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Nonsplit module extensions over the one-sided inverse of $k[x]$

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Let R be the associative k-algebra generated by two elements x and y with defining relation $yx = 1$. A complete description of simple modules over R is obtained by using the results of Irving and Gerritzen. We examine the short exact sequence $0 \to U \to E \to V \to 0$, where U and V are simple R-modules. It shows that nonsplit extension only occurs when both U and V are one-dimensional, or, under certain condition, U is infinite-dimensional and V is one-dimensional.

1. Introduction

In this short note, we study nonsplit extensions of simple modules over the associative algebra $R = k\{x, y\}/\langle yx - 1\rangle$ over a base field k of characteristic 0. The algebra R is also known as the one-sided inverse of the polynomial algebra $k[x]$ and appeared in [Bavula 2010; Gerritzen 2000; Jacobson 1950; Irving 1979]. Note that

$$y(1 - xy) = (1 - xy)x = 0.$$

The algebra R is not a domain, and $Z(R) = k$. As a k-vector space R has basis

$$\{x^i y^j \mid i, j = 0, 1, 2, \dots\}.$$

Moreover, R admits the involution $\eta : x \mapsto y$ and $y \mapsto x$. Hence, the left and right algebraic properties of R are the same.

Jacobson [1950] gave a faithful irreducible representation of R as follows. Let S be the infinite-dimensional k-vector space with the basis $\{e_1, e_2, \dots\}$ and let R act on S by assigning

$$\begin{aligned} xe_n &= e_{n+1}, \quad n > 0,\\ ye_n &= e_{n-1}, \quad n > 1,\\ ye_1 &= 0. \end{aligned}$$

It was proved by Bavula [2010] and Gerritzen [2000] that there is only one isomorphic class of infinite-dimensional simple R-modules. Note that there is an

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algebra monomorphism $R \to \mathrm{End}_k(k[x])$ such that $x \mapsto x$ and $y \mapsto H^{-1}\frac{d}{dx}$, where $H \in \mathrm{End}_k(k[x])$ is given by $H(f) = \frac{d}{dx}(xf)$ for any $f \in k[x]$. In particular,

$$\bigoplus_{i\geq 0} kx^i(1-xy) \cong k[x]$$

is a simple and faithful left R-module, where the left R-module structure on $k[x]$ is via the algebra map $R \to \mathrm{End}_k(k[x])$ discussed above. Following [Bavula 2010], R contains a subring which is canonically isomorphic to the ring (without identity) of infinite-dimensional matrices. Let

$$F = \bigoplus_{i,j\geq 0} kM_{ij} \cong M_\infty(k),$$

where $M_{ij} = x^i(1-xy)y^j$ can be identical to the matrix units of $M_\infty(k)$. In particular, we have

$$x \sim \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{pmatrix}, \quad y \sim \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix}. \tag{1}$$

As a left R-module,

$$F = \bigoplus_{i,j\geq 0} kx^i(1-xy)y^j \cong \bigoplus_{i\geq 0}\Bigl(\bigoplus_{t\geq 0} kx^t x^i(1-xy)y^i\Bigr) \cong \bigoplus_{i\geq 0} k[x]$$

is a direct sum of infinitely many simple R-modules. Hence R is neither left nor right noetherian. Similarly, we see that there is an ascending chain of left annihilators in R which is not stable. Then R is neither left nor right Goldie. Moreover, F is equal to the ideal of R generated by $\langle 1-xy\rangle$. Since $F^2 = F$, $\mathrm{lann}(F)$ and $\mathrm{rann}(F)$ are both zero, we have F is an essential left and right ideal of R, which equals the socle of left and right R-module R. Hence F is contained in any nonzero ideal of R and it follows that the set of proper (two-sided) ideals of R is

$$\{0, \langle 1-xy\rangle, \langle 1-xy, f(x)\rangle\},$$

where $f(x)$ is a monic polynomial in $k[x]$ which is not a monomial. In particular, the ideals of R satisfy the ascending chain condition.

It follows from [Bavula 2010; Gerritzen 2000; Irving 1979] that the prime ideals are given by

$$\mathrm{Spec}(R) = \{0, \langle 1-xy\rangle, \langle 1-xy, f(x)\rangle\},$$

where $f(x)$ is a monic irreducible polynomial in $k[x]$ which is not a monomial. In particular, $\langle 1-xy, f(x)\rangle$ are the maximal ideals of R. Therefore simple R-modules are isomorphic to $k[x]$ or $k[x^{\pm 1}]/\langle f(x)\rangle$. When k is algebraically closed, the simple R-modules are either one-dimensional or infinite-dimensional.

A discussion of how Jategaonkar's main lemma and a theorem of Stafford apply to this nonnoetherian R is given in Section 3.

2. Nonsplit extensions of simple R-modules

Throughout k is an algebraically closed field with $\operatorname{char}(k) = 0$. All modules are left modules. Then simple R-modules are isomorphic to $k[x]$ or $k[x^{\pm 1}]/\langle x-\lambda\rangle$ for $\lambda \in k^{\times}$. When a simple module is one-dimensional, i.e., isomorphic to k as a vector space, the x-action is multiplication by a scalar λ, and the y-action is multiplication by its inverse λ^{-1}. We denote such a simple R-module by k_λ. It is clear that $k_{\lambda_1} \cong k_{\lambda_2}$ as simple R-modules for any $\lambda_1, \lambda_2 \in k^{\times}$ if and only if $\lambda_1 = \lambda_2$.

We consider the R-module extension E with the short exact sequence (s.e.s.)

$$0 \to U \to E \to V \to 0 \tag{2}$$

of R-modules U and V. It is clear that E is isomorphic to $U \oplus V$, as k-vector spaces. The R-action on E is then given by the ring homomorphism

$$\rho_\delta : r \mapsto \begin{pmatrix} \alpha(r) & \delta(r) \\ 0 & \beta(r) \end{pmatrix},$$

where

$$\alpha : R \to \operatorname{End}_k(U) \quad \text{and} \quad \beta : R \to \operatorname{End}_k(V)$$

are ring homomorphisms, and $\delta(r)$ is a k-linear map in $\operatorname{Hom}_k(V, U)$ such that

$$\delta(r_1 r_2) = \alpha(r_1)\delta(r_2) + \delta(r_1)\beta(r_2)$$

for any $r_1, r_2 \in R$. In particular,

$$\alpha(y)\delta(x) + \delta(y)\beta(x) = \delta(yx) = \delta(1).$$

Since $\rho_\delta(1)$ must be the identity matrix, we have $\delta(1) = 0$. Therefore,

$$\alpha(y)\delta(x) + \delta(y)\beta(x) = 0. \tag{3}$$

That is, given α and β, the map δ is uniquely determined by the pair of k-linear maps $\delta(x), \delta(y) \in \operatorname{Hom}_k(V, U)$ satisfying the compatibility condition (3). If δ is the zero mapping, then $E \cong U \oplus V$. Let E_δ and $E_{\delta'}$ be two module extensions of U by V, equipped with ring homomorphisms ρ_δ and $\rho_{\delta'}$. Then $E_\delta \cong E_{\delta'}$ if and only if there is a k-vector space isomorphism $f : E_\delta \to E_{\delta'}$ such that $f \circ \rho_\delta(r) = \rho_{\delta'}(r) \circ f$. Note that R has the k-basis $\{x^i y^j \mid i, j = 0, 1, 2, \ldots\}$. Therefore, it is sufficient to verify $\rho_\delta(x) = f^{-1} \circ \rho_{\delta'}(x) \circ f$ and $\rho_\delta(y) = f^{-1} \circ \rho_{\delta'}(y) \circ f$.

Now consider another R-module extension E' with the s.e.s.

$$0 \to U' \to E' \to V' \to 0 \tag{4}$$

of R-modules U' and V'. We say that the two s.e.s. (2) and (4) are *equivalent* if there is an R-module isomorphism $f : E \to E'$ such that the restriction of f on U yields an isomorphism from U to U'.

We focus on the R-module extension E of a simple R-module U by another simple R-module V. We start with the case when V is infinite-dimensional. It is shown in the following lemma that the s.e.s in this case is always split. This result can be directly derived from Bavula's proof that the infinite-dimensional simple R-module $k[x]$ is projective. We include an alternative proof without using projectivity.

Lemma 2.1. *Suppose* $0 \to U \to E_\delta \to V \to 0$ *is an s.e.s., where* U *and* V *are simple* R*-modules and* $\dim_k(V) = \infty$. *Then the s.e.s. is always split.*

Proof. Let $\{b_0, b_1, b_2, \dots\}$ be a basis of V such that y and x are left and right shift operators, respectively. As vector spaces, $E_\delta \cong U \oplus V$. Consider the element

$$a := b_0 - x\delta(y)b_0$$

of E_δ. It is clear that $a \in E_\delta \setminus U$. Then the left cyclic submodule Ra of E_δ is distinct from 0 and U. For any element $r \in R$, we have

$$ra = \delta(r)b_0 + \beta(r)b_0 - rx\delta(y)b_0.$$

Hence $ra \in R_a \cap U$ only if $\beta(r)b_0 = 0$, that is, $r = sy$ for some $s \in R$. But

$$ya = yb_0 - yx\delta(y)b_0 = \delta(y)b_0 + \beta(y)b_0 - \delta(y)b_0 = 0.$$

That is, $R_a \cap U = 0$. Then $R_a \oplus U = E_\delta$ since $E_\delta/U \cong V$ is simple. Therefore $E_\delta \cong U \oplus V$ as left R-modules. □

The next case deals with the module extension when U is infinite-dimensional and V is one-dimensional.

Lemma 2.2. *Let* U *and* U' *be two infinite-dimensional simple* R*-modules,* k_λ *and* $k_{\lambda'}$ *be two one-dimensional* R*-modules for nonzero scalars* λ *and* λ'. *Suppose* E_δ *and* $E_{\delta'}$ *are two* R*-module extensions with the respective s.e.s.*

$$0 \to U \to E_\delta \to k_\lambda \to 0 \quad \textit{and} \quad 0 \to U' \to E_{\delta'} \to k_{\lambda'} \to 0.$$

Then $E_\delta \cong E_{\delta'}$ *if and only if* $\lambda = \lambda'$ *and* $\delta'(x) = c\delta(x)$ *for some nonzero* $c \in k$. *In this case the two s.e.s. are equivalent if and only if* $E_\delta \cong E_{\delta'}$. *As a consequence,* E_δ (*resp.* $E_{\delta'}$) *is nonsplit if and only if* $\delta \neq 0$ (*resp.* $\delta' \neq 0$).

Proof. We will fix a basis $\{e_0, e_1, e_2, \ldots, d\}$ for both E_δ and $E_{\delta'}$ as k-vector spaces, where $\{e_0, e_1, e_2, \ldots\}$ is a basis of U (and U') such that y and x are left and right shift operators, respectively. For any $r \in R$, we can identify the map $\delta(r)$, under the fixed basis, with an infinite-dimensional vector

$$\langle \delta(r)_0, \delta(r)_1, \delta(r)_2, \ldots \rangle$$

with only finitely many nonzero components. Note that $\alpha(y)\delta(x) + \delta(y)\beta(x) = 0$, where $\beta(x) = \lambda$ and y is the upper diagonal line matrix given in (1). It follows that

$$\delta(y)_i = \lambda^{-1}\delta(x)_{i+1} \quad \text{for } i \geq 1. \tag{5}$$

A similar result for $\delta'(x)$ and $\delta'(y)$ holds. Suppose that m is the smallest integer such that $\delta(y)_i = \delta'(y)_i = 0$ for any $i > m$. Consequently, $\delta(x)_i = \delta'(x)_i = 0$ for any $i > m+1$.

Suppose that f is an R-module isomorphism $E_{\delta'} \to E_\delta$; that is, f is a k-vector space isomorphism such that both $\rho_\delta(x) f = f \rho_{\delta'}(x)$ and $\rho_\delta(y) f = f \rho_{\delta'}(y)$. We will obtain necessary conditions on f through its images on the basis elements of the selected basis. Let

$$f(e_0) = ae_0 + a_1e_1 + a_2e_2 + \cdots + a'd$$

for some $a', a_i \in k$, $i = 1, 2, \ldots$, where only finitely many a_i's are nonzero. First,

$$f \circ \rho_{\delta'}(y)(e_0) = 0,$$
$$\rho_\delta(y) \circ f(e_0) = \sum_{i \geq 0} (a_{i+1} + a'\delta(y)_i)e_i + \frac{1}{\lambda}a'd.$$

Hence, $a' = a_i = 0$ for all $i = 1, 2, \ldots$, and so $f(e_0) = ae_0$. Moreover,

$$f(e_1) = f(xe_0) = xf(e_0) = x(ae_0) = ae_1$$

implies $f(e_1) = ae_1$. Inductively, $f(e_i) = ae_i$ for some $a \neq 0$ and all $i \geq 0$. Next, suppose that

$$f(d) = b_0e_0 + b_1e_1 + b_2e_2 + \cdots + bd,$$

where $b \neq 0$, $b_i \in k$ for $i \geq 0$, and only finitely many b_i's are nonzero. Then

$$\rho_\delta(y) \circ f(d) = \sum_{i \geq 0} b_{i+1}e_i + \sum_{i \geq 0} b\delta(y)_i e_i + \lambda^{-1}bd,$$
$$f \circ \rho_{\delta'}(y)(d) = \sum_{i \geq 0} \left(a\delta'(y)_i + \frac{1}{\lambda'}b_i\right)e_i + \frac{1}{\lambda'}bd.$$

Thus, we have

$$\lambda = \lambda', \quad b_{i+1} + b\delta(y)_i = a\delta'(y)_i + \lambda^{-1}b_i \quad \text{for } i \geq 0.$$

Since $\delta(y)_i = \delta'(y)_i = 0$ for any $i > m$, we have $b_{i+1} = \lambda^{-1} b_i$ for any $i > m$. But only finitely many b_i's are nonzero; it then follows inductively that

$$b_{m+1} = b_{m+2} = \cdots = 0.$$

Hence, we have the $m+1$ relations

$$\begin{aligned} b\delta(y)_m &= a\delta'(y)_m + \lambda^{-1} b_m, \\ b_{i+1} + b\delta(y)_i &= a\delta'(y)_i + \lambda^{-1} b_i \quad \text{for } i = 0, 1, \ldots, m-1. \end{aligned} \tag{6}$$

Similarly, we have

$$\begin{aligned} \rho_\delta(x) \circ f(d) &= \sum_{i \geq 1} b_{i-1} e_i + \sum_{i \geq 0} b\delta(x)_i e_i + \lambda b d, \\ f \circ \rho_{\delta'}(x)(d) &= \sum_{i \geq 0} (a\delta'(x)_i + \lambda' b_i) e_i + \lambda' b d. \end{aligned}$$

Note that $\delta(x)_j = \delta'(x)_j = 0$ for any $j > m+1$. It then follows that

$$\begin{aligned} b\delta(x)_0 &= a\delta'(x)_0 + \lambda b_0, \\ b_m + b\delta(x)_{m+1} &= a\delta'(x)_{m+1}, \\ b_{i-1} + b\delta(x)_i &= a\delta'(x)_i + \lambda b_i \quad \text{for } i = 1, 2, \ldots, m. \end{aligned} \tag{7}$$

Combining the relations (5) and (7), we have

$$\begin{aligned} b\delta(y)_m - a\delta'(y)_m &= -\lambda^{-1} b_m, \\ b\delta(y)_i - a\delta'(y)_i &= b_{i+1} - \lambda^{-1} b_i \quad \text{for } i = 0, 1, \ldots, m-1. \end{aligned}$$

From (6), we have

$$\begin{aligned} b\delta(y)_m - a\delta'(y)_m &= \lambda^{-1} b_m, \\ b\delta(y)_i - a\delta'(y)_i &= \lambda^{-1} b_i - b_{i+1} \quad \text{for } i = 0, 1, \ldots, m-1. \end{aligned}$$

Hence, $b_i = \lambda b_{i+1}$ for $0 \leq i \leq m-1$ and $b_m = 0$. Thus, $b_0 = b_1 = \cdots = b_m = 0$.

Therefore, $f(e_i) = a e_i$ and $f(d) = bd$ for some nonzero scalars $a, b \in k$ and all $i \geq 0$. Such a k-vector space isomorphism is an R-module isomorphism if and only if $\delta'(x) = \frac{b}{a}\delta(x)$ for the nonzero scalars $a, b \in k$ or equivalently, $\delta'(r) = \frac{b}{a}\delta(r)$ for any $r \in R$.

Therefore, any module extension E_δ such that $E_\delta / U \cong k_\lambda$ is nonsplit if and only if $\delta(x) \neq 0$. Let E_δ and $E_{\delta'}$ be nonsplit extensions such that

$$0 \to U \to E_\delta \to k_\lambda \to 0 \quad \text{and} \quad 0 \to U' \to E_{\delta'} \to k_{\lambda'} \to 0.$$

Then $E_\delta \cong E_{\delta'}$ if and only if $\lambda = \lambda'$ and $\delta'(x) = c\delta(x)$ for some nonzero scalar $c \in k$. Observe that the isomorphism f from E_δ to $E_{\delta'}$ yields an isomorphism from U to U'. Therefore, the two s.e.s. are equivalent if and only if $E_\delta \cong E_{\delta'}$. □

Now we can state our main result.

Theorem 2.3. *Suppose* $0 \to U \to E_\delta \to V \to 0$ *is an s.e.s. where* U *and* V *are simple* R*-modules and* E_δ *is associated with the* k*-linear map* δ *in* $\mathrm{Hom}_k(V, U)$. *Let* λ, λ' *be nonzero scalars*:

(i) *If* $\dim(V) = \infty$, *the s.e.s. is always split.*

(ii) *If* $\dim(U) = \infty$ *and* $V = k_\lambda$, *the s.e.s. is nonsplit if and only if* $\delta \neq 0$. *Any such two s.e.s. are equivalent if and only if* $\lambda = \lambda'$ *and the infinite vectors* $\delta(x)$ *and* $\delta'(x)$ *are proportional.*

(iii) *If* $U = k_\lambda$ *and* $V = k_{\lambda'}$ *are both one-dimensional, then the s.e.s. is nonsplit only if* $\delta \neq 0$ *and* $\lambda = \lambda'$. *Any such two nonsplit s.e.s. are equivalent if and only if the submodules* U *are the same.*

Proof. The first two cases are proved in Lemmas 2.1 and 2.2. We only need to consider the case when U and V are both one-dimensional. Suppose the two modules U and V are uniquely determined by nonzero scalars λ and λ'. Let

$$0 \to k_\lambda \to E_\delta \to k_{\lambda'} \to 0$$

be an s.e.s. Then δ is uniquely determined by $\delta(x)$ since $\delta(y) = -(\lambda\lambda')^{-1}\delta(x)$. Moreover, $\rho_\delta(y)$ is the inverse matrix of $\rho_\delta(x)$. Note that the 2×2 matrix $\rho_\delta(x)$ is similar to $\rho_0(x)$ if and only if $\lambda \neq \lambda'$. Hence, the s.e.s. is always split if $\lambda \neq \lambda'$, whether or not $\delta = 0$. Therefore, the nonsplit case occurs when $\delta \neq 0$ and $\lambda = \lambda'$. Consider two nonsplit s.e.s.

$$0 \to k_\lambda \to E_\delta \to k_\lambda \to 0 \quad \text{and} \quad 0 \to k_\gamma \to E_{\delta'} \to k_\gamma \to 0,$$

with nonzero δ and δ'. It is easy to see, by a linear transformation, that the two nonsplit s.e.s. are equivalent if and only if $E_\delta \cong E_{\delta'}$ if and only if the nonzero scalars λ and γ are equal. Thus, there is only one, up to equivalence, nonsplit s.e.s. $0 \to k_\lambda \to E_\delta \to k_\lambda \to 0$ for each one-dimensional simple R-module k_λ. □

3. Closing discussion

Let A be an associative ring. Recall a left (respectively, right) module M over A is called *torsion-free* if for any nonzero element m in M there is some $r \in A$ such that $rm \neq 0$ (respectively, $mr \neq 0$). Two prime ideals P and Q of an associative ring A are *linked*, denoted as $P \rightsquigarrow Q$, if there is an ideal I of A such that $(P \cap Q) > I \geq PQ$ and $(P \cap Q)/I$ is nonzero and torsion-free both as a left A/P-module and a right A/Q-module. The graph of links of A is a directed graph whose vertices are prime ideals of A, with an arrow from P to Q whenever $P \rightsquigarrow Q$. The vertex set of each connected component is called a *clique*.

Jategaonkar's main lemma [1986] states that if M is a (right) module over a noetherian ring A with a nonsplit short exact sequence $0 \to U \to M \to V \to 0$ and corresponding annihilators $Q = \operatorname{ann}_A(U)$ and $P = \operatorname{ann}_A(V)$, then exactly one of the following two alternatives occurs: (i) $P < Q$ and $PM = 0$; (ii) $P \rightsquigarrow Q$.

Now let $0 \to U \to E_\delta \to V \to 0$ be a nonsplit short exact sequence, where U and V are simple R-modules. Suppose $Q = \operatorname{ann}_R(U)$ and $P = \operatorname{ann}_R(V)$ are the affiliated primes. When $\dim U = \infty$ and $V \cong k_\lambda$, we have $Q = (0)$ and $P = \langle 1 - xy, x - \lambda \rangle$. There is no link between P and Q, and $P \not< Q$. When $U \cong V \cong k_\lambda$, we have $Q = P = \langle 1 - xy, x - \lambda \rangle$. There is no link between P and Q, and $P \not< Q$. This suggests that the noetherianess is necessary in the assumptions of Jategaonkar's main lemma.

On the other hand, [Stafford 1987, Corollary 3.13] states that all cliques of prime ideals in any noetherian ring are countable. When k is algebraically closed, the prime ideals of R are (0), $F = \langle 1 - xy \rangle$, and $P_\lambda = \langle 1 - xy, x - \lambda \rangle$, where $\lambda \in k^\times$. One can check that

$$F = F^2 = F \cap P_\lambda = F P_\lambda = P_\lambda F = P_\lambda \cap P_{\lambda'} = P_\lambda P_{\lambda'}$$

whenever $\lambda \neq \lambda'$. Moreover, $P_\lambda / P_\lambda^2 \cong (x - \lambda)/(x - \lambda)^2$ as in $k[x^{\pm 1}]$. Hence the cliques in the graph of links are

$$F, \qquad (0), \qquad \overset{\circlearrowright}{P_\lambda}, \qquad \overset{\circlearrowright}{P_{\lambda'}}.$$

This suggests that all cliques of R are countable.

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