

# involve

a journal of mathematics

Nonsplit module extensions over  
the one-sided inverse of  $k[x]$

Zheping Lu, Linhong Wang and Xingting Wang





# Nonsplit module extensions over the one-sided inverse of $k[x]$

Zheping Lu, Linhong Wang and Xingting Wang

(Communicated by Scott T. Chapman)

Let  $R$  be the associative  $k$ -algebra generated by two elements  $x$  and  $y$  with defining relation  $yx = 1$ . A complete description of simple modules over  $R$  is obtained by using the results of Irving and Gerritzen. We examine the short exact sequence  $0 \rightarrow U \rightarrow E \rightarrow V \rightarrow 0$ , where  $U$  and  $V$  are simple  $R$ -modules. It shows that nonsplit extension only occurs when both  $U$  and  $V$  are one-dimensional, or, under certain condition,  $U$  is infinite-dimensional and  $V$  is one-dimensional.

## 1. Introduction

In this short note, we study nonsplit extensions of simple modules over the associative algebra  $R = k\{x, y\}/\langle yx - 1 \rangle$  over a base field  $k$  of characteristic 0. The algebra  $R$  is also known as the one-sided inverse of the polynomial algebra  $k[x]$  and appeared in [Bavula 2010; Gerritzen 2000; Jacobson 1950; Irving 1979]. Note that

$$y(1 - xy) = (1 - xy)x = 0.$$

The algebra  $R$  is not a domain, and  $Z(R) = k$ . As a  $k$ -vector space  $R$  has basis

$$\{x^i y^j \mid i, j = 0, 1, 2, \dots\}.$$

Moreover,  $R$  admits the involution  $\eta : x \mapsto y$  and  $y \mapsto x$ . Hence, the left and right algebraic properties of  $R$  are the same.

Jacobson [1950] gave a faithful irreducible representation of  $R$  as follows. Let  $S$  be the infinite-dimensional  $k$ -vector space with the basis  $\{e_1, e_2, \dots\}$  and let  $R$  act on  $S$  by assigning

$$\begin{aligned} x e_n &= e_{n+1}, & n > 0, \\ y e_n &= e_{n-1}, & n > 1, \\ y e_1 &= 0. \end{aligned}$$

It was proved by Bavula [2010] and Gerritzen [2000] that there is only one isomorphic class of infinite-dimensional simple  $R$ -modules. Note that there is an

---

*MSC2010:* primary 16D60; secondary 16G99.

*Keywords:* simple modules, representations, module extensions.

algebra monomorphism  $R \rightarrow \text{End}_k(k[x])$  such that  $x \mapsto x$  and  $y \mapsto H^{-1} \frac{d}{dx}$ , where  $H \in \text{End}_k(k[x])$  is given by  $H(f) = \frac{d}{dx}(xf)$  for any  $f \in k[x]$ . In particular,

$$\bigoplus_{i \geq 0} kx^i(1 - xy) \cong k[x]$$

is a simple and faithful left  $R$ -module, where the left  $R$ -module structure on  $k[x]$  is via the algebra map  $R \rightarrow \text{End}_k(k[x])$  discussed above. Following [Bavula 2010],  $R$  contains a subring which is canonically isomorphic to the ring (without identity) of infinite-dimensional matrices. Let

$$F = \bigoplus_{i, j \geq 0} kM_{ij} \cong M_\infty(k),$$

where  $M_{ij} = x^i(1 - xy)y^j$  can be identical to the matrix units of  $M_\infty(k)$ . In particular, we have

$$x \sim \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{pmatrix}, \quad y \sim \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix}. \quad (1)$$

As a left  $R$ -module,

$$F = \bigoplus_{i, j \geq 0} kx^i(1 - xy)y^j \cong \bigoplus_{i \geq 0} \left( \bigoplus_{t \geq 0} kx^t x^i(1 - xy)y^i \right) \cong \bigoplus_{i \geq 0} k[x]$$

is a direct sum of infinitely many simple  $R$ -modules. Hence  $R$  is neither left nor right noetherian. Similarly, we see that there is an ascending chain of left annihilators in  $R$  which is not stable. Then  $R$  is neither left nor right Goldie. Moreover,  $F$  is equal to the ideal of  $R$  generated by  $\langle 1 - xy \rangle$ . Since  $F^2 = F$ ,  $\text{lann}(F)$  and  $\text{rann}(F)$  are both zero, we have  $F$  is an essential left and right ideal of  $R$ , which equals the socle of left and right  $R$ -module  $R$ . Hence  $F$  is contained in any nonzero ideal of  $R$  and it follows that the set of proper (two-sided) ideals of  $R$  is

$$\{0, \langle 1 - xy \rangle, \langle 1 - xy, f(x) \rangle\},$$

where  $f(x)$  is a monic polynomial in  $k[x]$  which is not a monomial. In particular, the ideals of  $R$  satisfy the ascending chain condition.

It follows from [Bavula 2010; Gerritzen 2000; Irving 1979] that the prime ideals are given by

$$\text{Spec}(R) = \{0, \langle 1 - xy \rangle, \langle 1 - xy, f(x) \rangle\},$$

where  $f(x)$  is a monic irreducible polynomial in  $k[x]$  which is not a monomial. In particular,  $\langle 1 - xy, f(x) \rangle$  are the maximal ideals of  $R$ . Therefore simple  $R$ -modules

are isomorphic to  $k[x]$  or  $k[x^{\pm 1}]/\langle f(x) \rangle$ . When  $k$  is algebraically closed, the simple  $R$ -modules are either one-dimensional or infinite-dimensional.

A discussion of how Jategaonkar's main lemma and a theorem of Stafford apply to this nonnoetherian  $R$  is given in Section 3.

## 2. Nonsplit extensions of simple $R$ -modules

Throughout  $k$  is an algebraically closed field with  $\text{char}(k) = 0$ . All modules are left modules. Then simple  $R$ -modules are isomorphic to  $k[x]$  or  $k[x^{\pm 1}]/\langle x - \lambda \rangle$  for  $\lambda \in k^\times$ . When a simple module is one-dimensional, i.e., isomorphic to  $k$  as a vector space, the  $x$ -action is multiplication by a scalar  $\lambda$ , and the  $y$ -action is multiplication by its inverse  $\lambda^{-1}$ . We denote such a simple  $R$ -module by  $k_\lambda$ . It is clear that  $k_{\lambda_1} \cong k_{\lambda_2}$  as simple  $R$ -modules for any  $\lambda_1, \lambda_2 \in k^\times$  if and only if  $\lambda_1 = \lambda_2$ .

We consider the  $R$ -module extension  $E$  with the short exact sequence (s.e.s.)

$$0 \rightarrow U \rightarrow E \rightarrow V \rightarrow 0 \quad (2)$$

of  $R$ -modules  $U$  and  $V$ . It is clear that  $E$  is isomorphic to  $U \oplus V$ , as  $k$ -vector spaces. The  $R$ -action on  $E$  is then given by the ring homomorphism

$$\rho_\delta : r \mapsto \begin{pmatrix} \alpha(r) & \delta(r) \\ 0 & \beta(r) \end{pmatrix},$$

where

$$\alpha : R \rightarrow \text{End}_k(U) \quad \text{and} \quad \beta : R \rightarrow \text{End}_k(V)$$

are ring homomorphisms, and  $\delta(r)$  is a  $k$ -linear map in  $\text{Hom}_k(V, U)$  such that

$$\delta(r_1 r_2) = \alpha(r_1) \delta(r_2) + \delta(r_1) \beta(r_2)$$

for any  $r_1, r_2 \in R$ . In particular,

$$\alpha(y) \delta(x) + \delta(y) \beta(x) = \delta(yx) = \delta(1).$$

Since  $\rho_\delta(1)$  must be the identity matrix, we have  $\delta(1) = 0$ . Therefore,

$$\alpha(y) \delta(x) + \delta(y) \beta(x) = 0. \quad (3)$$

That is, given  $\alpha$  and  $\beta$ , the map  $\delta$  is uniquely determined by the pair of  $k$ -linear maps  $\delta(x), \delta(y) \in \text{Hom}_k(V, U)$  satisfying the compatibility condition (3). If  $\delta$  is the zero mapping, then  $E \cong U \oplus V$ . Let  $E_\delta$  and  $E_{\delta'}$  be two module extensions of  $U$  by  $V$ , equipped with ring homomorphisms  $\rho_\delta$  and  $\rho_{\delta'}$ . Then  $E_\delta \cong E_{\delta'}$  if and only if there is a  $k$ -vector space isomorphism  $f : E_\delta \rightarrow E_{\delta'}$  such that  $f \circ \rho_\delta(r) = \rho_{\delta'}(r) \circ f$ . Note that  $R$  has the  $k$ -basis  $\{x^i y^j \mid i, j = 0, 1, 2, \dots\}$ . Therefore, it is sufficient to verify  $\rho_\delta(x) = f^{-1} \circ \rho_{\delta'}(x) \circ f$  and  $\rho_\delta(y) = f^{-1} \circ \rho_{\delta'}(y) \circ f$ .

Now consider another  $R$ -module extension  $E'$  with the s.e.s.

$$0 \rightarrow U' \rightarrow E' \rightarrow V' \rightarrow 0 \quad (4)$$

of  $R$ -modules  $U'$  and  $V'$ . We say that the two s.e.s. (2) and (4) are *equivalent* if there is an  $R$ -module isomorphism  $f : E \rightarrow E'$  such that the restriction of  $f$  on  $U$  yields an isomorphism from  $U$  to  $U'$ .

We focus on the  $R$ -module extension  $E$  of a simple  $R$ -module  $U$  by another simple  $R$ -module  $V$ . We start with the case when  $V$  is infinite-dimensional. It is shown in the following lemma that the s.e.s in this case is always split. This result can be directly derived from Bavula's proof that the infinite-dimensional simple  $R$ -module  $k[x]$  is projective. We include an alternative proof without using projectivity.

**Lemma 2.1.** *Suppose  $0 \rightarrow U \rightarrow E_\delta \rightarrow V \rightarrow 0$  is an s.e.s., where  $U$  and  $V$  are simple  $R$ -modules and  $\dim_k(V) = \infty$ . Then the s.e.s. is always split.*

*Proof.* Let  $\{b_0, b_1, b_2, \dots\}$  be a basis of  $V$  such that  $y$  and  $x$  are left and right shift operators, respectively. As vector spaces,  $E_\delta \cong U \oplus V$ . Consider the element

$$a := b_0 - x\delta(y)b_0$$

of  $E_\delta$ . It is clear that  $a \in E_\delta \setminus U$ . Then the left cyclic submodule  $Ra$  of  $E_\delta$  is distinct from 0 and  $U$ . For any element  $r \in R$ , we have

$$ra = \delta(r)b_0 + \beta(r)b_0 - rx\delta(y)b_0.$$

Hence  $ra \in R_a \cap U$  only if  $\beta(r)b_0 = 0$ , that is,  $r = sy$  for some  $s \in R$ . But

$$ya = yb_0 - yx\delta(y)b_0 = \delta(y)b_0 + \beta(y)b_0 - \delta(y)b_0 = 0.$$

That is,  $R_a \cap U = 0$ . Then  $R_a \oplus U = E_\delta$  since  $E_\delta/U \cong V$  is simple. Therefore  $E_\delta \cong U \oplus V$  as left  $R$ -modules.  $\square$

The next case deals with the module extension when  $U$  is infinite-dimensional and  $V$  is one-dimensional.

**Lemma 2.2.** *Let  $U$  and  $U'$  be two infinite-dimensional simple  $R$ -modules,  $k_\lambda$  and  $k_{\lambda'}$  be two one-dimensional  $R$ -modules for nonzero scalars  $\lambda$  and  $\lambda'$ . Suppose  $E_\delta$  and  $E_{\delta'}$  are two  $R$ -module extensions with the respective s.e.s.*

$$0 \rightarrow U \rightarrow E_\delta \rightarrow k_\lambda \rightarrow 0 \quad \text{and} \quad 0 \rightarrow U' \rightarrow E_{\delta'} \rightarrow k_{\lambda'} \rightarrow 0.$$

*Then  $E_\delta \cong E_{\delta'}$  if and only if  $\lambda = \lambda'$  and  $\delta'(x) = c\delta(x)$  for some nonzero  $c \in k$ . In this case the two s.e.s. are equivalent if and only if  $E_\delta \cong E_{\delta'}$ . As a consequence,  $E_\delta$  (resp.  $E_{\delta'}$ ) is nonsplit if and only if  $\delta \neq 0$  (resp.  $\delta' \neq 0$ ).*

*Proof.* We will fix a basis  $\{e_0, e_1, e_2, \dots, d\}$  for both  $E_\delta$  and  $E_{\delta'}$  as  $k$ -vector spaces, where  $\{e_0, e_1, e_2, \dots\}$  is a basis of  $U$  (and  $U'$ ) such that  $y$  and  $x$  are left and right shift operators, respectively. For any  $r \in R$ , we can identify the map  $\delta(r)$ , under the fixed basis, with an infinite-dimensional vector

$$\langle \delta(r)_0, \delta(r)_1, \delta(r)_2, \dots \rangle$$

with only finitely many nonzero components. Note that  $\alpha(y)\delta(x) + \delta(y)\beta(x) = 0$ , where  $\beta(x) = \lambda$  and  $y$  is the upper diagonal line matrix given in (1). It follows that

$$\delta(y)_i = \lambda^{-1} \delta(x)_{i+1} \quad \text{for } i \geq 1. \quad (5)$$

A similar result for  $\delta'(x)$  and  $\delta'(y)$  holds. Suppose that  $m$  is the smallest integer such that  $\delta(y)_i = \delta'(y)_i = 0$  for any  $i > m$ . Consequently,  $\delta(x)_i = \delta'(x)_i = 0$  for any  $i > m + 1$ .

Suppose that  $f$  is an  $R$ -module isomorphism  $E_{\delta'} \rightarrow E_\delta$ ; that is,  $f$  is a  $k$ -vector space isomorphism such that both  $\rho_\delta(x)f = f\rho_{\delta'}(x)$  and  $\rho_\delta(y)f = f\rho_{\delta'}(y)$ . We will obtain necessary conditions on  $f$  through its images on the basis elements of the selected basis. Let

$$f(e_0) = ae_0 + a_1e_1 + a_2e_2 + \dots + a'd$$

for some  $a', a_i \in k, i = 1, 2, \dots$ , where only finitely many  $a_i$ 's are nonzero. First,

$$f \circ \rho_{\delta'}(y)(e_0) = 0,$$

$$\rho_\delta(y) \circ f(e_0) = \sum_{i \geq 0} (a_{i+1} + a'\delta(y)_i)e_i + \frac{1}{\lambda}a'd.$$

Hence,  $a' = a_i = 0$  for all  $i = 1, 2, \dots$ , and so  $f(e_0) = ae_0$ . Moreover,

$$f(e_1) = f(xe_0) = xf(e_0) = x(ae_0) = ae_1$$

implies  $f(e_1) = ae_1$ . Inductively,  $f(e_i) = ae_i$  for some  $a \neq 0$  and all  $i \geq 0$ . Next, suppose that

$$f(d) = b_0e_0 + b_1e_1 + b_2e_2 + \dots + bd,$$

where  $b \neq 0$ ,  $b_i \in k$  for  $i \geq 0$ , and only finitely many  $b_i$ 's are nonzero. Then

$$\begin{aligned} \rho_\delta(y) \circ f(d) &= \sum_{i \geq 0} b_{i+1}e_i + \sum_{i \geq 0} b\delta(y)_i e_i + \lambda^{-1}bd, \\ f \circ \rho_{\delta'}(y)(d) &= \sum_{i \geq 0} \left( a\delta'(y)_i + \frac{1}{\lambda'}b_i \right) e_i + \frac{1}{\lambda'}bd. \end{aligned}$$

Thus, we have

$$\lambda = \lambda', \quad b_{i+1} + b\delta(y)_i = a\delta'(y)_i + \lambda^{-1}b_i \quad \text{for } i \geq 0.$$

Since  $\delta(y)_i = \delta'(y)_i = 0$  for any  $i > m$ , we have  $b_{i+1} = \lambda^{-1}b_i$  for any  $i > m$ . But only finitely many  $b_i$ 's are nonzero; it then follows inductively that

$$b_{m+1} = b_{m+2} = \cdots = 0.$$

Hence, we have the  $m + 1$  relations

$$\begin{aligned} b\delta(y)_m &= a\delta'(y)_m + \lambda^{-1}b_m, \\ b_{i+1} + b\delta(y)_i &= a\delta'(y)_i + \lambda^{-1}b_i \quad \text{for } i = 0, 1, \dots, m-1. \end{aligned} \quad (6)$$

Similarly, we have

$$\begin{aligned} \rho_\delta(x) \circ f(d) &= \sum_{i \geq 1} b_{i-1}e_i + \sum_{i \geq 0} b\delta(x)_i e_i + \lambda bd, \\ f \circ \rho_{\delta'}(x)(d) &= \sum_{i \geq 0} (a\delta'(x)_i + \lambda' b_i) e_i + \lambda' bd. \end{aligned}$$

Note that  $\delta(x)_j = \delta'(x)_j = 0$  for any  $j > m + 1$ . It then follows that

$$\begin{aligned} b\delta(x)_0 &= a\delta'(x)_0 + \lambda b_0, \\ b_m + b\delta(x)_{m+1} &= a\delta'(x)_{m+1}, \\ b_{i-1} + b\delta(x)_i &= a\delta'(x)_i + \lambda b_i \quad \text{for } i = 1, 2, \dots, m. \end{aligned} \quad (7)$$

Combining the relations (5) and (7), we have

$$\begin{aligned} b\delta(y)_m - a\delta'(y)_m &= -\lambda^{-1}b_m, \\ b\delta(y)_i - a\delta'(y)_i &= b_{i+1} - \lambda^{-1}b_i \quad \text{for } i = 0, 1, \dots, m-1. \end{aligned}$$

From (6), we have

$$\begin{aligned} b\delta(y)_m - a\delta'(y)_m &= \lambda^{-1}b_m, \\ b\delta(y)_i - a\delta'(y)_i &= \lambda^{-1}b_i - b_{i+1} \quad \text{for } i = 0, 1, \dots, m-1. \end{aligned}$$

Hence,  $b_i = \lambda b_{i+1}$  for  $0 \leq i \leq m-1$  and  $b_m = 0$ . Thus,  $b_0 = b_1 = \cdots = b_m = 0$ .

Therefore,  $f(e_i) = ae_i$  and  $f(d) = bd$  for some nonzero scalars  $a, b \in k$  and all  $i \geq 0$ . Such a  $k$ -vector space isomorphism is an  $R$ -module isomorphism if and only if  $\delta'(x) = \frac{b}{a}\delta(x)$  for the nonzero scalars  $a, b \in k$  or equivalently,  $\delta'(r) = \frac{b}{a}\delta(r)$  for any  $r \in R$ .

Therefore, any module extension  $E_\delta$  such that  $E_\delta/U \cong k_\lambda$  is nonsplit if and only if  $\delta(x) \neq 0$ . Let  $E_\delta$  and  $E_{\delta'}$  be nonsplit extensions such that

$$0 \rightarrow U \rightarrow E_\delta \rightarrow k_\lambda \rightarrow 0 \quad \text{and} \quad 0 \rightarrow U' \rightarrow E_{\delta'} \rightarrow k_{\lambda'} \rightarrow 0.$$

Then  $E_\delta \cong E_{\delta'}$  if and only if  $\lambda = \lambda'$  and  $\delta'(x) = c\delta(x)$  for some nonzero scalar  $c \in k$ . Observe that the isomorphism  $f$  from  $E_\delta$  to  $E_{\delta'}$  yields an isomorphism from  $U$  to  $U'$ . Therefore, the two s.e.s. are equivalent if and only if  $E_\delta \cong E_{\delta'}$ .  $\square$



Now we can state our main result.

**Theorem 2.3.** *Suppose  $0 \rightarrow U \rightarrow E_\delta \rightarrow V \rightarrow 0$  is an s.e.s. where  $U$  and  $V$  are simple  $R$ -modules and  $E_\delta$  is associated with the  $k$ -linear map  $\delta$  in  $\text{Hom}_k(V, U)$ . Let  $\lambda, \lambda'$  be nonzero scalars:*

- (i) *If  $\dim(V) = \infty$ , the s.e.s. is always split.*
- (ii) *If  $\dim(U) = \infty$  and  $V = k_\lambda$ , the s.e.s. is nonsplit if and only if  $\delta \neq 0$ . Any such two s.e.s. are equivalent if and only if  $\lambda = \lambda'$  and the infinite vectors  $\delta(x)$  and  $\delta'(x)$  are proportional.*
- (iii) *If  $U = k_\lambda$  and  $V = k_{\lambda'}$  are both one-dimensional, then the s.e.s. is nonsplit only if  $\delta \neq 0$  and  $\lambda = \lambda'$ . Any such two nonsplit s.e.s. are equivalent if and only if the submodules  $U$  are the same.*

*Proof.* The first two cases are proved in Lemmas 2.1 and 2.2. We only need to consider the case when  $U$  and  $V$  are both one-dimensional. Suppose the two modules  $U$  and  $V$  are uniquely determined by nonzero scalars  $\lambda$  and  $\lambda'$ . Let

$$0 \rightarrow k_\lambda \rightarrow E_\delta \rightarrow k_{\lambda'} \rightarrow 0$$

be an s.e.s. Then  $\delta$  is uniquely determined by  $\delta(x)$  since  $\delta(y) = -(\lambda\lambda')^{-1}\delta(x)$ . Moreover,  $\rho_\delta(y)$  is the inverse matrix of  $\rho_\delta(x)$ . Note that the  $2 \times 2$  matrix  $\rho_\delta(x)$  is similar to  $\rho_0(x)$  if and only if  $\lambda \neq \lambda'$ . Hence, the s.e.s. is always split if  $\lambda \neq \lambda'$ , whether or not  $\delta = 0$ . Therefore, the nonsplit case occurs when  $\delta \neq 0$  and  $\lambda = \lambda'$ . Consider two nonsplit s.e.s.

$$0 \rightarrow k_\lambda \rightarrow E_\delta \rightarrow k_\lambda \rightarrow 0 \quad \text{and} \quad 0 \rightarrow k_\gamma \rightarrow E_{\delta'} \rightarrow k_\gamma \rightarrow 0,$$

with nonzero  $\delta$  and  $\delta'$ . It is easy to see, by a linear transformation, that the two nonsplit s.e.s. are equivalent if and only if  $E_\delta \cong E_{\delta'}$  if and only if the nonzero scalars  $\lambda$  and  $\gamma$  are equal. Thus, there is only one, up to equivalence, nonsplit s.e.s.  $0 \rightarrow k_\lambda \rightarrow E_\delta \rightarrow k_\lambda \rightarrow 0$  for each one-dimensional simple  $R$ -module  $k_\lambda$ .  $\square$

### 3. Closing discussion

Let  $A$  be an associative ring. Recall a left (respectively, right) module  $M$  over  $A$  is called *torsion-free* if for any nonzero element  $m$  in  $M$  there is some  $r \in A$  such that  $rm \neq 0$  (respectively,  $mr \neq 0$ ). Two prime ideals  $P$  and  $Q$  of an associative ring  $A$  are *linked*, denoted as  $P \rightsquigarrow Q$ , if there is an ideal  $I$  of  $A$  such that  $(P \cap Q) > I \geq PQ$  and  $(P \cap Q)/I$  is nonzero and torsion-free both as a left  $A/P$ -module and a right  $A/Q$ -module. The graph of links of  $A$  is a directed graph whose vertices are prime ideals of  $A$ , with an arrow from  $P$  to  $Q$  whenever  $P \rightsquigarrow Q$ . The vertex set of each connected component is called a *clique*.

Jategaonkar's main lemma [1986] states that if  $M$  is a (right) module over a noetherian ring  $A$  with a nonsplit short exact sequence  $0 \rightarrow U \rightarrow M \rightarrow V \rightarrow 0$  and corresponding annihilators  $Q = \text{ann}_A(U)$  and  $P = \text{ann}_A(V)$ , then exactly one of the following two alternatives occurs: (i)  $P < Q$  and  $PM = 0$ ; (ii)  $P \rightsquigarrow Q$ .

Now let  $0 \rightarrow U \rightarrow E_\delta \rightarrow V \rightarrow 0$  be a nonsplit short exact sequence, where  $U$  and  $V$  are simple  $R$ -modules. Suppose  $Q = \text{ann}_R(U)$  and  $P = \text{ann}_R(V)$  are the affiliated primes. When  $\dim U = \infty$  and  $V \cong k_\lambda$ , we have  $Q = (0)$  and  $P = \langle 1 - xy, x - \lambda \rangle$ . There is no link between  $P$  and  $Q$ , and  $P \not\prec Q$ . When  $U \cong V \cong k_\lambda$ , we have  $Q = P = \langle 1 - xy, x - \lambda \rangle$ . There is no link between  $P$  and  $Q$ , and  $P \not\prec Q$ . This suggests that the noetherianess is necessary in the assumptions of Jategaonkar's main lemma.

On the other hand, [Stafford 1987, Corollary 3.13] states that all cliques of prime ideals in any noetherian ring are countable. When  $k$  is algebraically closed, the prime ideals of  $R$  are  $(0)$ ,  $F = \langle 1 - xy \rangle$ , and  $P_\lambda = \langle 1 - xy, x - \lambda \rangle$ , where  $\lambda \in k^\times$ . One can check that

$$F = F^2 = F \cap P_\lambda = F P_\lambda = P_\lambda F = P_\lambda \cap P_{\lambda'} = P_\lambda P_{\lambda'}$$

whenever  $\lambda \neq \lambda'$ . Moreover,  $P_\lambda / P_\lambda^2 \cong (x - \lambda) / (x - \lambda)^2$  as in  $k[x^{\pm 1}]$ . Hence the cliques in the graph of links are

$$F, \quad (0), \quad \begin{array}{c} \frown \\ P_\lambda \end{array}, \quad \begin{array}{c} \frown \\ P_{\lambda'} \end{array}.$$

This suggests that all cliques of  $R$  are countable.

### Acknowledgements

L. Wang would like to express her gratitude to Professor E. Letzter for his valuable suggestions. Lu was a math major at the University of Pittsburgh who participated in an undergraduate research project that was related to this short article. We are thankful to the math department of the University of Pittsburgh for its support. We also would like to express our gratitude to the referee for careful reading and helpful suggestions.

### References

- [Bavula 2010] V. V. Bavula, "The algebra of one-sided inverses of a polynomial algebra", *J. Pure Appl. Algebra* **214**:10 (2010), 1874–1897. MR Zbl
- [Gerritzen 2000] L. Gerritzen, "Modules over the algebra of the noncommutative equation  $yx = 1$ ", *Arch. Math. (Basel)* **75**:2 (2000), 98–112. MR Zbl
- [Irving 1979] R. S. Irving, "Prime ideals of Ore extensions over commutative rings, II", *J. Algebra* **58**:2 (1979), 399–423. MR Zbl
- [Jacobson 1950] N. Jacobson, "Some remarks on one-sided inverses", *Proc. Amer. Math. Soc.* **1** (1950), 352–355. MR Zbl

- [Jategaonkar 1986] A. V. Jategaonkar, *Localization in Noetherian rings*, London Mathematical Society Lecture Note Series **98**, Cambridge University Press, 1986. MR Zbl
- [Stafford 1987] J. T. Stafford, “The Goldie rank of a module”, pp. 1–20 in *Noetherian rings and their applications* (Oberwolfach, 1983), edited by L. W. Small, Math. Surveys Monogr. **24**, Amer. Math. Soc., Providence, RI, 1987. MR Zbl

Received: 2019-05-08      Revised: 2019-09-08      Accepted: 2019-09-09

|                          |                                                                                               |
|--------------------------|-----------------------------------------------------------------------------------------------|
| zl2965@pitt.edu          | <i>Tandon School of Engineering, New York University,<br/>Brooklyn, NY, United States</i>     |
| lhwang@pitt.edu          | <i>Department of Mathematics, University of Pittsburgh,<br/>Pittsburgh, PA, United States</i> |
| xingting.wang@howard.edu | <i>Department of Mathematics, Howard University,<br/>Washington, DC, United States</i>        |



# involve

msp.org/involve

## INVOLVE YOUR STUDENTS IN RESEARCH

*Involve* showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

## MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

## BOARD OF EDITORS

|                      |                                           |                        |                                           |
|----------------------|-------------------------------------------|------------------------|-------------------------------------------|
| Colin Adams          | Williams College, USA                     | Robert B. Lund         | Clemson University, USA                   |
| Arthur T. Benjamin   | Harvey Mudd College, USA                  | Gaven J. Martin        | Massey University, New Zealand            |
| Martin Bohner        | Missouri U of Science and Technology, USA | Mary Meyer             | Colorado State University, USA            |
| Amarjit S. Budhiraja | U of N Carolina, Chapel Hill, USA         | Frank Morgan           | Williams College, USA                     |
| Pietro Cerone        | La Trobe University, Australia            | Mohammad Sal Moslehian | Ferdowsi University of Mashhad, Iran      |
| Scott Chapman        | Sam Houston State University, USA         | Zuhair Nashed          | University of Central Florida, USA        |
| Joshua N. Cooper     | University of South Carolina, USA         | Ken Ono                | Univ. of Virginia, Charlottesville        |
| Jem N. Corcoran      | University of Colorado, USA               | Yuval Peres            | Microsoft Research, USA                   |
| Toka Diagana         | Howard University, USA                    | Y.-F. S. Pétermann     | Université de Genève, Switzerland         |
| Michael Dorff        | Brigham Young University, USA             | Jonathon Peterson      | Purdue University, USA                    |
| Sever S. Dragomir    | Victoria University, Australia            | Robert J. Plemmons     | Wake Forest University, USA               |
| Joel Foisy           | SUNY Potsdam, USA                         | Carl B. Pomerance      | Dartmouth College, USA                    |
| Errin W. Fulp        | Wake Forest University, USA               | Vadim Ponomarenko      | San Diego State University, USA           |
| Joseph Gallian       | University of Minnesota Duluth, USA       | Bjorn Poonen           | UC Berkeley, USA                          |
| Stephan R. Garcia    | Pomona College, USA                       | Józeph H. Przytycki    | George Washington University, USA         |
| Anant Godbole        | East Tennessee State University, USA      | Richard Rebarber       | University of Nebraska, USA               |
| Ron Gould            | Emory University, USA                     | Robert W. Robinson     | University of Georgia, USA                |
| Sat Gupta            | U of North Carolina, Greensboro, USA      | Javier Rojo            | Oregon State University, USA              |
| Jim Haglund          | University of Pennsylvania, USA           | Filip Saidak           | U of North Carolina, Greensboro, USA      |
| Johnny Henderson     | Baylor University, USA                    | Hari Mohan Srivastava  | University of Victoria, Canada            |
| Glenn H. Hurlbert    | Virginia Commonwealth University, USA     | Andrew J. Sterge       | Honorary Editor                           |
| Charles R. Johnson   | College of William and Mary, USA          | Ann Trenk              | Wellesley College, USA                    |
| K. B. Kulasekera     | Clemson University, USA                   | Ravi Vakil             | Stanford University, USA                  |
| Gerry Ladas          | University of Rhode Island, USA           | Antonia Vecchio        | Consiglio Nazionale delle Ricerche, Italy |
| David Larson         | Texas A&M University, USA                 | John C. Wierman        | Johns Hopkins University, USA             |
| Suzanne Lenhart      | University of Tennessee, USA              | Michael E. Zieve       | University of Michigan, USA               |
| Chi-Kwong Li         | College of William and Mary, USA          |                        |                                           |

## PRODUCTION

Silvio Levy, Scientific Editor

Cover: Alex Scorpan

See inside back cover or [msp.org/involve](http://msp.org/involve) for submission instructions. The subscription price for 2019 is US \$195/year for the electronic version, and \$260/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1444-4184 electronic, 1444-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2019 Mathematical Sciences Publishers

# involve

2019 vol. 12 no. 8

|                                                                                                                                |      |
|--------------------------------------------------------------------------------------------------------------------------------|------|
| On the zero-sum group-magicness of cartesian products                                                                          | 1261 |
| ADAM FONG, JOHN GEORGES, DAVID MAURO, DYLAN SPAGNUOLO,<br>JOHN WALLACE, SHUFAN WANG AND KIRSTI WASH                            |      |
| The variable exponent Bernoulli differential equation                                                                          | 1279 |
| KAREN R. RÍOS-SOTO, CARLOS E. SEDA-DAMIANI AND ALEJANDRO<br>VÉLEZ-SANTIAGO                                                     |      |
| The supersingularity of Hurwitz curves                                                                                         | 1293 |
| ERIN DAWSON, HENRY FRAUENHOFF, MICHAEL LYNCH, AMETHYST<br>PRICE, SEAMUS SOMERSTEP, ERIC WORK, DEAN BISOGNO AND RACHEL<br>PRIES |      |
| Multicast triangular semilattice network                                                                                       | 1307 |
| ANGELINA GROSSO, FELICE MANGANIELLO, SHIWANI VARAL AND<br>EMILY ZHU                                                            |      |
| Edge-transitive graphs and combinatorial designs                                                                               | 1329 |
| HEATHER A. NEWMAN, HECTOR MIRANDA, ADAM GREGORY AND<br>DARREN A. NARAYAN                                                       |      |
| A logistic two-sex model with mate-finding Allee effect                                                                        | 1343 |
| ELIZABETH ANDERSON, DANIEL MAXIN, JARED OTT AND GWYNETH<br>TERRETT                                                             |      |
| Unoriented links and the Jones polynomial                                                                                      | 1357 |
| SANDY GANZELL, JANET HUFFMAN, LESLIE MAVRAKIS, KAITLIN<br>TADAMY AND GRIFFIN WALKER                                            |      |
| Nonsplit module extensions over the one-sided inverse of $k[x]$                                                                | 1369 |
| ZHEPING LU, LINHONG WANG AND XINGTING WANG                                                                                     |      |
| Split Grothendieck rings of rooted trees and skew shapes via monoid<br>representations                                         | 1379 |
| DAVID BEERS AND MATT SZCZESNY                                                                                                  |      |
| On the classification of Specht modules with one-dimensional summands                                                          | 1399 |
| AUBREY PIPER COLLINS AND CRAIG J. DODGE                                                                                        |      |
| The monochromatic column problem with a prime number of colors                                                                 | 1415 |
| LORAN CROWELL AND STEVE SZABO                                                                                                  |      |
| Total Roman domination edge-critical graphs                                                                                    | 1423 |
| CHLOE LAMPMAN, KIEKA (C. M.) MYNHARDT AND SHANNON OGDEN                                                                        |      |



1944-4176(2019)12:8;1-Y