
  - $K_9$ with edges cloned $t \geq 2$ times gives a $(8t, 2)$-biregular subgraph of $K_{9,36t}$.
  - $C_9$ with edges cloned $t \geq 2$ times gives a $(4, 2)$-biregular subgraph of $K_{9,9t}$.
  - $C_3 \times C_3$ doubled gives a $(4t, 2)$-biregular subgraph of $K_{9,18t}$.
  - $K_{3,3,3}$ doubled gives a $(8, 2t)$-biregular subgraph of $K_{9,36t}$.

We can also state a result of a general nature. For every positive integer $n$, $K_n$ and $C_n$ are arc-transitive graphs. As a result, we can double $K_n$ to obtain an $(n-1, 2)$-biregular subgraph of $K_{n,n^2-n}$ and double $C_n$ to obtain a $(4, 2)$-biregular subgraph of $K_{n,2n}$. For even $n$ we can double $K_{\frac{n}{2},\frac{n}{2}}$ to obtain an $(n, 2)$-biregular subgraph of $K_{n,2n^2}$. Other graphs will depend on the prime factorization of $n$.

**2.2.** ***Edge-transitive graphs and combinatorial designs.*** We now explore regular and biregular edge-transitive bipartite graphs, where the valences can be larger than 2. In fact we will provide constructions of edge-transitive bipartite graphs where the valences can be made arbitrarily large. We investigate connections between biregular bipartite edge-transitive graphs and combinatorial designs. Here the edge incidences arise directly from the combinatorial structure. We begin by recalling the definition of a balanced incomplete block design (BIBD).

**Definition 16.** A $(v, b, r, k, \lambda)$-BIBD is an arrangement of $v$ objects (varieties) into $b$ blocks such that

(i) each object appears in exactly $r$ blocks,

(ii) each block contains exactly $k$ $(k < v)$ objects, and

(iii) each pair of distinct objects appear together in exactly $\lambda$ blocks.

A *partially balanced incomplete block design* is a design where $\lambda$ is not fixed.

A BIBD is called symmetric if $v = b$. Connections are known between the existence of symmetric BIBDs and edge-transitive graphs [Levi 1942; Yang et al. 2016]. A symmetric BIBD is defined for any block design $(P, B)$, where $P$ is the set of points and $B$ is the set of blocks with every edge representing an incident point-block pair $(p, B)$. We note that a projective plane of order $n$ is equivalent to a bipartite graph with two parts each of size $n^2 + n + 1$, where every vertex has degree $n+1$, and every two vertices in the same part have a unique common neighbor. The edge-transitive Levi graphs are incidence graphs of the projective plane. Yang, W. Liu, H. Liu, and Feng [Yang et al. 2016] proved a relationship between incidence graphs and BIBDs. These showed a connection between edge-transitive regular bipartite graphs and flag transitive symmetric block designs.

We note here that connections also exist between nonsymmetric $(v, b, r, k, \lambda)$-balanced incomplete block designs and edge-transitive graphs.

**Example 17.** Consider the $(4, 6, 3, 2, 1)$-block design with blocks

$$\{y_1, y_2\}, \quad \{y_1, y_3\}, \quad \{y_1, y_4\}, \quad \{y_2, y_3\}, \quad \{y_2, y_4\}, \quad \{y_3, y_4\}.$$

This corresponds to the graph in Figure 2 where the edges connect vertices corresponding to the different points in $P$ and different elements of the blocks.

The edge-transitivity of this graph follows from the symmetry as the neighborhoods of the vertices on the left side are the $\binom{4}{2}$ different pairs of the vertices $y_1, y_2, y_3$, and $y_4$. As a result the $(4, 6, 3, 2, 1)$-block design corresponds to an edge-transitive $(2, 3)$-biregular subgraph of the complete bipartite graph $K_{6,4}$.

**Example 18.** Consider the $(5, 10, 4, 2, 1)$-block design with blocks

$$\begin{array}{lllll} \{y_1, y_2\}, & \{y_1, y_3\}, & \{y_1, y_4\}, & \{y_1, y_5\}, & \{y_2, y_3\}, \\ \{y_2, y_4\}, & \{y_2, y_5\}, & \{y_3, y_4\}, & \{y_3, y_5\}, & \{y_4, y_5\}. \end{array}$$

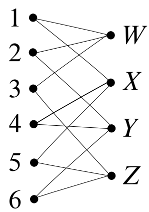


**Figure 2.** The bipartite graph from Example 17.

This corresponds to an edge-transitive $(2,4)$-biregular subgraph of the complete bipartite graph $K_{10,5}$.

We can generalize the past two examples in the following theorem, where we consider the different subsets of size $k$ from the set $\{y_1, y_2, \dots, y_t\}$.

**Theorem 19.** *For any $k \in \mathbb{Z}^+$, a $\left(t, \binom{t}{k}, r, k, 1\right)$-balanced incomplete block design forms the incidences of an edge-transitive $K_{\binom{t}{k},t}$ graph.*

*Proof.* The edge-transitivity of the graph follows from the fact that neighbors of the vertices on the left are the different subsets of $k$ vertices on the right. □

Theorem 19 can be further generalized by replacing each $y_i$ with multiple elements.

**Example 20.** Using the design from Example 17, we replace each $y_i$ with the elements $y_{i,1}$ and $y_{i,2}$. This creates the design

$$\begin{aligned}&\{\{y_{1,1}, y_{1,2}\}, \{y_{2,1}, y_{2,2}\}\}, && \{\{y_{1,1}, y_{1,2}\}, \{y_{3,1}, y_{3,2}\}\},\\ &\{\{y_{1,1}, y_{1,2}\}, \{y_{4,1}, y_{4,2}\}\}, && \{\{y_{2,1}, y_{2,2}\}, \{y_{3,1}, y_{3,2}\}\},\\ &\{\{y_{2,1}, y_{2,2}\}, \{y_{4,1}, y_{4,2}\}\}, && \{\{y_{3,1}, y_{3,2}\}, \{y_{4,1}, y_{4,2}\}\}.\end{aligned}$$

This will correspond to a $\left(2t, \binom{t}{k}, \frac{(t-1)!}{(k-1)!\,(t-k)!}, 2k, \lambda\right)$-partially balanced incomplete block design which forms the incidences of an edge-transitive $K_{\binom{t}{k},2t}$ graph.

In general we can replace each $y_i$ with the elements $y_{i,1}, y_{i,2}, \dots, y_{i,s}$ to form a larger class of edge-transitive graphs.

**Theorem 21.** *For integers $k \geq 1$ and $s \geq 0$ a $\left(st, \binom{t}{k}, \frac{(t-1)!}{(k-1)!\,(t-k)!}, sk, \lambda\right)$-partially balanced incomplete block design forms the incidences of an edge-transitive $K_{\binom{t}{s},kt}$ graph.*

In Example 18 we provided an example of a $(2,4)$-biregular subgraph of the complete bipartite graph $K_{10,5}$ that corresponded a $(5,10,4,2,1)$-block design. We can form a second $(2,4)$-biregular subgraph of the complete bipartite graph $K_{10,5}$ (nonisomorphic to the first) by starting with a different block design. Let the blocks of this design be

$$B_1=\{y_1,y_3\},\quad B_3=\{y_1,y_5\},\quad B_5=\{y_2,y_3\},\quad B_7=\{y_2,y_4\},\quad B_9=\{y_4,y_5\},$$
$$B_2=\{y_1,y_3\},\quad B_4=\{y_1,y_5\},\quad B_6=\{y_2,y_3\},\quad B_8=\{y_2,y_4\},\quad B_{10}=\{y_4,y_5\}.$$

This is a $(5,10,4,2,\lambda)$-block design whose structure represents the incidences of the Folkman graph.

For designs where there is an initial block and other blocks can be obtained by a linear transformation, it is straightforward to show that the resulting graph is edge-transitive. However if this is not the case, the graph may not be edge-transitive as shown below.

Consider a $(9,18,8,4,3)$-BIBD whose incidences form a biregular bipartite graph, but the resulting graph is not edge-transitive. Consider the design

$$\begin{array}{lll}
(0,1,2,4), & (6,7,8,1), & (3,6,7,1),\\
(1,2,3,5), & (7,8,0,2), & (4,7,8,2),\\
(2,3,4,6), & (8,0,1,3), & (5,8,0,3),\\
(3,4,5,7), & (0,3,4,7), & (6,0,1,4),\\
(4,5,6,8), & (1,4,5,8), & (7,1,2,5),\\
(5,6,7,0), & (2,5,6,0), & (8,2,3,6)
\end{array}$$

[Bose 1939]. This corresponds to a $(4,8)$-biregular subgraph of $K_{18,9}$ with the incidences

$$\begin{array}{lll}
x_1: y_0,y_1,y_2,y_4, & x_7: y_6,y_7,y_8,y_1, & x_{13}: y_3,y_6,y_7,y_1,\\
x_2: y_1,y_2,y_3,y_5, & x_8: y_7,y_8,y_0,y_2, & x_{14}: y_4,y_7,y_8,y_2,\\
x_3: y_2,y_3,y_4,y_6, & x_9: y_8,y_0,y_1,y_3, & x_{15}: y_5,y_8,y_0,y_3,\\
x_4: y_3,y_4,y_5,y_7, & x_{10}: y_0,y_3,y_4,y_7, & x_{16}: y_6,y_0,y_1,y_4,\\
x_5: y_4,y_5,y_6,y_8, & x_{11}: y_1,y_4,y_5,y_8, & x_{17}: y_7,y_1,y_2,y_5,\\
x_6: y_5,y_6,y_7,y_0, & x_{12}: y_2,y_5,y_6,y_0, & x_{18}: y_8,y_2,y_3,y_6,
\end{array}$$

However, the graph $G$ is not edge-transitive, as $G-x_1y_1$ is not isomorphic to $G-x_{18}y_6$. Verification of this fact is far from trivial. Using Mathematica we found that $G-x_1y_1$ has 172924 cycles of length 10 and $G-x_{18}y_6$ has 172926 cycles of length 10. Hence by Theorem 1, $G$ is not edge-transitive.

We also note that there can exist an edge-transitive $(r,k)$-biregular subgraph of the complete bipartite graph $K_{v,b}$ where the incidences are not a $(v,b,r,k,\lambda)$-BIBD design.

For example, consider the blocks

$$B_1=\{y_1,y_2,y_7,y_8\},\quad B_3=\{y_5,y_6,y_7,y_8\},\quad B_5=\{y_1,y_3,y_5,y_7\},$$
$$B_2=\{y_3,y_4,y_5,y_6\},\quad B_4=\{y_1,y_2,y_3,y_4\},\quad B_6=\{y_2,y_4,y_6,y_8\}.$$

This is not a design as the pair $\{y_1, y_5\}$ does not appear in any block. However the incidences give rise to an edge-transitive $(4, 3)$-biregular subgraph of the complete bipartite graph $K_{6,8}$. We used Mathematica to show that this graph is edge-transitive and is nonisomorphic to the graph in Example 17. Both graphs are noted in the online supplement.

## 3. Complements of Cartesian products

Recall that some previously known infinite families of vertex-transitive graphs are wreath graphs and Kneser graphs. We identify an additional infinite family of edge-transitive graphs that are vertex-transitive, stated in terms of Cartesian products.

**Theorem 22.** *The graph $\overline{K_m \times K_n}$ is edge-transitive.*

*Proof.* It may be helpful to refer to Figure 3. The graph $\overline{K_m \times K_n}$ is precisely the graph $\overline{L(K_{m,n})}$, that is, the complement of the line graph of $K_{m,n}$ [Weisstein and Wagon]. First, we observe the structure of $L(K_{m,n})$. Let the partite sets of $K_{m,n}$ be $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$. The graph $L(K_{m,n})$ consists of $m$ sets of $n$ vertices, which we denote by $V_1, V_2, \dots, V_m$. The $n$ vertices in each $V_i$ correspond to the edges incident to $a_i$ in the graph of $K_{m,n}$. Specifically, $V_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,n}\}$, where $v_{i,k}$ corresponds to the edge $a_i b_k$ in the graph $K_{m,n}$. By construction, all of the vertices in a given set $V_i$ are adjacent to each other, since these vertices correspond to all edges incident to $a_i$ in $K_{m,n}$. Additionally, each $v_{i,k}$ is adjacent to $v_{j,k}$ for all $j \neq i$, since these vertices correspond to all edges incident to $b_k$ in $K_{m,n}$. This completes the construction of $L(K_{m,n})$. To construct $\overline{L(K_{m,n})}$, we retain the vertex sets $V_1, \dots, V_m$. However, now we have an $m$-partite graph, since none of the edges in $V_i$ are connected to each other in $\overline{L(K_{m,n})}$. Each $v_{i,k}$ is connected to $v_{j,l}$ for all $j \neq i$ and all $l \neq k$. In other words, all possible edges of the $m$-partite graph exist except for edges of the form $v_{i,k} v_{j,k}$. It is clear from this description that $\overline{L(K_{m,n})}$ is edge-transitive. This follows from

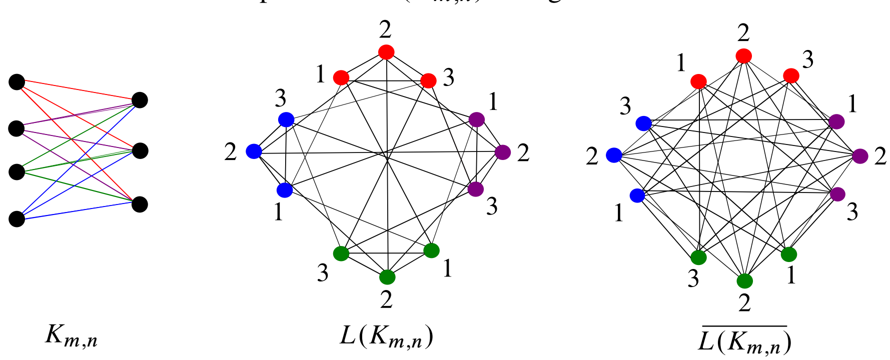


**Figure 3.** An example of the construction in the proof of Theorem 22 for $m = 4,\ n = 3$.

the fact every vertex in a given partite set is indistinguishable from every other vertex in that set, and the fact that each partite set is indistinguishable from every other partite set. Hence $\overline{K_m \times K_n} = \overline{L(K_{m,n})}$. □

## 4. Conclusion

In Section 2.2 we explored $(r, 2)$-bipartite subgraphs of $K_{m,n}$. More results of this type can be obtained by determining all arc-transitive graphs of order larger than 9. It would be an interesting but challenging problem to explore the family of $(r, k)$ and determine which graphs are edge-transitive and determine the number of nonisomorphic graphs of this form.

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