involve
a journal of mathematics

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msp

2019 vol. 12, no. 8

INVOLVE 12:8 (2019)
dx.doi.org/10.2140/involve.2019.12.1423

Total Roman domination edge-critical graphs

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(Communicated by Anant Godbole)

A total Roman dominating function on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ is adjacent to some vertex u with $f(u) = 2$, and the subgraph of G induced by the set of all vertices w such that $f(w) > 0$ has no isolated vertices. The weight of f is $\sum_{v \in V(G)} f(v)$. The total Roman domination number $\gamma_{tR}(G)$ is the minimum weight of a total Roman dominating function on G. A graph G is k-γ_{tR}-edge-critical if $\gamma_{tR}(G+e) < \gamma_{tR}(G) = k$ for every edge $e \in E(\overline{G}) \neq \varnothing$, and k-γ_{tR}-edge-supercritical if it is k-γ_{tR}-edge-critical and $\gamma_{tR}(G+e) = \gamma_{tR}(G) - 2$ for every edge $e \in E(\overline{G}) \neq \varnothing$. We present some basic results on γ_{tR}-edge-critical graphs and characterize certain classes of γ_{tR}-edge-critical graphs. In addition, we show that, when k is small, there is a connection between k-γ_{tR}-edge-critical graphs and graphs which are critical with respect to the domination and total domination numbers.

1. Introduction

We consider the behaviour of the total Roman domination number of a graph G upon the addition of edges to G. A *dominating set* S in a graph G is a set of vertices such that every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set in G. A *total dominating set* S (abbreviated by *TD-set*) in a graph G with no isolated vertices is a set of vertices such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* $\gamma_t(G)$ (abbreviated by *TD-number*) is the cardinality of a minimum total dominating set in G. For $S \subseteq V(G)$ and a function $f : S \to \mathbb{R}$, define $f(S) = \sum_{s \in S} f(s)$. A *Roman dominating function* (abbreviated by *RD-function*) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that every

MSC2010: 05C69.

Keywords: Roman domination, total Roman domination, total Roman domination edge-critical graphs.

Lampman was supported by an Undergraduate Student Research Award from the Natural Sciences and Engineering Research Council of Canada. Mynhardt was supported by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada. Ogden was supported by a Science Undergraduate Research Award from the University of Victoria.

vertex v with $f(v)=0$ is adjacent to some vertex u with $f(u)=2$. The *weight* of f, denoted by $\omega(f)$, is defined as $f(V(G))$. The *Roman domination number* $\gamma_R(G)$ (abbreviated by *RD-number*) is defined as $\min\{\omega(f) : f$ is an RD-function on $G\}$. For an RD-function f, let $V_f^i = \{v \in V(G) : f(v)=i\}$ and $V_f^+ = V_f^1 \cup V_f^2$. Thus, we can uniquely express an RD-function f as $f=(V_f^0, V_f^1, V_f^2)$.

As defined by Ahangar, Henning, Samodivkin and Yero [2016], a *total Roman dominating function* (abbreviated by *TRD-function*) on a graph G with no isolated vertices is a Roman dominating function with the additional condition that $G[V_f^+]$ has no isolated vertices. The *total Roman domination number* $\gamma_{tR}(G)$ (abbreviated by *TRD-number*) is the minimum weight of a TRD-function on G; that is, $\gamma_{tR}(G) = \min\{\omega(f) : f$ is a TRD-function on $G\}$. A TRD-function f such that $\omega(f) = \gamma_{tR}(G)$ is called a $\gamma_{tR}(G)$*-function*, or a γ_{tR}*-function* if the graph G is clear from the context; γ_R*-functions* are defined analogously.

The addition of an edge to a graph has the potential to change its total domination or Roman domination number. Van der Merwe, Mynhardt and Haynes [1998b] studied γ_t*-edge-critical graphs*, that is, graphs G for which $\gamma_t(G+e) < \gamma_t(G)$ for each $e \in E(\overline{G})$ and $E(\overline{G}) \neq \varnothing$. We consider the same concept for total Roman domination. A graph G is *total Roman domination edge-critical*, or simply γ_{tR}*-edge-critical*, if $\gamma_{tR}(G+e) < \gamma_{tR}(G)$ for every edge $e \in E(\overline{G})$ and $E(\overline{G}) \neq \varnothing$. We say that G is k*-*γ_{tR}*-edge-critical* if $\gamma_{tR}(G) = k$ and G is γ_{tR}-edge-critical. If $\gamma_{tR}(G+e) \leq \gamma_{tR}(G) - 2$ for every edge $e \in E(\overline{G})$ and $E(\overline{G}) \neq \varnothing$, we say that G is γ_{tR}*-edge-supercritical*. If $\gamma_{tR}(G+e) = \gamma_{tR}(G)$ for all $e \in E(\overline{G})$, or $E(\overline{G}) = \varnothing$, we say that G is *stable*.

Pushpam and Padmapriea [2017] established bounds on the total Roman domination number of a graph in terms of its order and girth. Total Roman domination in trees was studied by Amjadi, Nazari-Moghaddam, Sheikholeslami and Volkmann [2017], as well as by Amjadi, Sheikholeslami and Soroudi [2019]. Amjadi, Sheikholeslami, and Soroudi [2018] also studied Nordhaus–Gaddum bounds for total Roman domination. Campanelli and Kuziak [2019] considered total Roman domination in the lexicographic product of graphs. We refer the reader to the well-known books [Chartrand and Lesniak 2016; Haynes, Hedetniemi, and Slater 1998] for graph theory concepts not defined here. Frequently used or lesser known concepts are defined where needed.

We begin with some general results regarding the addition of an edge $e \in E(\overline{G})$ to a graph G in Section 2. In Section 3, we characterize n-γ_{tR}-edge-critical graphs of order n. We characterize 4-γ_{tR}-edge-critical graphs in Section 4, and, after investigating γ_{tR}-edge-supercritical graphs in Section 5, we present a necessary condition for 5-γ_{tR}-edge-critical graphs in Section 6. In Section 7, we determine the total Roman domination number of spiders and characterize γ_{tR}-edge-critical spiders. As can be expected, every graph G with $\gamma_{tR}(G) = k \geq 4$ is a spanning subgraph of a k-$\gamma_{tR}(G)$-edge-critical graph; a short proof is given in Section 8, where we also show that for any $k \geq 4$, there exists a k-γ_{tR}-edge-critical graph of diameter 2. We conclude in Section 9 with ideas for future research.

2. Adding an edge

We begin with a result from [Van der Merwe, Mynhardt, and Haynes 1998a] which bounds the effect the addition of an edge can have on the total domination number of a graph and show that the same bounds hold with respect to the total Roman domination number.

Proposition 2.1 [Van der Merwe, Mynhardt, and Haynes 1998a]. *For a graph G with no isolated vertices, if $uv \in E(\overline{G})$, then $\gamma_t(G) - 2 \leq \gamma_t(G + uv) \leq \gamma_t(G)$.*

An edge $uv \in E(\overline{G})$ is *critical* if $\gamma_{tR}(G+uv) < \gamma_{tR}(G)$. The following proposition restricts the possible values assigned to the vertices of a critical edge uv by a $\gamma_{tR}(G+uv)$-function f, which will be useful in proving subsequent results. For a graph G and a vertex $v \in V(G)$, the *open neighbourhood* of v in G is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, and the *closed neighbourhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$. When $G \neq K_2$, the unique neighbour of an end-vertex of G is called a *support vertex*.

Proposition 2.2. *Given a graph G with no isolated vertices, if $uv \in E(\overline{G})$ is a critical edge and f is a $\gamma_{tR}(G+uv)$-function, then*

$$\{f(u), f(v)\} \in \{\{2, 2\}, \{2, 1\}, \{2, 0\}, \{1, 1\}\}.$$

If, in addition, $\deg(u) = \deg(v) = 1$, *then there exists a $\gamma_{tR}(G+uv)$-function f such that $f(u) = f(v) = 1$.*

Proof. Let G be a graph with no isolated vertices, $uv \in E(\overline{G})$ such that $\gamma_{tR}(G+uv) < \gamma_{tR}(G)$, and f a γ_{tR}-function on $G + uv$. Suppose for a contradiction that $\{f(u), f(v)\} \notin \{\{2, 2\}, \{2, 1\}, \{2, 0\}, \{1, 1\}\}$. Then $\{f(u), f(v)\} \in \{\{0, 0\}, \{0, 1\}\}$. Note that, in either case, the edge uv cannot affect whether u and v are dominated or whether, in the case where (say) $f(v) = 1$, v is isolated. Hence f is a TRD-function of G, contradicting $\gamma_{tR}(G + uv) < \gamma_{tR}(G)$. Therefore $\{f(u), f(v)\} \in \{\{2, 2\}, \{2, 1\}, \{2, 0\}, \{1, 1\}\}$.

Now, suppose in addition that $\deg(u) = \deg(v) = 1$, and let f be a $\gamma_{tR}(G+uv)$-function such that $|V_f^2|$ is as small as possible. Let w and x be the unique neighbours of u and v, respectively, noting that possibly $w = x$. Suppose for a contradiction that $f(u) = 2$ (without loss of generality). If $f(v) = 0$, then $f(w) > 0$, otherwise u would be isolated in $G[V_f^+]$. Thus, regardless of whether $w = x$ or not, consider the function $f' : V(G) \to \{0, 1, 2\}$ defined by $f'(u) = f'(v) = 1$ and $f'(y) = f(y)$ for all other $y \in V(G)$. Otherwise, if $f(v) \geq 1$, then clearly $f(w) = 0$. Thus, regardless of whether $w = x$ or not, consider the function $f' : V(G) \to \{0, 1, 2\}$ defined by $f'(u) = f'(w) = 1$ and $f'(y) = f(y)$ for all other $y \in V(G)$. In either case, f' is a γ_{tR}-function on $G + uv$. However, $|V^2_{f'}| < |V^2_f|$, contradicting $|V^2_f|$ being as small as possible. Hence $f(u) \neq 2$, and thus $f(u) = f(v) = 1$. □

Proposition 2.3. *Given a graph G with no isolated vertices, if $uv \in E(\overline{G})$, then $\gamma_{tR}(G) - 2 \leq \gamma_{tR}(G + uv) \leq \gamma_{tR}(G)$.*

Proof. Let G be a graph with no isolated vertices. Clearly, adding an edge cannot increase the total Roman domination number; hence the upper bound holds. Now, let $uv \in E(\overline{G})$. Note that when $\gamma_{tR}(G + uv) = \gamma_{tR}(G)$, the lower bound clearly holds. So assume $\gamma_{tR}(G + uv) < \gamma_{tR}(G)$ and let f be a $\gamma_{tR}(G+uv)$-function. By Proposition 2.2, $\{f(u), f(v)\} \in \{\{2, 2\}, \{2, 1\}, \{2, 0\}, \{1, 1\}\}$.

First assume $\{f(u), f(v)\} \in \{\{2, 2\}, \{2, 1\}, \{1, 1\}\}$. Then f is an RD-function of G, and the only possible isolated vertices in $G[V^+_f]$ are u and v. Consider the function $f' : V(G) \to \{0, 1, 2\}$ defined as follows: If u is isolated in $G[V^+_f]$, choose $u' \in N_G(u)$ and let $f'(u') = 1$. Similarly, if v is isolated in $G[V^+_f]$, choose $v' \in N_G(v)$ and let $f'(v') = 1$. Let $f'(x) = f(x)$ for all other $x \in V(G)$. Now, assume instead that $f(u) = 2$ and $f(v) = 0$ (without loss of generality). Since u is not isolated in $G[V^+_f]$, f is a TRD-function of $G - v$. Consider the function $f' : V(G) \to \{0, 1, 2\}$ defined as follows: Let $f'(v) = 1$. Then, if v is isolated in $G[V^+_{f'}]$, choose $v' \in N_G(v)$ and let $f'(v') = 1$. Let $f'(x) = f(x)$ for all other $x \in V(G)$. In either case, f' is a TRD-function of G and $\omega(f') \leq \gamma_{tR}(G + uv) + 2$. Thus $\gamma_{tR}(G) \leq \gamma_{tR}(G + uv) + 2$, and hence the lower bound holds. □

3. γ_{tR}-edge-critical graphs with large TRD-numbers

We now investigate the γ_{tR}-edge-critical graphs G which have the largest TRD-number, namely $|V(G)|$. A *subdivided star* is a tree obtained from a star on at least three vertices by subdividing each edge exactly once. A *double star* is a tree obtained from two disjoint nontrivial stars by joining the two central vertices (choosing either central vertex in the case of K_2). The *corona* $\mathrm{cor}(G)$ (sometimes denoted by $G \circ K_1$) of G is obtained by joining each vertex of G to a new end-vertex.

Connected graphs G for which $\gamma_{tR}(G) = |V(G)|$ were characterized in [Ahangar, Henning, Samodivkin, and Yero 2016]. There $\mathcal{G}$ was defined as the family of connected graphs obtained from a 4-cycle v_1, v_2, v_3, v_4, v_1 by adding $k_1 + k_2 \geq 1$ vertex-disjoint paths P_2, and joining v_i to the end of k_i such paths for $i \in \{1, 2\}$. Note that possibly $k_1 = 0$ or $k_2 = 0$. Furthermore, they defined $\mathcal{H}$ to be the family of graphs obtained from a double star by subdividing each pendant edge once and the nonpendant edge $r \geq 0$ times. For $r \geq 0$, we define $\mathcal{H}_r \subseteq \mathcal{H}$ as the family of graphs in $\mathcal{H}$ where the nonpendant edge was subdivided r times.

Proposition 3.1 [Ahangar, Henning, Samodivkin, and Yero 2016]. *If G is a connected graph of order $n \geq 2$, then $\gamma_{tR}(G) = n$ if and only if one of the following holds*:

(i) *G is a path or a cycle.*

(ii) *G is the corona of a graph.*

(iii) *G is a subdivided star.*

(iv) $G \in \mathcal{G} \cup \mathcal{H}$.

Using Proposition 3.1, we characterize connected n-γ_{tR}-edge-critical graphs as follows.

Theorem 3.2. *A connected graph G of order $n \geq 4$ is n-γ_{tR}-edge-critical if and only if G is one of the following graphs*:

(i) C_n, $n \geq 4$.

(ii) $\mathrm{cor}(K_r)$, $r \geq 3$.

(iii) *a subdivided star of order $n \geq 7$.*

(iv) $G \in \mathcal{G}$.

(v) $G \in \mathcal{H} - \mathcal{H}_0 - \mathcal{H}_2$.

Proof. Let G be a connected graph of order $n \geq 4$ with $\gamma_{tR}(G) = n$. First, suppose G is any of the graphs listed in (i)–(v) above. Then, for any $e \in E(\overline{G})$, $G + e$ is not one of the graphs listed in Proposition 3.1. Therefore $\gamma_{tR}(G + e) < n$ for all $e \in E(\overline{G})$, and thus G is γ_{tR}-edge-critical.

Otherwise, suppose G is not one of the graphs listed in (i)–(v) above. Note that since $\gamma_{tR}(G) = n$, G is still listed in Proposition 3.1(i)–(iv). If $G \cong P_n : v_1, \dots, v_n$, $n \geq 4$, then $G + v_1 v_n \cong C_n$ and $\gamma_{tR}(G) = \gamma_{tR}(C_n) = n$. If $G \cong \mathrm{cor}(F)$, where F is not a complete graph of order at least 3, then $\gamma_{tR}(G) = \gamma_{tR}(G + uv)$ for any $uv \in E(\overline{F})$. If G is a subdivided star of order less than 7, then $G = P_5$. In each of these cases, G is clearly not γ_{tR}-edge-critical.

Now consider $G \in \mathcal{H}$. Let $w_1, \dots, w_k$ be the leaves of G, $u_1, \dots, u_k$ be their respective support vertices, and $v_1, \dots, v_m$ be the path such that v_1 and v_m are the two support vertices in the original double star S, labelled so that w_1 is adjacent, in S, to v_1. Note that $m = r + 2$, and therefore $m \geq 2$. If $G \in \mathcal{H}_0$, consider the graph $G + v_2 w_1$, and note that $G + v_2 w_1 \in \mathcal{G}$. Therefore, by Proposition 3.1, $\gamma_{tR}(G + v_2 w_1) = n$, and thus G is not γ_{tR}-edge-critical. Similarly, if $G \in \mathcal{H}_2$, consider the graph $G + v_1 v_4$, and note that $G + v_1 v_4 \in \mathcal{G}$. Therefore, by Proposition 3.1, $\gamma_{tR}(G + v_1 v_4) = n$, and again G is not γ_{tR}-edge-critical. □

4. 4-γ_{tR}-edge-critical graphs

Before we characterize the graphs G such that $\gamma_{tR}(G) = 4$ and $\gamma_{tR}(G+e) = 3$ for any $e \in E(\overline{G})$ (that is, the graphs which are 4-γ_{tR}-edge-critical), we present the following result from [Pushpam and Padmapriea 2017] which characterizes the graphs with a total Roman domination number of 3, the smallest possible TRD-number. Note that while the authors required that G has girth 3, the result actually holds in general for any graph G on at least three vertices, as we now show. A *universal vertex* of G is a vertex that is adjacent to all other vertices of G.

Proposition 4.1. *For a graph G of order $n \geq 3$ with no isolated vertices, $\gamma_{tR}(G) = 3$ if and only if $\Delta(G) = n-1$, that is, G has a universal vertex.*

Proof. Suppose $\gamma_{tR}(G) = 3$ and let $f = (V_f^0, V_f^1, V_f^2)$ be a $\gamma_{tR}(G)$-function. If $V_f^2 = \varnothing$, then $|V_f^1| = 3$, and thus $n = 3$. Since G has no isolated vertices, this implies that $G = K_3$ or P_3, both of which have a universal vertex. Otherwise, assume $|V_f^2| = 1$ and $|V_f^1| = 1$. Pick $u, v \in V(G)$ so that $f(u) = 1$ and $f(v) = 2$. Since $G[V_f^+]$ has no isolated vertices, $uv \in E(G)$. Furthermore, since $\gamma_{tR}(G) = 3$, $f(x) = 0$ for all other $x \in V(G)$. Therefore $N_G[v] = V(G)$, and thus v is a universal vertex.

Conversely, suppose G has a universal vertex v, and take any $u \in N_G(v)$. Consider the TRD-function $f : V(G) \to \{0, 1, 2\}$ defined by $f(v) = 2$, $f(u) = 1$, and $f(x) = 0$ for all other $x \in V(G)$. Since G has at least three vertices, $\gamma_{tR}(G) > 2$. Therefore, since $\omega(f) = 3$, we conclude that $\gamma_{tR}(G) = 3$. □

A *galaxy* is defined as the disjoint union of two or more nontrivial stars. The characterization of 4-γ_{tR}-edge-critical graphs follows; note that this class of graphs is exactly the class of 2-γ-edge-critical graphs, as characterized in [Sumner and Blitch 1983].

Theorem 4.2. *A graph G with no isolated vertices is 4-γ_{tR}-edge-critical if and only if $\overline{G}$ is a galaxy.*

Proof. Let G be a graph of order n with no isolated vertices. Suppose first that G is 4-γ_{tR}-edge-critical. Then for any $e \in E(\overline{G})$, we have $\gamma_{tR}(G+e) = 3$, and thus Proposition 4.1 implies that the addition of any edge to G creates a universal vertex. Therefore, for each edge $uv \in E(\overline{G})$, one of u and v has degree $n-2$ in G; that is, one of u and v is a leaf in $\overline{G}$. Since each edge of $\overline{G}$ connects a leaf to either a support vertex or another leaf, the components of $\overline{G}$ are nontrivial stars. Moreover, $\overline{G}$ has at least two components, otherwise G has an isolated vertex.

Conversely, suppose $\overline{G}$ is a galaxy. Since $\overline{G}$ has no isolated vertices, G has no universal vertices, and thus, by Proposition 4.1, $\gamma_{tR}(G) > 3$. Let u and v be vertices in different components of $\overline{G}$, and define $f : V(G) \to \{0, 1, 2\}$ by $f(u) = f(v) = 2$ and $f(x) = 0$ for all other $x \in V(G)$. Clearly f is a TRD-function on G, and hence

$\gamma_{tR}(G) = 4$. Since the deletion of any edge in $\overline{G}$ produces an isolated vertex, the addition of any edge to G creates a universal vertex. Therefore, by Proposition 4.1, $\gamma_{tR}(G+e) = 3$ for all $e \in E(\overline{G})$, and hence G is 4-γ_{tR}-edge-critical. □

Corollary 4.3. *If G is a connected $(n-2)$-regular graph, then G is 4-γ_{tR}-edge-critical.*

Having characterized 4-γ_{tR}-edge-critical graphs, our next result demonstrates the existence of stable graphs with total Roman domination number 4.

Proposition 4.4. *If G is an $(n-3)$-regular graph of order $n \geq 6$, then $\gamma_{tR}(G) = 4$. Moreover, G is stable.*

Proof. We prove that $\gamma(G) = 2$. Since G is $(n-3)$-regular, its complement $\overline{G}$ is 2-regular. If $\overline{G}$ is disconnected, let u and v be vertices in different components of $\overline{G}$. Otherwise, if $\overline{G}$ is connected, then $\overline{G} \cong C_n$, $n \geq 6$, and thus we can choose $u, v \in V(\overline{G})$ such that $d_{\overline{G}}(u, v) \geq 3$. In either case, $N_{\overline{G}}[u] \cap N_{\overline{G}}[v] = \varnothing$. In G, u dominates all vertices in $G - N_{\overline{G}}(u)$ and v dominates all vertices in $G - N_{\overline{G}}(v)$. Therefore $\{u, v\}$ dominates G, and thus, since G has no universal vertex, $\gamma(G) = 2$.

Now, define $f : V(G) \to \{0, 1, 2\}$ by $f(u) = f(v) = 2$ and $f(y) = 0$ for all other $y \in V(G)$. Since $uv \in E(G)$, f is a TRD-function on G and $\omega(f) = 4$, so $\gamma_{tR}(G) \leq 4$. Since G has no universal vertex, $\gamma_{tR}(G) > 3$ by Proposition 4.1, and thus $\gamma_{tR}(G) = 4$, as required. Furthermore, since the addition of any edge to G does not create a universal vertex, it follows from Proposition 4.1 that $\gamma_{tR}(G+e) = \gamma_{tR}(G)$ for all $e \in E(\overline{G})$. Therefore G is stable. □

5. γ_{tR}-edge-supercritical graphs

We now consider the graphs G which attain the lower bound in Proposition 2.3 for all $e \in E(\overline{G})$, that is, γ_{tR}-edge-supercritical graphs. An edge $uv \in E(\overline{G})$ is *supercritical* if $\gamma_{tR}(G+uv) = \gamma_{tR}(G) - 2$. Van der Merwe, Mynhardt, and Haynes [1998a] defined a graph G to be *γ_t-edge-supercritical* if $\gamma_t(G+e) = \gamma_t(G) - 2$ for all $e \in E(\overline{G})$. We begin with their characterization of γ_t-edge-supercritical graphs.

Proposition 5.1 [Van der Merwe, Mynhardt, and Haynes 1998a]. *A graph G is γ_t-edge-supercritical if and only if G is the union of two or more nontrivial complete graphs.*

The proof of the previous result relies on the fact that, if u and v are vertices of a graph G with $d(u, v) = 2$, then $\gamma_t(G) - 1 \leq \gamma_t(G+uv)$. However, the analogous result does not hold with respect to the total Roman domination number, as we now show. Consider the graph $G = \mathrm{cor}(K_3)$. By Proposition 3.1, $\gamma_{tR}(G) = 6$. Consider any two nonadjacent vertices u and v in G such that $\deg(u) = 1$ and $\deg(v) = 3$. Clearly uv is a supercritical edge with $d(u, v) = 2$, and thus $d(u, v) = 2$ does not always imply that $\gamma_{tR}(G) - 1 \leq \gamma_{tR}(G+uv)$.

As a result, the classification of γ_{tR}-edge-supercritical graphs will be less straightforward than that of γ_t-edge-supercritical graphs. However, it is easy to see that there are no 5-γ_{tR}-edge-supercritical graphs, where 5 is the smallest possible TRD-number of a γ_{tR}-edge-supercritical graph, and that the disjoint union of two or more complete graphs of order at least 3 is γ_{tR}-edge-supercritical.

Proposition 5.2. (i) *There are no 5-γ_{tR}-edge-supercritical graphs.*

(ii) *If G is the disjoint union of $k \geq 2$ complete graphs, each of order at least 3, then G is $3k$-γ_{tR}-edge-supercritical.*

Proof. (i) Suppose for a contradiction that G is a 5-γ_{tR}-edge-supercritical graph. Then $\gamma_{tR}(G + uv) = 3$ for any edge $uv \in E(\overline{G})$. However, as in the proof of Theorem 4.2, this implies that $\overline{G}$ is a galaxy, that is, G is 4-γ_{tR}-edge-critical, a contradiction.

(ii) It follows from Proposition 4.1 that $\gamma_{tR}(G) = 3k$. Moreover, joining any two vertices in different components of G results in a graph with TRD-number $3k-2$. □

6. 5-γ_{tR}-edge-critical graphs

We now investigate the graphs which are 5-γ_{tR}-edge-critical. We begin with the following results, which bound $\gamma_{tR}(G)$ in terms of $\gamma_t(G)$.

Proposition 6.1 [Ahangar, Henning, Samodivkin, and Yero 2016]. *If G is a graph with no isolated vertices, then $\gamma_t(G) \leq \gamma_{tR}(G) \leq 2\gamma_t(G)$. Furthermore, $\gamma_{tR}(G) = \gamma_t(G)$ if and only if G is the disjoint union of copies of K_2.*

Note that Amjadi, Nazari-Moghaddam, Sheikholeslami, and Volkmann [2017] characterized the trees which attain the upper bound in Proposition 6.1.

Proposition 6.2 [Ahangar, Henning, Samodivkin, and Yero 2016]. *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{tR}(G) = \gamma_t(G)+1$ if and only if $\Delta(G) = n-1$, that is, G has a universal vertex.*

By Proposition 4.1, Proposition 6.2 implies that, if G is a connected graph of order $n \geq 3$, then $\gamma_{tR}(G) = \gamma_t(G) + 1$ if and only if $\gamma_{tR}(G) = 3$. These results lead to the following observation.

Observation 6.3. If G is a connected graph of order $n \geq 3$ such that $\Delta(G) \leq n-2$, then $\gamma_t(G) + 2 \leq \gamma_{tR}(G) \leq 2\gamma_t(G)$.

We now provide a result characterizing graphs with $\gamma_{tR} \in \{3, 4\}$ in terms of their domination and total domination numbers that will be useful in describing 5-γ_{tR}-edge-critical graphs.

Proposition 6.4. *If G is a connected graph of order $n \geq 3$, then $\gamma_{tR}(G) \in \{3, 4\}$ if and only if $\gamma_t(G) = 2$. Moreover, $\gamma(G) = 1$ when $\gamma_{tR}(G) = 3$, and $\gamma(G) = 2$ when $\gamma_{tR}(G) = 4$.*

Proof. Suppose first that $\gamma_t(G) = 2$. By Proposition 6.1, $2 \leq \gamma_{tR}(G) \leq 4$. Clearly $\gamma_{tR}(G) \neq 2$, since $n \geq 3$. Therefore $\gamma_{tR}(G) \in \{3, 4\}$.

Conversely, suppose $\gamma_{tR}(G) \in \{3, 4\}$. First, if $\gamma_{tR}(G) = 3$, then Proposition 4.1 implies that G has a universal vertex. Therefore $\gamma_t(G) = 2$ and $\gamma(G) = 1$. Otherwise, if $\gamma_{tR}(G) = 4$, then Proposition 4.1 implies that G has no universal vertex. Therefore, by Observation 6.3, $\gamma_t(G) + 2 \leq 4$, and thus $\gamma_t(G) = 2$. Furthermore, since $\gamma(G) \leq \gamma_t(G)$ and G has no universal vertex, $\gamma(G) = 2$. □

Proposition 6.5. *For any graph G, if G is 5-γ_{tR}-edge-critical, then G is either 3-γ_t-edge-critical or $G = K_2 \cup K_n$ for $n \geq 3$, in which case G is 4-γ_t-edge-supercritical.*

Proof. Suppose G is 5-γ_{tR}-edge-critical. By Proposition 6.4, $\gamma_t(G) > 2$ and $\gamma_t(G + e) = 2$ for any $e \in E(\overline{G})$. Therefore, by Proposition 2.1, G is either 3-γ_t-edge-critical or 4-γ_t-edge-supercritical. If G is 4-γ_t-edge-supercritical, then by Proposition 5.1, G is the union of two or more nontrivial complete graphs. Since $\gamma_{tR}(G) = 5$, this implies that $G = K_2 \cup K_n$ for $n \geq 3$. □

Note that if we add the condition that G is not 6-γ_{tR}-edge-supercritical, then the above becomes a necessary and sufficient condition. Clearly $G = K_2 \cup K_n$ is 5-γ_{tR}-edge-critical for any $n \geq 3$. Otherwise, if G is 3-γ_t-edge-critical, then by Proposition 6.4, $\gamma_{tR}(G) > 4$ and $\gamma_{tR}(G + e) \in \{3, 4\}$ for any $e \in E(\overline{G})$. By Proposition 6.1, $\gamma_{tR}(G) \leq 6$, and thus, since G is not 6-γ_{tR}-edge-supercritical, $\gamma_{tR}(G) = 5$. Hence G is 5-γ_{tR}-edge-critical, as required.

7. γ_{tR}-edge-critical spiders

A (*generalized*) *spider* $\mathrm{Sp}(l_1, \ldots, l_k)$, $l_i \geq 1$, $k \geq 2$, is a tree obtained from the star $K_{1,k}$ with centre u and leaves $v_1, \ldots, v_k$ by subdividing the edge uv_i exactly $l_i - 1$ times, $i = 1, \ldots, k$. Thus, a spider $\mathrm{Sp}(2, \ldots, 2)$ is a subdivided star. The $u - v_i$ paths (of length l_i) are called the *legs* of the spider, while u is its *head*. We now investigate the spiders which are γ_{tR}-edge-critical. Note that when $k = 2$, $\mathrm{Sp}(l_1, \ldots, l_k) \cong P_n$ for $n \geq 3$, which, by Theorem 3.2, is not γ_{tR}-edge-critical. We begin with two propositions restricting γ_{tR}-edge-criticality in general graphs, which will be useful in classifying γ_{tR}-edge-critical spiders.

Proposition 7.1. *For a graph G with no isolated vertices, if G has an end-vertex w with support vertex x such that $G[N(x) - \{w\}]$ is not complete, then G is not γ_{tR}-edge-critical.*

Proof. Suppose $u, v \in N_G(x) - \{w\}$ such that $uv \in E(\overline{G})$. We claim $\gamma_{tR}(G + uv) = \gamma_{tR}(G)$. Suppose for a contradiction that $\gamma_{tR}(G + uv) < \gamma_{tR}(G)$, and consider a γ_{tR}-function $f = (V_f^0, V_f^1, V_f^2)$ on $G + uv$. Note that, since w is an end-vertex, $f(x) > 0$. By Proposition 2.2, $\{f(u), f(v)\} \in \{\{2, 2\}, \{2, 1\}, \{2, 0\}, \{1, 1\}\}$. Since $ux, vx \in E(G)$ and at least one of $f(u)$ and $f(v)$ is positive, we can assume without loss of generality that $f(x) = 2$. In any case, f is also a TRD-function on G, contradicting $\gamma_{tR}(G+uv) < \gamma_{tR}(G)$. Therefore $\gamma_{tR}(G+uv) = \gamma_{tR}(G)$ and G is not γ_{tR}-edge-critical. □

In a tree, the support vertex of a leaf is called a *stem*. A stem is called *weak* if it is adjacent to exactly one leaf, and *strong* if it is adjacent to two or more leaves. A vertex b of a tree such that $\deg(b) \geq 3$ is called a *branch vertex*. An *endpath* in a tree is a path from a branch vertex to a leaf, where all of the internal vertices of the path have degree 2. The next result follows immediately from Proposition 7.1.

Corollary 7.2. *If T is a γ_{tR}-edge-critical tree, then T contains no stems of degree at least* 3, *and hence no strong stems.*

Proposition 7.3. *For a graph G with no isolated vertices, if G has two endpaths $v_0, v_1, \dots, v_k$ and $u_0, u_1, \dots, u_m$, where $k, m \geq 3$ and v_k and u_m are leaves, then G is not γ_{tR}-edge-critical.*

Proof. We claim that $\gamma_{tR}(G + v_k u_m) = \gamma_{tR}(G)$. Suppose for a contradiction that $\gamma_{tR}(G + v_k u_m) < \gamma_{tR}(G)$, and let f be a γ_{tR}-function on $G + v_k u_m$. Then, by Proposition 2.2, we may assume $f(u_m) = f(v_k) = 1$. Define $f' : V(G) \to \{0, 1, 2\}$ as follows: If $f(v_{k-1}) = 0$, then clearly $f(v_{k-2}) = 2$ and $f(v_{k-3}) \geq 1$, so let $f'(v_{k-1}) = f'(v_{k-2}) = 1$. Otherwise, let $f'(v_{k-1}) = f(v_{k-1})$ and $f'(v_{k-2}) = f(v_{k-2})$. Similarly, if $f(u_{m-1}) = 0$, then clearly $f(u_{m-2}) = 2$ and $f(u_{m-3}) \geq 1$, so let $f'(u_{m-1}) = f'(u_{m-2}) = 1$. Otherwise, let $f'(u_{m-1}) = f(u_{m-1})$ and $f'(u_{m-2}) = f(u_{m-2})$. Finally, let $f'(y) = f(y)$ for all other $y \in V(G)$. Clearly f' is a TRD-function on G and $\omega(f') = \omega(f)$, contradicting $\gamma_{tR}(G + v_k u_m) < \gamma_{tR}(G)$. Therefore $\gamma_{tR}(G + v_k u_m) = \gamma_{tR}(G)$, and thus G is not γ_{tR}-edge-critical. □

Let S be a spider with $k \geq 3$ legs. In what follows, let c be the head of S, and let the k legs be labelled $c, v_{i1}, v_{i2}, \dots, v_{im_i}$, where $i \in \{1, 2, \dots, k\}$, in order of increasing length. Let $m = m_k$; that is, m is the length of a longest leg of S. We begin by determining the TRD-number of spiders.

Proposition 7.4. *If S is a spider of order n with $k \geq 3$ legs such that S has y legs of length* 2, *then*

$$\gamma_{tR}(S) = \begin{cases} n & \text{if } y \geq k-1, \\ n-k+y+1 & \text{if } 1 \leq y < k-1, \\ n-k+2 & \text{if } y = 0. \end{cases}$$

Proof. Suppose S has x legs of length 1, and consider a γ_{tR}-function f on S such that $|V_f^2|$ is as small as possible. First, suppose $y \geq k-1$. If $y = k$, then S is a subdivided star. Otherwise, if $y = k-1$, then S has exactly one leg that is not of length 2, and thus either $x = 1$ or $x = 0$. If $x = 1$, then S is the corona of a graph (specifically, $S = \mathrm{cor}(K_{1,k-1})$). Otherwise, if $x = 0$, then $m = m_k \geq 3$, and $S \in \mathcal{H}_r$, where $r = m - 3$. In any case, by Proposition 3.1, $\gamma_{tR}(S) = n$.

Assume therefore that $y < k - 1$. Then S has at least two legs that are not of length 2. Therefore S is not one of the graphs listed in Proposition 3.1, and thus $\gamma_{tR}(S) < n$. Hence there is some vertex $u \in V(S)$ such that $f(u) = 2$ and $f(w) = 0$ for at least two vertices w adjacent to u. Furthermore, since f is a TRD-function, such a vertex u is not isolated in $S[V_f^+]$, and thus $\deg(u) \geq 3$. Since c is the only vertex in S with degree at least 3, $f(c) = 2$. Therefore c Roman dominates each v_{i1}, and we need $f(v_{i1})$ to be positive for at least one i to ensure that $S[V_f^+]$ has no isolated vertices.

Consider an arbitrary leg $c, v_{i1}, v_{i2}, \ldots, v_{im_i}$ of S. If $m_i = 1$, then $f(v_{i1}) \in \{0, 1\}$ in order for f to totally Roman dominate c and v_{i1}. If $m_i = 2$, a total weight of 2 on v_{i1} and v_{i2} is required in order for f to total Roman dominate $\{v_{i1}, v_{i2}\}$. Since $|V_f^2|$ is as small as possible, $f(v_{i1}) = f(v_{i2}) = 1$. Finally, if $m_i > 2$, by Proposition 3.1 and since $f(c) = 2$, a total weight of at least $m_i - 1$ on $v_{i1}, \ldots, v_{im_i}$ is required in order for f to totally Roman dominate c and $\{v_{i1}, \ldots, v_{im_i}\}$. Moreover, by the choice of f, $f(v_{i1}) \in \{0, 1\}$ and $f(v_{i2}) = \cdots = f(v_{im}) = 1$. Therefore $\omega(f) \geq n - k + y + 1$.

Now, if $y > 0$, where (say) $m_j = 2$, then $f(v_{j1}) = 1$. By minimality and since c is adjacent to v_{j1}, $f(v_{i1}) = 0$ for each i such that $m_i \neq 2$. Then $\gamma_{tR}(S) = \omega(f) = n - k + y + 1$, as required. Otherwise, if $y = 0$, then $f(v_{i1}) = 1$ for some i to ensure that c is not isolated in $S[V_f^+]$. By minimality, $f(v_{j1}) = 0$ for each $j \neq i$. Therefore $\gamma_{tR}(S) = \omega(f) = n - k + 2$. □

The characterization of γ_{tR}-edge-critical spiders follows. Our result also shows that a spider of order n is γ_{tR}-edge-critical if and only if it is n-γ_{tR}-edge-critical.

Theorem 7.5. *A spider $S = \mathrm{Sp}(l_1, \ldots, l_k)$, $k \geq 3$, is γ_{tR}-edge-critical if and only if $l_i = 2$ for each i, $1 \leq i \leq k - 1$, and $l_k \in \{2, m\}$, where $m = 4$ or $m \geq 6$.*

Proof. Suppose S has order n. If $l_i = 2$ for each $i = 1, \ldots, k$, then S is a subdivided star and, by Theorem 3.2, S is n-γ_{tR}-edge-critical. Now, suppose S has exactly one leg of length $m \neq 2$. If $m = 1$, then by Proposition 7.1, S is not γ_{tR}-edge-critical. If $m = 3$ or $m = 5$, then $S \in \mathcal{H}_r$ with $r = 0$ or 2, respectively, and thus, by Theorem 3.2, S is not γ_{tR}-edge-critical. If $m = 4$ or $m \geq 6$, then $S \in \mathcal{H}_r$ with $r = m - 3$, and therefore, by Theorem 3.2, S is n-γ_{tR}-edge-critical. Finally, suppose S has at least two legs that are not of length 2. Again, by Proposition 7.1, if S has a leg of length 1, S is not γ_{tR}-edge-critical. Otherwise, assume S has at least two legs of length at least 3. Then, by Proposition 7.3, S is not γ_{tR}-edge-critical. □

8. k-γ_{tR}-edge-critical graphs with minimum diameter

We now consider the minimum diameter possible in a k-γ_{tR}-edge-critical graph for $k \geq 4$. There are no γ_{tR}-edge-critical graphs with diameter 1, as the only graphs with diameter 1 are nontrivial complete graphs, which are clearly not γ_{tR}-edge-critical since $E(\overline{G})=\varnothing$. Therefore, the minimum possible diameter for a γ_{tR}-edge-critical graph is 2. Asplund, Loizeaux and Van der Merwe [2018] constructed families of 3-γ_t-edge-critical graphs with diameter 2. We will show that, for any $k \geq 4$, there exists a k-γ_{tR}-edge-critical graph of diameter 2. We first present the following proposition which shows that every graph G without a dominating vertex is a spanning subgraph of a $\gamma_{tR}(G)$-edge-critical graph with the same total Roman domination number, which will be useful in proving our result.

Proposition 8.1. *For a graph G with no isolated vertices, if $\gamma_{tR}(G)=k\geq 4$, then G is a spanning subgraph of a k-$\gamma_{tR}(G)$-edge-critical graph.*

Proof. Suppose $\gamma_{tR}(G)=k\geq 4$. If G is k-$\gamma_{tR}(G)$-edge-critical, then we are done. Otherwise, there is, by definition, some edge $e_1\in E(\overline{G})$ such that $\gamma_{tR}(G+e_1)=\gamma_{tR}(G)$. Let $G_1=G+e_1$. If G_1 is k-$\gamma_{tR}(G)$-edge-critical, then we are done. Otherwise, there is some edge $e_2\in E(\overline{G}_1)$ such that $\gamma_{tR}(G_1+e_2)=\gamma_{tR}(G_1)$. Let $G_2=G_1+e_2$. Continuing in this way, we eventually obtain a graph G_i such that for all $e\in E(\overline{G}_i)$, $\gamma_{tR}(G_i+e)<\gamma_{tR}(G_i)$ and $\gamma_{tR}(G_i)=\gamma_{tR}(G_{i-1})=\cdots=\gamma_{tR}(G_1)=\gamma_{tR}(G)$. Since $k\geq 4$, G_i is not complete and thus $E(G_i)\neq\varnothing$. Therefore, G_i is a k-$\gamma_{tR}(G)$-edge-critical graph, of which G is a spanning subgraph. □

Before demonstrating the existence of k-γ_{tR}-edge-critical graphs of diameter 2 for any $k\geq 4$, we determine the TRD-number of $K_n\,\Box\,K_m$, where $n, m\geq 2$. Consider the vertices of $K_n\,\Box\,K_m$ as an $n\times m$ matrix, where vertices v_{ij} and v_{st} are adjacent if and only if $i=s$ or $j=t$. The rows and columns of the matrix form disjoint copies of K_m and K_n, respectively. If v_{ij} and v_{st} are nonadjacent, then v_{sj} is adjacent to both v_{ij} and v_{st}, and hence $\mathrm{diam}(K_n\,\Box\,K_m)=2$.

Proposition 8.2. *If m and n are integers such that $m\geq n\geq 2$, then $\gamma_{tR}(K_n\,\Box\,K_m)=2n$.*

Proof. Let $G=K_n\,\Box\,K_m$. To see that $\gamma_{tR}(G)\leq 2n$, consider the TRD-function $g=(V_g^0, V_g^1, V_g^2)$ on G where $V_g^1=\varnothing$ and $V_g^2=\{v_{i1}: 1\leq i\leq n\}$.

Now, suppose for a contradiction that $\gamma_{tR}(G)\leq 2n-1$ and consider a TRD-function $f=(V_f^0, V_f^1, V_f^2)$ on G with $\omega(f)=2n-1$. Each vertex v dominates one row and one column of G, so if $|V_f^2|=x$ (note that $x\leq n-1$), then at most x rows and at most x columns are dominated by elements of V_f^2. Let S be the set of vertices undominated by V_f^2. Then $|S|\geq (n-x)(m-x)\geq (n-x)^2$. Moreover, $|V_f^1|=(2n-1)-2x$ since $\omega(f)=2n-1$ and $|V_f^2|=x$.

If $x=n-1$, then $|V_f^1|=1$. Since f is a TRD-function and $|S|\geq (n-x)^2$, we have $|S|=1$; say $S=\{w\}$. Hence $V_f^1=\{w\}$. However, V_f^2 does not dominate w, and thus w is isolated in $G[V_f^+]$, which is a contradiction. Therefore, there is no TRD-function on G with weight $2n-1$ when $x=n-1$.

Otherwise, if $x < n-1$, then

$$\begin{aligned} |S| - |V_f^1| &\geq (n-x)^2 - (2n-1-2x) \\ &= x^2 - 2(n-1)x + (n-1)^2 \\ &= (n-1-x)^2 > 0. \end{aligned}$$

Therefore, the number of vertices undominated by V_f^2 is greater than $|V_f^1|$, contradicting f being a TRD-function. Thus there is no TRD-function on G with weight $2n-1$ when $x < n-1$. We conclude that $\gamma_{tR}(G) = 2n$. □

Theorem 8.3. *If $k \geq 4$, then there exists a k-γ_{tR}-edge-critical graph of diameter* 2.

Proof. First, assume that k is even; say $k = 2l$ for some $l \geq 2$. Let $G_l = K_l \square K_l$. By Proposition 8.2, $\gamma_{tR}(G_l) = 2l$, and, by Proposition 8.1, G_l is a spanning subgraph of a k-γ_{tR}-edge-critical graph G_l'. Since $k > 3$, Proposition 4.1 implies that G_l' has no dominating vertex, and hence $2 \leq \operatorname{diam}(G_l') \leq \operatorname{diam}(G_l) = 2$.

Now, consider the case where k is odd; say $k = 2l+1$ for some $l \geq 2$. Let G_l^d be the graph formed by taking $K_{l+1} \square K_{l+1}$ and deleting the vertices in the set $\{v_{j1} : \lfloor \frac{l}{2} \rfloor + 2 \leq j \leq l+1\}$. Similarly to G_l, $\operatorname{diam}(G_l^d) = 2$. See Figure 1.

We claim that $\gamma_{tR}(G_l^d) = 2l+1$. To see that $\gamma_{tR}(G_l^d) \leq 2l+1$, consider the following TRD-function on G_l^d: If l is even, place two 2's in each of the first $\frac{l}{2} - 1$ rows, and one 2 in each of rows $\frac{l}{2}$ and $\frac{l}{2}+1$, such that they span columns 2 through $l+1$. At this point, every vertex in G_l^d is dominated. However, the 2's in rows $\frac{l}{2}$ and $\frac{l}{2}+1$ are isolated, so place a 1 in row $\frac{l}{2}$ such that it shares a column with the 2 in row $\frac{l}{2}+1$. Otherwise, if l is odd, place two 2's in each of the first $\frac{l-1}{2}$ rows, and one 2 in row $\frac{l+1}{2}$, such that they span columns 2 through $l+1$. Similarly to the even case, every vertex in G_l^d is now dominated. However, the 2 in row $\frac{l+1}{2}$ is isolated, so place a 1 in row $\frac{l-1}{2}$ such that it shares a column with that 2. In either case, we have a TRD-function on G_l^d with weight $2l+1$; hence $\gamma_{tR}(G_l^d) \leq 2l+1$.

Now, suppose for a contradiction that $\gamma_{tR}(G_l^d) < 2l+1$, and consider a TRD-function $f = (V_f^0, V_f^1, V_f^2)$ on G_l^d with $\omega(f) = 2l$. We claim that $f(v_{j1}) = 0$ for all $1 \leq j \leq \lfloor \frac{l}{2} \rfloor + 1$. If $f(v_{j1}) = 2$ for $x \geq 1$ vertices in column 1, the undominated vertices in columns 2 through $l+1$ form the graph $K_l \square K_{l+1-x}$. By Proposition 8.2, a TRD-function on $K_l \square K_{l+1-x}$ requires a weight of $2\min\{l, l+1-x\} = 2(l+1-x)$. However, since $2x + 2(l+1-x) > 2l$, this is impossible. Therefore $f(v_{j1}) \neq 2$ for all $1 \leq j \leq \lfloor \frac{l}{2} \rfloor + 1$. If $f(v_{j1}) = 1$ for $x \geq 1$ vertices in column 1, the undominated vertices in columns 2 through $l+1$ (that is, those for which f could be assigned a 2) form the graph $K_l \square K_{l+1}$. Again by Proposition 8.2, a TRD-function on $K_l \square K_{l+1}$ requires a weight of $2\min\{l, l+1\} = 2l$. However, $x + 2l > 2l$ for $x \geq 1$, so this is also not possible. Therefore, $f(v_{j1}) = 0$ for all $1 \leq j \leq \lfloor \frac{l}{2} \rfloor + 1$.

As a result, in order to totally Roman dominate the first column, there must be a 2 in each of the first $\lfloor \frac{l}{2} \rfloor + 1$ rows, none of which can be in the first column. That

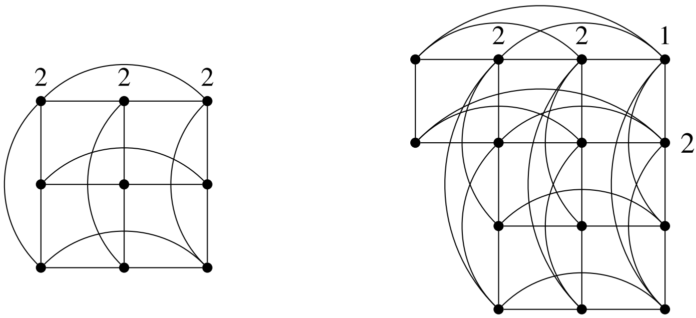

Figure 1. The graphs G_3 and G_3^d with a γ_{tR}-function.

is, for each $1 \le s \le \lfloor \frac{l}{2} \rfloor + 1$, $f(v_{st}) = 2$ for some $2 \le t \le l+1$. Let S be the set of these vertices. Note that, thus far, we have accounted for a total weight of

$$2\left(\left\lfloor \tfrac{l}{2} \right\rfloor + 1\right) = \begin{cases} l+2 & \text{if } l \text{ is even,} \\ l+1 & \text{if } l \text{ is odd,} \end{cases}$$

which leaves a weight of $l-2$ if l is even and $l-1$ if l is odd to be assigned. That is, a weight of $2\left(\left\lceil \frac{l}{2} \right\rceil - 1\right)$ remains to be accounted for. We now claim that no two vertices in S can be in the same column. If the vertices in S span fewer than $\lfloor \frac{l}{2} \rfloor + 1$ columns, then the vertices which are undominated by S induce a graph containing $K_{\lceil l/2 \rceil} \Box K_{\lceil l/2 \rceil}$ as subgraph. If $l = 2$, then no weight remains to dominate this vertex, as $2\left(\left\lceil \frac{l}{2} \right\rceil - 1\right) = 0$. Otherwise, if $l > 2$, Proposition 8.2 implies that $\gamma_{tR}(K_{\lceil l/2 \rceil} \Box K_{\lceil l/2 \rceil}) = 2\left(\left\lceil \frac{l}{2} \right\rceil\right)$. However, $2\left(\left\lceil \frac{l}{2} \right\rceil\right) > 2\left(\left\lceil \frac{l}{2} \right\rceil - 1\right)$. In either case, this contradicts f being a TRD-function, and thus no vertices of S share a column.

Therefore, the vertices left undominated by S induce a graph $T \cong K_{\lceil l/2 \rceil} \Box K_{\lceil l/2 \rceil - 1}$, with $\lceil \frac{l}{2} \rceil$ rows and $\lceil \frac{l}{2} \rceil - 1$ columns. Moreover, the vertices in S are all isolated, as none share a row or column. By Proposition 8.2, $\gamma_{tR}(T) = 2\left(\left\lceil \frac{l}{2} \right\rceil - 1\right)$. Thus the entire remaining weight is required in order to dominate T; necessarily, the vertices in $V_f^+ - S$ belong to rows and columns that do not contain vertices in S. However, this still leaves the vertices in S isolated, which contradicts f being a TRD-function on G_l^d. Therefore $\gamma_{tR}(G_l^d) \ge 2l+1$ and we conclude that $\gamma_{tR}(G_l^d) = 2l+1$. As in the case where k is even, G_l^d is a spanning subgraph of a k-γ_{tR}-edge-critical graph with diameter 2. □

9. Future work

We showed in Section 5 that the disjoint union of two or more complete graphs, each having order at least 3, is γ_{tR}-edge-supercritical. We also explained that a proof similar to that of Proposition 5.1 does not work for total Roman domination. Hence we pose the following question.

Question 1. *Are the disjoint unions of two or more complete graphs, each having order at least* 3, *the only* γ_{tR}*-edge-supercritical graphs*?

Note that if this is the case, Proposition 6.5 automatically becomes a necessary and sufficient condition for a graph to be 5-γ_{tR}-edge-critical.

Now consider, for a moment, Roman dominating functions, and suppose a graph G has nonadjacent vertices u and v such that $f(u) = f(v) = 0$ for every γ_R-function f on G. We claim that $\gamma_R(G+uv) = \gamma_R(G)$. Suppose $\gamma_R(G+uv) < \gamma_R(G)$ and let f be a γ_R-function on $G + uv$. Similar to Proposition 2.2, we may assume without loss of generality that $f(u) = 2$ and $f(v) = 0$, otherwise f is an RD-function on G such that $\omega(f) < \gamma_R(G)$. However, the function f' defined by $f'(v) = 1$ and $f'(y) = f(y)$ for all other $y \in V(G)$ is a γ_R-function on G such that $f'(v) > 0$, contrary to our assumption. The situation for total Roman domination is different.

For a graph G, we define $u \in V(G)$ to be a *dead vertex* if every γ_{tR}-function f on G has $f(u) = 0$. Not only do there exist graphs G containing nonadjacent dead vertices u and v such that $\gamma_{tR}(G + uv) < \gamma_{tR}(G)$, but it is possible to find such a graph G with $\gamma_{tR}(G + uw) < \gamma_{tR}(G)$ for every edge $uw \in E(\overline{G})$; that is, every edge in $E(\overline{G})$ incident with the dead vertex u is critical. We define the graph D_n below and show that D_n is such a graph.

Let D_n be the graph composed of $n \geq 2$ copies of $K_4 - e$ sharing a single central vertex as follows: let c be the central vertex, $w_1, \ldots, w_n$ be the degree-2 vertices, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be the remaining vertices (where u_i and v_i are adjacent for each i) such that c, u_i, w_i, v_i, c is a 4-cycle in D_n for each $1 \leq i \leq n$. See Figure 2.

Proposition 9.1. *If* $n \geq 2$, *then* $\gamma_{tR}(D_n) = 2n + 1$. *Moreover*, w_i *is a dead vertex for each* $1 \leq i \leq n$.

Proof. To see that $\gamma_{tR}(D_n) \leq 2n + 1$, consider the TRD-function $g : V(D_n) \to \{0, 1, 2\}$ on D_n defined by $g(c) = 1$, $g(u_i) = 2$ for $1 \leq i \leq n$, and $g(y) = 0$ for all other $y \in V(D_n)$.

We claim that, if f is a TRD-function on D_n with $\omega(f) \leq 2n+1$, then $f(c) = 1$. If $f(c) = 2$, then the only vertices that remain undominated in D_n are w_i for $1 \leq i \leq n$. However, since $d(w_i, w_j) = 4$ for all $i \neq j$, a weight of $2n$ is required in order to totally Roman dominate these vertices, contradicting $\omega(f) \leq 2n + 1$. If $f(c) = 0$, then since $D_n - c$ is the disjoint union of n triangles, Proposition 3.1 implies that a weight of $3n$ is required to totally Roman dominate the remaining vertices, contradicting $\omega(f) \leq 2n + 1$. Therefore $f(c) = 1$. Since a weight of at least $2n$ is required to totally Roman dominate the remaining disjoint union of n triangles, we conclude that $\gamma_{tR}(D_n) = 2n + 1$.

Now, let f be any γ_{tR}-function on D_n. Then $\omega(f) = 2n + 1$ and $f(c) = 1$. To dominate each triangle of $D_n - c$ with a weight of 2, $\{f(u_i), f(v_i)\} = \{0, 2\}$ and $f(w_i) = 0$ for each $1 \leq i \leq n$. Hence each w_i is a dead vertex. □

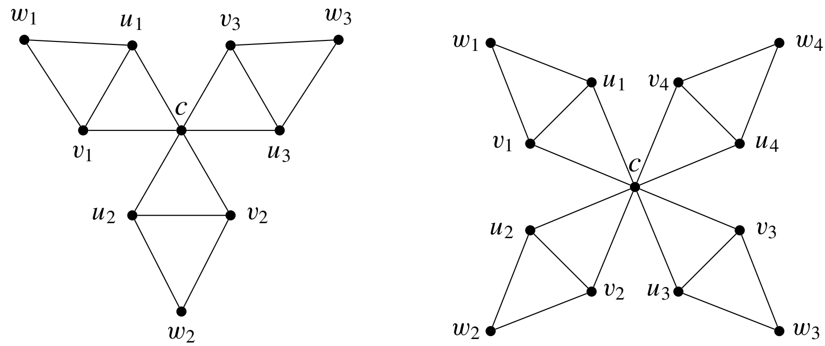

Figure 2. The graphs D_3 and D_4.

The following result shows that, for $n \geq 3$, every edge in $E(\overline{D}_n)$ incident with w_i is critical.

Proposition 9.2. *If $n \geq 3$, $i \in \{1, \dots, n\}$, and $w_i v \in E(\overline{D}_n)$, then $\gamma_{tR}(D_n + w_i v) < \gamma_{tR}(D_n)$.*

Proof. Without loss of generality, consider an edge $w_1 v \in E(\overline{D}_n)$. Then (without loss of generality) $v \in \{w_2, u_2, c\}$. If $v = w_2$, define $f : V(D_n + w_1 v) \to \{0, 1, 2\}$ by $f(w_1) = f(w_2) = 1$, $f(c) = f(u_3) = \cdots = f(u_n) = 2$, and $f(y) = 0$ for all other $y \in V(D_n)$. Otherwise, if $v \in \{u_2, c\}$, define $f : V(D_n + w_1 v) \to \{0, 1, 2\}$ by $f(c) = f(u_2) = f(u_3) = \cdots = f(u_n) = 2$ and $f(y) = 0$ for all other $y \in V(D_n)$. In either case, f is a TRD-function on $D_n + w_1 v$ and $\omega(f) = 2n$. Therefore, by Proposition 9.1, every edge $w_i v \in E(\overline{D}_n)$ is critical. □

However, for $n \geq 3$, the graph D_n is not γ_{tR}-edge-critical since (for example) $\gamma_{tR}(D_n + u_1 u_2) = 2n + 1$. Furthermore, the graph D_2 is not γ_{tR}-edge-critical since (for example) $\gamma_{tR}(D_2 + w_1 w_2) = 5$. However, adding edges to D_n until a $(2n+1)$-γ_{tR}-edge-critical graph D'_n is obtained results in D'_n having no dead vertices. Hence we pose the following question.

Question 2. *Do there exist γ_{tR}-edge-critical graphs containing dead vertices*?

We characterized γ_{tR}-edge-critical spiders in Theorem 7.5. Finding other classes of γ_{tR}-edge-critical trees and, indeed, characterizing γ_{tR}-edge-critical trees, remain open problems.

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Received: 2019-07-23 Accepted: 2019-09-26

chloe.lampman3@gmail.com — *Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada*

kieka@uvic.ca — *Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada*

sogden@uvic.ca — *Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada*

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

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nonprofit scientific publishing

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2019 vol. 12 no. 8