involve
a journal of mathematics

Unoriented links and the Jones polynomial

Sandy Ganzell, Janet Huffman, Leslie Mavrakis,
Kaitlin Tademy and Griffin Walker

msp

2019 vol. 12, no. 8

INVOLVE 12:8 (2019)
dx.doi.org/10.2140/involve.2019.12.1357

Unoriented links and the Jones polynomial

Sandy Ganzell, Janet Huffman, Leslie Mavrakis,
Kaitlin Tademy and Griffin Walker

(Communicated by Joel Foisy)

The Jones polynomial is an invariant of oriented links with $n \geq 1$ components. When $n = 1$, the choice of orientation does not affect the polynomial, but for $n > 1$, changing orientations of some (but not all) components can change the polynomial. Here we define a version of the Jones polynomial that is an invariant of *unoriented* links; i.e., changing orientation of any sublink does not affect the polynomial. This invariant shares some, but not all, of the properties of the Jones polynomial.

The construction of this invariant also reveals new information about the original Jones polynomial. Specifically, we show that the Jones polynomial of a knot is never the product of a nontrivial monomial with another Jones polynomial.

1. Introduction

Jones' original construction [1985] of the polynomial $V_L = V_L(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ was through the skein relation

$$t^{-1}V_{L_-} - tV_{L_+} = (t^{1/2} - t^{-1/2})V_{L_0},$$

where L_+, L_- and L_0 are three oriented links that are identical except inside a ball that contains respectively, a positive crossing, a negative crossing, and two uncrossed strands. It is easy to see that when L is a knot (i.e., a link of one component), the polynomial $V_L(t)$ is unchanged by reversing the orientation on L, since crossing signs are preserved by such a change in orientation.

For links of more than one component, however, the Jones polynomial may change depending on the choice of orientation for each component. The Hopf link is the simplest example. The oriented Hopf link with linking number $+1$ has Jones polynomial $-t^{1/2} - t^{5/2}$, but reversing the orientation of one component gives us $-t^{-5/2} - t^{-1/2}$. A complete list of oriented links up to nine crossings, together with their polynomials can be found in [Doll and Hoste 1991].

MSC2010: 57M25, 57M27.
Keywords: Jones polynomial, unoriented link.
Supported by NSF grant DMS-1560301.

Based on the skein-relation definition, it is a surprising result that a change in orientation of some components of L simply multiplies V_L by a power of t. Let $L = M \cup N$ be an oriented link with components $M_1, \ldots, M_r$, $N_1, \ldots, N_s$, and write $L_N = M \cup -N$ for the link formed by reversing the orientations on $N_1, \ldots, N_s$. Morton [1986] proved that

$$V_L(t) = t^{3\lambda} V_{L_N}(t),$$

where λ is the linking number of M with N, defined as

$$\lambda = \mathrm{lk}(M, N) = \sum_{i,j} \mathrm{lk}(M_i, N_j).$$

A much simpler proof using Kauffman's bracket polynomial construction of the Jones polynomial appears below.

Recall [Kauffman 1988] the bracket polynomial $\langle L \rangle \in \mathbb{Z}[A, A^{-1}]$ is defined recursively:

$$\left\langle \times \right\rangle = A\left\langle \asymp \right\rangle + A^{-1}\left\langle \right) \left(\right\rangle,$$
$$\langle \bigcirc L \rangle = (-A^2 - A^{-2})\langle L \rangle,$$
$$\langle \bigcirc \rangle = 1.$$

The bracket polynomial is invariant under Reidemeister moves R2 and R3, but not under move R1. Define $X_L(A) = (-A^3)^{-w}\langle L \rangle$, where $w = w(L)$ is the writhe (sum of all crossing signs) of L, to obtain a link invariant. Under the change of variables $A = t^{-1/4}$, we have $X_L(A) = V_L(t)$. We will often write $d = -A^2 - A^{-2}$; thus $\langle \bigcirc L \rangle = d\langle L \rangle$.

Now it is clear that changing the orientations of some components of L multiplies the Jones polynomial by a power of t, since only the writhe (but not the bracket polynomial) is affected by such a change. Using the notation above, if $L = M \cup N$, then the crossing signs that change to produce L_N are the ones that involve some crossing of component M_i with component N_j. Since the linking number of M with N involves precisely the same crossings, we have

$$\begin{aligned} w(L_N) &= w(L) - 2\sum(\text{crossing signs of } M_i \text{ with } N_j) \\ &= w(L) - 4 \cdot \mathrm{lk}(M, N). \end{aligned}$$

Thus

$$\begin{aligned} V_L(t) = X_L(A) &= (-A^3)^{-w(L)}\langle L \rangle \\ &= (-A^3)^{-4\cdot \mathrm{lk}(M,N) - w(L_N)}\langle L \rangle \\ &= (-A)^{-12\cdot \mathrm{lk}(M,N)}(-A^3)^{-w(L_N)}\langle L \rangle \\ &= (A^4)^{-3\lambda} X_{L_N}(A) = t^{3\lambda} V_{L_N}(t), \end{aligned}$$

confirming Morton's result.

Given an unoriented link of n components, there may be up to 2^{n-1} associated Jones polynomials for the links obtained by choosing an orientation for each component. (Note: it is not up to 2^n, since changing *all* orientations does not affect the Jones polynomial.) None of these is a natural choice to be the Jones polynomial of the unoriented link since there is no preferred orientation. In the next section we define a version of the Jones polynomial that is an invariant of unoriented links.

2. The Jones polynomial for unoriented links

We begin by defining the *self-writhe* of a link diagram.

Definition 1. For a link diagram L with components $K_1, \ldots, K_n$, we define the self-writhe of L, denoted by $\psi(L)$, to be the sum of the writhes of each component of L, ignoring the other components when computing each writhe. That is,

$$\psi(L) = \sum_{j=1}^{n} w(K_j).$$

Equivalently, the self-writhe can be defined as the sum of the signs of those crossings of L for which both the under and over strands are from the same component.

Reidemeister moves affect the self-writhe exactly as they do the writhe. Both are invariant under moves R2 and R3. This is because the two crossings involved in move R2 are of opposite sign regardless of orientation, and the crossing signs ε_i are unchanged by R3 moves regardless of orientations and components. See Figure 1. Under move R1, both the writhe and self-writhe change by ± 1, since move R1 always involves a single component of the link. See Figure 2.

Unlike the writhe, however, the self-writhe of a link L is independent of the choice of orientations of the components of L. This is because changing the orientation of a component K of L does not affect the writhe of K, and hence does not affect $\psi(L)$.

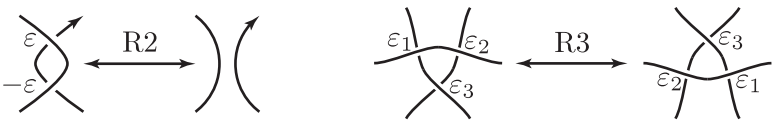


Figure 1. Crossing signs and Reidemeister moves.

Figure 2. Crossing signs and move R1.

Thus we can define $U_L(A) = (-A^3)^{-\psi}\langle L\rangle$. This modified Jones polynomial is an invariant for the same reason that $X_L(A)$ is: both $\langle L\rangle$ and $\psi(L)$ are invariant under moves R2 and R3, and $\langle L\rangle$ changes by a factor of $(-A^3)^{\pm 1}$ with each R1 move.

But since $\psi(L)$ is unaffected by changing orientations of any components of L, the polynomial $U_L(A)$ is also unaffected by such changes. We can thus make the same change of variables $A = t^{-1/4}$ to obtain $W_L(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$, noting that $W_L(t) = U_L(A)$.

Definition 2. Let L be an unoriented link with self-writhe ψ. The Laurent polynomial $U_L(A) = (-A^3)^{-\psi}\langle L\rangle$ (or equivalently $W_L(t)$) for any choice of orientation of components of L is the unoriented Jones polynomial of L. We will refer to $U_L(A)$ as the U-polynomial of L.

3. Properties of the unoriented Jones polynomial

For knots we have $W_K(t) = V_K(t)$ since $w(K) = \psi(K)$. Thus we will examine the properties of the unoriented Jones polynomial for links of at least two components. Jones [1985] established that if the link L has an odd number of components, then $V_L(t)$ is a Laurent polynomial over the integers; if the number of components of L is even then $V_L(t)$ is $\sqrt{t}$ times a Laurent polynomial. $W_L(t)$ does not share these properties. For example, if L is the Hopf link, then $W_L(t) = -t^{-1} - t$, since $\langle \bigcirc\!\bigcirc \rangle = -A^4 - A^{-4}$, and $\psi(\bigcirc\!\bigcirc) = 0$.

On the other hand, if L is link 5_1^2 (Figure 3), then $\psi(L) = w(L) = -1$, regardless of orientation. Therefore,

$$W_L(t) = V_L(t) = t^{-7/2} - 2t^{-5/2} + t^{-3/2} - 2t^{-1/2} + t^{1/2} - t^{3/2}.$$

There are two different oriented links corresponding to 7_1^3 (Figure 3), both of which have integral exponents for the original Jones polynomials, but the unoriented Jones polynomial is

$$t^{-5/2} - t^{-3/2} + 4t^{-1/2} - 3t^{1/2} + 4t^{3/2} - 3t^{5/2} + 3t^{7/2} - t^{9/2}.$$

For the remainder of this paper we use A as the indeterminate. This is simply to avoid fractional exponents.

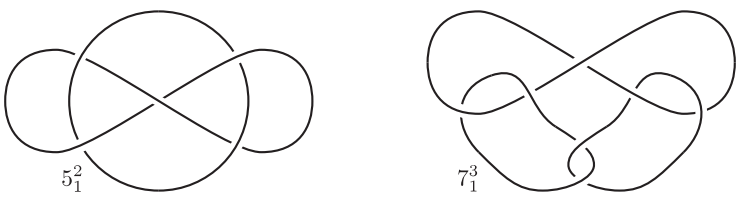


Figure 3. Links 5_1^2 and 7_1^3.

Figure 4. *L#M.*

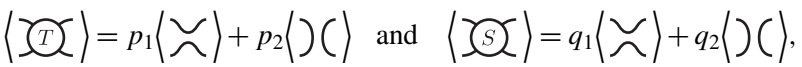
Some properties of the Jones polynomial do carry over to $U_L(A)$. Let L^* denote the mirror image of L.

Proposition 3. $$U_{L^*}(A) = U_L(A^{-1}).$$

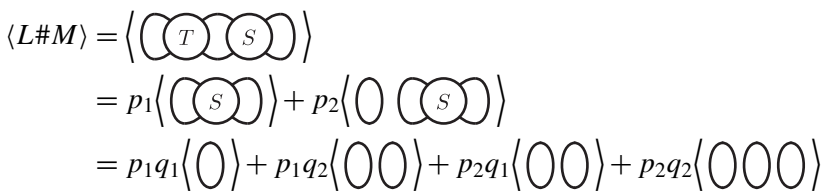
Proof. This follows immediately from Definition 2, since $\psi(L) = -\psi(L^*)$. □

Theorem 4. *If L and M are links, then $U_{L\#M} = U_L U_M$.*

Proof. Observe that for diagrams L and M, the self-writhe of $L\#M$ is just $\psi(L) + \psi(M)$. Now take diagrams for L, M and $L\#M$ as in Figure 4. Let



where p_1, p_2, q_1 and q_2 are polynomials in A. Then we have $\langle L\rangle = p_1 + p_2 d$, and $\langle M\rangle = q_1 + q_2 d$. Moreover,



$$= p_1q_1 + p_1q_2d + p_2q_1d + p_2q_2d^2.$$

Thus,

$$\begin{aligned} U_{L\#M}(A) &= (-A^3)^{-\psi(L\#M)}(p_1q_1 + p_1q_2d + p_2q_1d + p_2q_2d^2) \\ &= (-A^3)^{-\psi(L)}(-A^3)^{-\psi(M)}(p_1 + p_2d)(q_1 + q_2d) \\ &= U_L(A)U_M(A). \end{aligned}$$ □

When two links have the same number of components, their U-polynomials are related algebraically. Specifically, if L and L' are both n-component links, then $U(L) - U(L')$ is divisible by a certain fixed polynomial $C(A)$, independent of L, L' and n. Equivalently, we may say $U(L)$ and $U(L')$ are equal in the quotient ring $\mathbb{Z}[A, A^{-1}]/\langle C(A)\rangle$. For convenience, we will write $U(L) \equiv U(L') \pmod{C(A)}$.

Theorem 5. *Let L and L' be two links with the same number of components. Then $U_L(A) \equiv U_{L'}(A) \pmod{A^6 - 1}$.*

Proof. Suppose L and L' are two links that differ by a crossing change. We will show that $U_L(A) - U_{L'}(A)$ is divisible by $A^6 - 1$. Since any link can be transformed

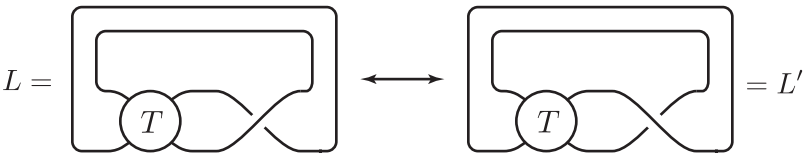


Figure 5. Two links that differ by a crossing change.

by crossing changes to any other link with the same number of components, the theorem follows.

Draw L and L' as the numerator closures of tangles that differ by a crossing as in Figure 5. Take the self-writhes of L and L' to be ψ and ψ' respectively. Therefore ψ' will equal ψ, $\psi+2$, or $\psi-2$, depending on the orientation of the strands in the crossing change, and whether they are from the same component. We compute $U_L(A) - U_{L'}(A)$. Write

$$\left\langle \boxed{T} \right\rangle = p_1 \left\langle \times \right\rangle + p_2 \left\langle) (\right\rangle,$$

where p_1 and p_2 are polynomials in A. Then

$$\begin{aligned}
\left\langle \boxed{T} \right\rangle &= A \left\langle \boxed{T} (\right\rangle + A^{-1} \left\langle \boxed{T} \right\rangle \\
&= Ap_1 \left\langle) (\right\rangle + Ap_2 d \left\langle) (\right\rangle + A^{-1} p_1 \left\langle \times \right\rangle + A^{-1} p_2 \left\langle) (\right\rangle, \\
\langle L \rangle &= Ap_1 + Ap_2 d + A^{-1} p_1 d + A^{-1} p_2 \\
&= p_1(A + A^{-1} d) + p_2(Ad + A^{-1}) \\
&= p_1(-A^{-3}) + p_2(-A^3), \\
U_L(A) &= (-A^3)^{-\psi} [p_1(-A^{-3}) + p_2(-A^3)].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle L' \rangle &= p_1(-A^3) + p_2(-A^{-3}), \\
U_{L'}(A) &= (-A^3)^{-\psi'} [p_1(-A^3) + p_2(-A^{-3})].
\end{aligned}$$

Since $\psi' \in \{\psi, \psi + 2, \psi - 2\}$, either

$$\begin{aligned}
U_L(A) - U_{L'}(A) &= (-A^3)^{-\psi} [p_1(-A^{-3} + A^3) - p_2(A^3 - A^{-3})] \\
&= (-1)^{-\psi} (A^3)^{-\psi-1} (p_1 - p_2)(A^6 - 1),
\end{aligned}$$

or

$$\begin{aligned}
U_L(A) - U_{L'}(A) &= (-A^3)^{-\psi} p_2(-A^3 + A^{-9}) \\
&= (-A^3)^{-\psi-3} p_2(A^6 + 1)(A^6 - 1),
\end{aligned}$$

or

$$U_L(A) - U_{L'}(A) = (-A^3)^{-\psi} p_1(-A^{-3} + A^9)$$
$$= (-A^3)^{-\psi-1} p_1(A^6 + 1)(A^6 - 1). \qquad \square$$

Corollary 6. *Let L be a link with n components. Then $U_L(1) = (-2)^{n-1}$.*

Proof. Let $\bigcirc^n$ be the unlink of n components. Then

$$U_{\bigcirc^n}(A) = d^{n-1} = (-A^2 - A^{-2})^{n-1}.$$

Therefore by Theorem 5, we can write $U_L(A) = (A^6 - 1)q(A) + (-A^2 - A^{-2})^{n-1}$, where q is some polynomial in A. Thus $U_L(1) = (-2)^{n-1}$. $\square$

Theorem 5 establishes that $A^6 - 1$ divides the difference of any two U-polynomials of links with the same number of components. However, $A^6 - 1$ does not appear to be the highest-degree such polynomial. In all examples known to the authors, the difference is a multiple of $A^8 - A^6 - A^2 + 1$, which equals $(A^6 - 1)(A^2 - 1)$. We conjecture this is always the case.

Conjecture 7. Let L and L' be two links with the same number of components. Then $U_L(A) \equiv U_{L'}(A) \pmod{A^8 - A^6 - A^2 + 1}$.

We prove Conjecture 7 for links of three or fewer components.

Theorem 8. *Let L and L' be two n-component links, where $n \leq 3$. Then $U_L(A) \equiv U_{L'}(A) \pmod{A^8 - A^6 - A^2 + 1}$.*

Proof. It is shown in [Ganzell 2014] that when L is a knot (i.e., $n = 1$), then $X_L(A) - X_{L'}(A)$ (and hence $U_L(A) - U_{L'}(A)$) is always divisible by $A^{16} - A^{12} - A^4 + 1$, which equals $(A^8 - A^6 - A^2 + 1)(A^8 + A^6 + A^2 + 1)$.

For $n = 2$, we proceed as follows. It is proved in [Murakami and Nakanishi 1989] that the link L can be transformed into the link L' by Δ-moves (Figure 6) if and only if L and L' have the same number of components and the pairwise linking numbers of the components of L equal those of L'. That is, if $L = K_1 \cup \cdots \cup K_n$ and $L' = K'_1 \cup \cdots \cup K'_{n'}$, then L can be transformed into L' by Δ-moves if and only if $n = n'$ and $\mathrm{lk}(K_i, K_j) = \mathrm{lk}(K'_i, K'_j)$ for $1 \leq i < j \leq n$. In this case we say L and L' are Δ-move equivalent. Thus every 2-component link is Δ-move equivalent to a link of the form in Figure 7, where $k \in \mathbb{Z}$ is the linking number.

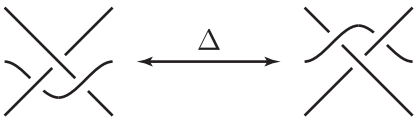


Figure 6. Δ-move.

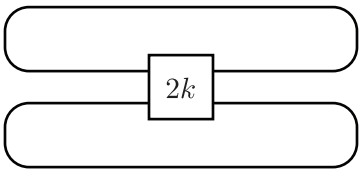


Figure 7. L_{2k}, a 2-component link with linking number k.

It is shown in [Ganzell 2014] that two links that differ by a sequence of Δ-moves have bracket polynomials that are congruent mod $A^8 - A^6 - A^2 + 1$ (in fact mod $A^{16} - A^{12} - A^4 + 1$). Since Δ-moves do not affect the self-writhe, the U-polynomials are also congruent mod $A^8 - A^6 - A^2 + 1$. Now, let L_{2k} be the link in Figure 7. We will show that $\langle L_{2k}\rangle - \langle \bigcirc\bigcirc\rangle$ is also a multiple of $A^8 - A^6 - A^2 + 1$. Thus every 2-component link has bracket polynomial congruent to $\langle \bigcirc\bigcirc\rangle \pmod{A^8 - A^6 - A^2 + 1}$. Since L_{2k} has self-writhe equal to 0, this will complete the proof.

We first compute $\langle L_{2k}\rangle$. We have

$$\left\langle \; \right\rangle = p_1 \left\langle \; \right\rangle + p_2 \left\langle \; \right\rangle,$$

where $p_1 = A^{2k}$ and $p_2 = \sum_{m=1}^{2k} \binom{2k}{m} A^{2k-2m} d^{m-1}$. Now observe that

$$\begin{aligned}\sum_{m=0}^{2k} \binom{2k}{m} A^{2k-2m} d^m &= \sum_{m=0}^{2k} \binom{2k}{m} A^{2k-2m} (-A^2 - A^{-2})^m \\ &= \sum_{m=0}^{2k} \binom{2k}{m} A^{2k-m} (-A - A^{-3})^m = [(-A - A^{-3}) + A]^{2k}\end{aligned}$$

by the binomial theorem. The last expression simplifies to A^{-6k}. Therefore $p_2 = (A^{-6k} - A^{2k})/d$, and

$$\begin{aligned}\langle L_{2k}\rangle &= A^{2k} d + \frac{A^{-6k} - A^{2k}}{d} \\ &= \frac{A^{2k}(A^4 + 2 + A^{-4}) + A^{-6k} - A^{2k}}{-A^2 - A^{-2}} \\ &= \frac{-A^{2k+6} - A^{2k+2} - A^{2k-2} - A^{-6k+2}}{A^4 + 1}.\end{aligned}$$

Thus,

$$\begin{aligned}\langle L_{2k}\rangle - \langle \bigcirc\bigcirc\rangle &= \frac{-A^{2k+6} - A^{2k+2} - A^{2k-2} - A^{-6k+2}}{A^4 + 1} + A^2 + A^{-2} \\ &= \frac{A^{8k+4} + A^{8k} + A^{8k-4} - A^{6k+4} - 2A^{6k} - A^{6k-4} + 1}{-A^{6k-2}(A^4 + 1)}. \qquad (1)\end{aligned}$$

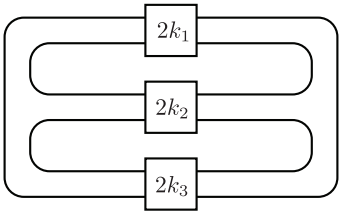


Figure 8. A 3-component link with linking numbers k_1, k_2, k_3.

Let $N(A)$ be the numerator of (1). Since

$$A^8 - A^6 - A^2 + 1 = (A+1)^2(A-1)^2(A^2+A+1)(A^2-A+1),$$

we must show that $N(A)$ has these factors. (Actually, we only need to prove that 1 and -1 are double roots, since we have already established Theorem 5. But it is not hard to show directly.) Rewrite $N(A)$ in the form

$$N(A) = (A^{8k+4} + A^{8k} + A^{8k-4}) - (A^{6k+4} + A^{6k} + A^{6k-4}) - (A^{6k} - 1).$$

Observe that

$$A^{8k+4} + A^{8k} + A^{8k-4} = A^{8k-4}(A^4 - A^2 + 1)(A^2 + A + 1)(A^2 - A + 1),$$
$$A^{6k+4} + A^{6k} + A^{6k-4} = A^{6k-4}(A^4 - A^2 + 1)(A^2 + A + 1)(A^2 - A + 1),$$

and

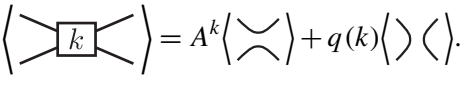
$$A^{6k} - 1 = (A^2 + A + 1)(A^2 - A + 1) \sum_{m=0}^{6k-6} (A^{m+2} - A^m).$$

It remains to show that $(A+1)^2$ and $(A-1)^2$ are factors of $N(A)$. It is straightforward to verify that 1 and -1 are both roots of $N(A)$ and of the derivative $N'(A)$, completing the proof for 2-component links.[1]

The proof for $n = 3$ is similar. Observe that every 3-component link is Δ-move equivalent to a link of the form in Figure 8. Define

$$q(k) = \sum_{m=1}^{k} \binom{k}{m} A^{k-2m} d^{m-1},$$

so that



[1]Note that $N(A)$ must also be divisible by $A^4 + 1$, since bracket polynomials are Laurent polynomials. We can see this directly by writing $N(A) = (A^{8k} + A^{8k-4}) - (A^{6k+4} + A^{6k}) - (A^{6k} + A^{6k-4}) + (A^{8k+4} + 1)$. The first three binomials are multiples of $A^4 + 1$, and $A^{8k+4} + 1 = (A^4 + 1)(A^{8k} - A^{8k-4} + A^{8k-8} - \cdots + 1)$.

Then if L is the link in Figure 8, we have

$$\begin{aligned}\langle L\rangle = {} & A^{2k_1+2k_2+2k_3}d^2 \\ & + A^{2k_1+2k_2}q(2k_3)d + A^{2k_1+2k_3}q(2k_2)d + A^{2k_2+2k_3}q(2k_1)d \\ & + A^{2k_1}q(2k_2)q(2k_3) + A^{2k_2}q(2k_1)q(2k_3) + A^{2k_3}q(2k_1)q(2k_2) \\ & + q(2k_1)q(2k_2)q(2k_3)d,\end{aligned}$$

and we must verify that $\langle L\rangle - \langle \bigcirc\bigcirc\bigcirc\rangle$ is divisible by $A^8 - A^6 - A^2 + 1$. The proof is tedious but elementary, and follows the same outline as for 2-component links. □

Corollary 9. *For n-component links L, L' with $n \leq 3$, the U-polynomial of L can never be a nontrivial monomial times the U-polynomial of L'. That is, if $U_{L'}(A) = rA^kU_L(A)$, then $r = 1$ and $k = 0$.*

Proof. Let $p(A) = A^8 - A^6 - A^2 + 1$, so that $U_L(A) - U_{L'}(A) = p(A)g(A)$ for some Laurent polynomial g. Now suppose $U_{L'}(A) = rA^kU_L(A)$. Then

$$U_L(A) - rA^kU_L(A) = p(A)g(A). \tag{2}$$

Setting $A = 1$, we obtain

$$(-2)^{n-1} - r(-2)^{n-1} = 0$$

from Corollary 6. Thus $r = 1$.

Differentiating (2) with respect to A and setting $r = 1$, we obtain

$$(1 - A^k)U_L'(A) - kA^{k-1}U_L(A) = p'(A)g(A) + p(A)g'(A).$$

Again, setting $A = 1$ produces

$$kU_L(A) = 0.$$

Thus $k = 0$. □

Corollary 9 does not hold for the original Jones polynomial. Example 10 below, shows a pair of 2-component links whose Jones polynomials do not satisfy the conclusion of the corollary. However, since the U-polynomial for a *knot* is identical to the original Jones polynomial, Corollary 9 does apply. Hence, the Jones polynomial of a knot cannot be the product of a nontrivial monomial with another Jones polynomial.

Example 10. In [Eliahou et al. 2003], examples are given of n-component links (for $n \geq 2$) that have the same Jones polynomial as $\bigcirc^n$. The link in Figure 9 (left) is the first of an infinite family of such links. Those examples all have $w = \psi = 0$, and therefore satisfy $U_L = U_{\bigcirc^n}$. Other examples are given in that paper of links

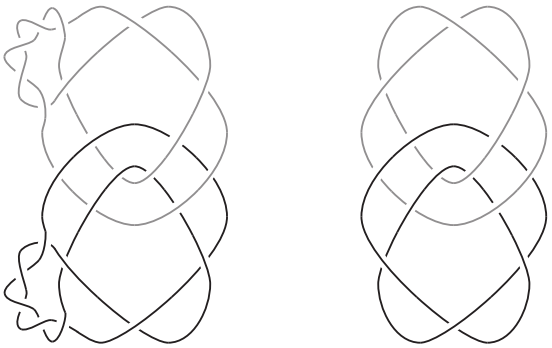


Figure 9. Links with $U_L = U_{\bigcirc^n}$.

whose Jones polynomial has the form $t^k d^{n-1}$, as in Figure 9 (right). These links have $\psi = 0$, and as a result, $U(A) = d^{n-1}$.

References

[Doll and Hoste 1991] H. Doll and J. Hoste, "A tabulation of oriented links", *Math. Comp.* **57**:196 (1991), 747–761. MR Zbl

[Eliahou et al. 2003] S. Eliahou, L. H. Kauffman, and M. B. Thistlethwaite, "Infinite families of links with trivial Jones polynomial", *Topology* **42**:1 (2003), 155–169. MR Zbl

[Ganzell 2014] S. Ganzell, "Local moves and restrictions on the Jones polynomial", *J. Knot Theory Ramifications* **23**:2 (2014), art. id. 1450011. MR Zbl

[Jones 1985] V. F. R. Jones, "A polynomial invariant for knots via von Neumann algebras", *Bull. Amer. Math. Soc.* (*N.S.*) **12**:1 (1985), 103–111. MR Zbl

[Kauffman 1988] L. H. Kauffman, "New invariants in the theory of knots", *Amer. Math. Monthly* **95**:3 (1988), 195–242. MR Zbl

[Morton 1986] H. R. Morton, "The Jones polynomial for unoriented links", *Quart. J. Math. Oxford Ser.* (2) **37**:145 (1986), 55–60. MR Zbl

[Murakami and Nakanishi 1989] H. Murakami and Y. Nakanishi, "On a certain move generating link-homology", *Math. Ann.* **284**:1 (1989), 75–89. MR Zbl

Received: 2019-04-10 Revised: 2019-06-19 Accepted: 2019-07-06

sganzell@smcm.edu *Department of Mathematics and Computer Science, St. Mary's College of Maryland, St. Mary's City, MD, United States*

janet.huffman@uky.edu *University of Kentucky, Lexington, KY, United States*

lmavrakis@math.ucsb.edu *University of California, Santa Barbara, CA, United States*

kaitlin.tademy@huskers.unl.edu *University of Nebraska, Lincoln, NE, United States*

griffin.ea.walker@gmail.com *Wheaton College, Wheaton, IL, United States*

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

BOARD OF EDITORS

Colin Adams	Williams College, USA	Robert B. Lund	Clemson University, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Gaven J. Martin	Massey University, New Zealand
Martin Bohner	Missouri U of Science and Technology, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of N Carolina, Chapel Hill, USA	Frank Morgan	Williams College, USA
Pietro Cerone	La Trobe University, Australia	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Scott Chapman	Sam Houston State University, USA	Zuhair Nashed	University of Central Florida, USA
Joshua N. Cooper	University of South Carolina, USA	Ken Ono	Univ. of Virginia, Charlottesville
Jem N. Corcoran	University of Colorado, USA	Yuval Peres	Microsoft Research, USA
Toka Diagana	Howard University, USA	Y.-F. S. Pétermann	Université de Genève, Switzerland
Michael Dorff	Brigham Young University, USA	Jonathon Peterson	Purdue University, USA
Sever S. Dragomir	Victoria University, Australia	Robert J. Plemmons	Wake Forest University, USA
Joel Foisy	SUNY Potsdam, USA	Carl B. Pomerance	Dartmouth College, USA
Errin W. Fulp	Wake Forest University, USA	Vadim Ponomarenko	San Diego State University, USA
Joseph Gallian	University of Minnesota Duluth, USA	Bjorn Poonen	UC Berkeley, USA
Stephan R. Garcia	Pomona College, USA	Józeph H. Przytycki	George Washington University, USA
Anant Godbole	East Tennessee State University, USA	Richard Rebarber	University of Nebraska, USA
Ron Gould	Emory University, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Javier Rojo	Oregon State University, USA
Jim Haglund	University of Pennsylvania, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Johnny Henderson	Baylor University, USA	Hari Mohan Srivastava	University of Victoria, Canada
Glenn H. Hurlbert	Virginia Commonwealth University, USA	Andrew J. Sterge	Honorary Editor
Charles R. Johnson	College of William and Mary, USA	Ann Trenk	Wellesley College, USA
K. B. Kulasekera	Clemson University, USA	Ravi Vakil	Stanford University, USA
Gerry Ladas	University of Rhode Island, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
David Larson	Texas A&M University, USA	John C. Wierman	Johns Hopkins University, USA
Suzanne Lenhart	University of Tennessee, USA	Michael E. Zieve	University of Michigan, USA
Chi-Kwong Li	College of William and Mary, USA		

PRODUCTION

Silvio Levy, Scientific Editor

Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2019 is US $195/year for the electronic version, and $260/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2019 Mathematical Sciences Publishers

involve

2019 vol. 12 no. 8

On the zero-sum group-magicness of cartesian products 1261
ADAM FONG, JOHN GEORGES, DAVID MAURO, DYLAN SPAGNUOLO, JOHN WALLACE, SHUFAN WANG AND KIRSTI WASH

The variable exponent Bernoulli differential equation 1279
KAREN R. RÍOS-SOTO, CARLOS E. SEDA-DAMIANI AND ALEJANDRO VÉLEZ-SANTIAGO

The supersingularity of Hurwitz curves 1293
ERIN DAWSON, HENRY FRAUENHOFF, MICHAEL LYNCH, AMETHYST PRICE, SEAMUS SOMERSTEP, ERIC WORK, DEAN BISOGNO AND RACHEL PRIES

Multicast triangular semilattice network 1307
ANGELINA GROSSO, FELICE MANGANIELLO, SHIWANI VARAL AND EMILY ZHU

Edge-transitive graphs and combinatorial designs 1329
HEATHER A. NEWMAN, HECTOR MIRANDA, ADAM GREGORY AND DARREN A. NARAYAN

A logistic two-sex model with mate-finding Allee effect 1343
ELIZABETH ANDERSON, DANIEL MAXIN, JARED OTT AND GWYNETH TERRETT

Unoriented links and the Jones polynomial 1357
SANDY GANZELL, JANET HUFFMAN, LESLIE MAVRAKIS, KAITLIN TADEMY AND GRIFFIN WALKER

Nonsplit module extensions over the one-sided inverse of $k[x]$ 1369
ZHEPING LU, LINHONG WANG AND XINGTING WANG

Split Grothendieck rings of rooted trees and skew shapes via monoid representations 1379
DAVID BEERS AND MATT SZCZESNY

On the classification of Specht modules with one-dimensional summands 1399
AUBREY PIPER COLLINS AND CRAIG J. DODGE

The monochromatic column problem with a prime number of colors 1415
LORAN CROWELL AND STEVE SZABO

Total Roman domination edge-critical graphs 1423
CHLOE LAMPMAN, KIEKA (C. M.) MYNHARDT AND SHANNON OGDEN

1944-4176(2019)12:8;1-Y