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Generalized Cantor functions: random function iteration

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We provide a generalization of the classical Cantor function. One characterization of the Cantor function is generated by a sequence of real numbers that starts with a seed value and at each step randomly applies one of two different linear functions. The Cantor function is defined as the probability that this sequence approaches infinity. We generalize the Cantor function to instead use a set of any number of linear functions with integer coefficients. We completely describe the resulting probability function and give a full explanation of which intervals of seed values lead to a constant probability function value.

Generalizing the Cantor function

Most analysis books include a discussion of the Cantor set (or Cantor middle-thirds set) as a closed subset of the interval [0, 1] whose complement has total length 1 but has an uncountably infinite number of points; see, for example, [Royden 1988]. The classical Cantor function is defined on the complement of the Cantor set. There are many different descriptions of the Cantor function; for a very complete overview, see [Dovgoshey et al. 2006]. In its simplest form, it can be described iteratively as the function $f(x)$ that has height $\frac{1}{2}$ on the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$, height $\frac{1}{4}$ on the interval $\left(\frac{1}{9}, \frac{2}{9}\right)$ $\left(\text{which is the middle third of the interval } \left(0, \frac{1}{3}\right)\right)$, height $\frac{3}{4}$ on the middle-third of the interval $\left(\frac{2}{3}, 1\right)$, or equivalently height $\frac{3}{4}$ on interval $\left(\frac{7}{9}, \frac{8}{9}\right)$. Continue in this manner, defining the height of $f(x)$ on the middle-third of a remaining interval to be the average of the heights on either side. So for example, since $f(x)$ has a height of $\frac{1}{2}$ on $\left(\frac{1}{3}, \frac{2}{3}\right)$, and a height of $\frac{3}{4}$ on $\left(\frac{7}{9}, \frac{8}{9}\right)$, it must have a height of $\frac{5}{8}$ on the middle third of the interval $\left(\frac{2}{3}, \frac{7}{9}\right)$, or the interval $\left(\frac{19}{27}, \frac{20}{27}\right)$. Continue this process indefinitely to get a function that has steps of height $\gamma/2^k$ for any integer $\gamma < 2^k$ and $k \in \mathbb{N}$. See Figure 1 for a graph of the Cantor function.

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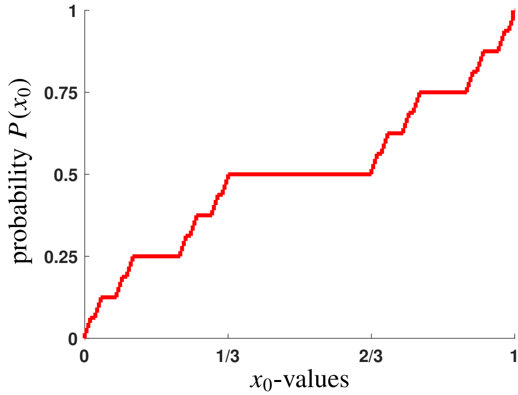
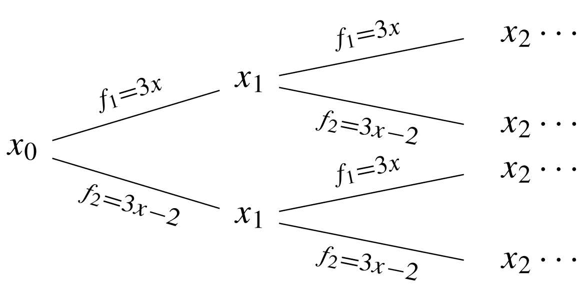


Figure 1. Original Cantor function from [Stankewitz and Rolf 2012] computed using $f_1(x) = 3x$ and $f_2(x) = 3x - 2$.

An alternative description of the Cantor function is found in [Stankewitz and Rolf 2012], in which the function arises from a random iterative process. We recall the fundamentals of single function iteration: Starting with a seed value x_0, compute $x_1 = f(x_0)$ and $x_2 = f(x_1) = f(f(x_0))$, and so forth, so that

$$x_n = \underbrace{f(f(\cdots f}_{n}(x_0))).$$

We can then examine the long-term behavior of the sequence $\{x_n\}$. To get to the description of the Cantor function found in [Stankewitz and Rolf 2012], we instead start with a seed value x_0, randomly choose a function from a finite set of functions, apply that function to x_0 to find x_1, and then repeat the process indefinitely. In particular, for the Cantor function, at each step we choose to compute either $f_1(x_j)$ or $f_2(x_j)$, with $f_1(x) = 3x$ and $f_2(x) = 3x - 2$. This is diagrammed in the figure below, where the tree would continue to the right:



Now we determine, out of all possible branches on the tree, what proportion of them will diverge to positive infinity. Equivalently, what is the probability of picking a sequence of functions such that $\{x_j\} \to \infty$? We describe this process formally.

Definition 1 (Cantor function). Let $f_1(x) = 3x$ and $f_2(x) = 3x - 2$. Given a seed value x_0, let $x_1 = f_{k_1}(x_0)$, $x_2 = f_{k_2}(x_1)$, and so forth, so that $x_j = f_{k_j}(x_{j-1})$, where $k_j \in \{1, 2\}$ is chosen with equal probability. For each initial value x_0, we define $P(x_0)$ as the probability that the sequence $\{x_j\}$ diverges to ∞ using the seed value x_0.

In this work, we generalize Definition 1 to n linear functions instead of the two functions $f_1(x) = 3x$ and $f_2(x) = 3x - 2$. We begin with a concrete example and the computation of a few values of $P(x_0)$.

Example 2. Consider the functions $f_1(x) = 20x - 1$, $f_2(x) = 6x - 3$, and $f_3(x) = 10x - 8$. We evaluate $P(x_0)$ for various values of x_0.

(a) Suppose $x_0 > 1$. Then $f_1(x_0) > 19$, $f_2(x_0) > 3$, and $f_3(x_0) > 2$. No matter which f_k we choose, we have $x_1 = f_k(x_0) > 2$. At the next step, if $x_2 = f_1(x_1)$, then $x_2 > 20(2) - 1 = 39$, if $x_2 = f_2(x_1)$, then $x_2 > 6(2) - 3 = 9$, and if $x_2 = f_3(x_1)$, then $x_3 > 10(2) - 8 = 12$. Each successive x_j is larger than x_{j-1}, thus $\{x_j\} \to \infty$ and $P(x_0) = 1$.

(b) Suppose $x_0 = 0.25$. If $x_1 = f_1(x_0) = 20x_0 - 1$, then $x_1 = 4 > 1$, and the rest of the sequence will diverge to infinity as in part (a) of this example. However, if $x_1 = f_2(x_0)$ then $x_1 = -1.5$ and if $x_1 = f_3(x_0)$ then $x_1 = -5.5$. We note that all of the functions f_1, f_2, and f_3 have a negative y-intercept and a positive slope, meaning that a negative input into each of these functions will result in a negative output. Thus if $x_1 = f_2(0.25)$ or if $x_1 = f_3(0.25)$, there is no way for the sequence $\{x_j\}$ to have any positive values after x_0, and thus the sequence cannot diverge to infinity. Combining the three different choices for x_1, we see that one of the choices leads to $\{x_j\} \to \infty$, while the other two do not, so $P(0.25) = \frac{1}{3}$.

For this set of functions, the graph of $P(x_0)$ is shown in Figure 2 (see next page). In the figure, we can see long intervals of constancy for $P(x_0)$ at heights of $\frac{1}{3}$ and $\frac{2}{3}$, and shorter intervals of constancy at heights of $\gamma/9$ for $\gamma \in \mathbb{N}$ and $\gamma < 9$.

We note that in Example 2, the three functions were chosen such that any $x_j > 1$ would result in that sequence diverging to infinity, and any $x_j < 0$ would result in the remainder of that sequence being negative. Thus the interval of interest for those three functions is the interval [0, 1], since the probability function is $P(x_0) = 1$ for all $x_0 > 1$ and $P(x_0) = 0$ for all $x_0 < 0$. Furthermore, if the input of the function is $x_0 \in [0, 1]$, then the outputs are ordered such that $f_3(x_0) < f_2(x_0) < f_1(x_0)$, as can be seen in Example 2(b) with $x_0 = 0.25$. That is, there is some ordering to our functions over the interval [0, 1]. The following definition does not quite have the same consequence of having $f_3(x_0) < f_2(x_0) < f_1(x_0)$ for all $x_0 \in [0, 1]$, but applies a restriction on the functions that forces an order in which sequences involving the different functions diverge to infinity.

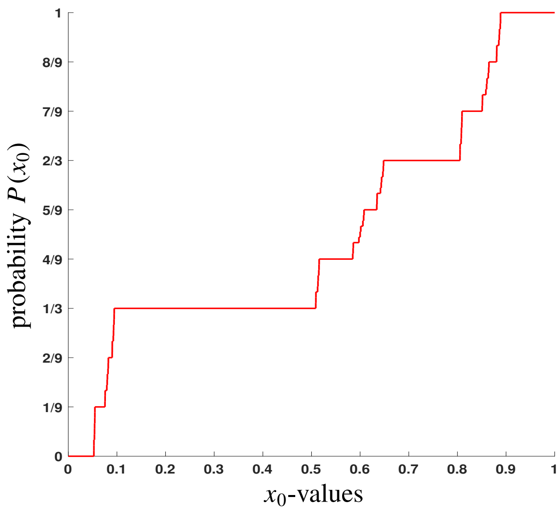


Figure 2. $P(x_0)$ over the interval [0, 1] given $f_1(x) = 20x - 1$, $f_2(x) = 6x - 3$, and $f_3(x) = 10x - 8$.

Definition 3. Define the functions $f_k(x) = a_k x - b_k$ for $k \in \{1, 2, \ldots, n\}$, with $a_k, b_k \in \mathbb{N} \cup \{0\}$, $a_k > b_k$, and

$$\frac{b_k + 1}{a_k} < \frac{b_{k+1}}{a_{k+1}} \tag{1}$$

for all $k \in \{1, 2, \ldots, n-1\}$.

In much of the work that follows, the notions of preimages and of fixed points will be critical.

Definition 4. Let f_k be a function as in Definition 3:

(a) The *preimage* of a number $y \in \mathbb{R}$ is the value x such that $f_k(x) = y$, and is denoted by $x = f_k^{-1}(y)$.

(b) For a given function $f_k(x)$, a *fixed point* of the function is a point p_k such that $f(p_k) = p_k$.

The following lemma formalizes the consequences of the restrictions on a_k and b_k in Definition 3.

Lemma 5. *Let f_k be given as in Definition 3. Then $f_k^{-1}(1) < f_{k+1}^{-1}(0)$ and consequently, if $0 < f_k(x) < 1$ then $f_{k+1}(x) < 0 < f_k(x) < 1 < f_{k-1}(x)$.*

Proof. Given $f_k(x) = a_k x - b_k$, we compute $f_k^{-1}(0)$ by solving $0 = a_k x - b_k$ to get $x = b_k/a_k$. Thus $f_k^{-1}(0) = b_k/a_k$. A similar computation where we solve $1 = a_k x - b_k$ gives $f_k^{-1}(1) = (b_k + 1)/a_k$. Thus condition (1) ensures

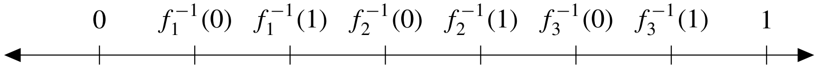


Figure 3. The ordering the preimages of 0 and 1 for three functions in Definition 3.

that $f_k^{-1}(1) < f_{k+1}^{-1}(0)$. Figure 3 illustrates these observations for three functions.

The condition that $0 < f_k(x) < 1$ is equivalent to $f_k^{-1}(0) < x < f_k^{-1}(1)$. Furthermore, the condition that $f_k^{-1}(1) < f_{k+1}^{-1}(0)$ ensures that

$$f_{k-1}^{-1}(1) < f_k^{-1}(0) < x < f_k^{-1}(1) < f_{k+1}^{-1}(0).$$

The left-hand part of the equation ensures that $f_{k-1}^{-1}(1) < x$, or equivalently, that $1 < f_{k-1}(x)$. The right-hand part of the equation ensures that $x < f_{k+1}^{-1}(0)$ or that $f_{k+1}(x) < 0$ as desired. □

The following definition mimics the definition of the Cantor function, but gives the definition of a sequence and a probability function $P(x_0)$ for the more general case of n functions described in Definition 3.

Definition 6. Let f_k be given as in Definition 3. Given a seed value x_0, let $x_1 = f_{k_1}(x_0)$, $x_2 = f_{k_2}(x_1)$, and so forth, so that $x_j = f_{k_j}(x_{j-1})$, where $k_j \in \{1, 2, \ldots, n\}$ is chosen with equal probability. For each initial value x_0, we define $P(x_0)$ as the probability that the sequence $\{x_j\}$ diverges to ∞ using the seed value x_0.

The following lemma formalizes the behavior of $P(x_0)$ as seen in the examples above. In particular, part (c) of Lemma 7 is particularly useful when we want to find $P(x_0)$, since it permits us to find the largest m for which the sequence $\{f_m(x_{j-1})\}$ diverges to infinity, and then draw conclusions about the behavior of sequences where the index of the function is restricted to the set $\{1, 2, \ldots, m\}$.

Lemma 7. *Let f_k be given as in Definition 3 and the sequence $\{x_j\}$ be given as in Definition 6. Then the following are true*:

(a) *If $x_0 \leq 0$, then $P(x_0) = 0$.*

(b) *If $x_0 \geq 1$, then $P(x_0) = 1$.*

(c) *Suppose a seed value x_0 has the property that if the sequence $\{x_j\}$ is formed by $x_j = f_m(x_{j-1})$ for a fixed value of m, then $\lim_{j\to\infty} x_j = \infty$. Then any sequence $\{x_j\}$ formed by $x_j = f_{k_j}(x_{j-1})$, where $k_j \in \{1, 2, \ldots, m\}$, will also have $\lim_{j\to\infty} x_j = \infty$.*

Proof. We prove each part individually.

Proof of (a): Suppose $x_0 \leq 0$. By Definition 3, $x_1 \leq -b_k < 0$, so any sequence $\{x_j\}$ will have all negative terms. Thus $P(x_0) = 0$.

Proof of (b): Suppose $x_0 \geq 1$. By definition, $x_1 \geq a_{k_1} - b_{k_1} \geq 1$. Applying Definition 3 gives $x_j > 1$ for all j. We now show that $x_{j-1} < x_j$, or equivalently that $x_{j-1} < a_{k_j} x_{j-1} - b_{k_j}$. By way of contradiction, suppose instead that $x_{j-1} \geq a_{k_j} x_{j-1} - b_{k_j}$, which is equivalent to $x_{j-1} \leq b_{k_j}/(a_{k_j} - 1) < 1$, and this contradicts the assumption of $x_{j-1} \geq 1$. Thus the sequence $\{x_j\}$ is increasing. Since the functions are linear with positive integer slope, the sequence increases without bound, and $P(x_0) = 1$.

Proof of (c): Suppose that x_0 has the property that if the sequence $\{x_j\}$ is formed by $x_j = f_m(x_{j-1})$ for a fixed value of m, then $\lim_{j\to\infty} x_j = \infty$. By part (a), we know that all terms of the sequence are positive. Since $f_m(x_0) > 0$, we have $f_k(x_0) > 1$ for all $k \in \{1, 2, \ldots, m-1\}$ by Lemma 5. Thus any sequence $\{x_j\} = \{f_{k_j}(x_{j-1})\}$ with $k_j \in \{1, 2, \ldots, m\}$ would have $\lim_{j\to\infty} x_j = \infty$. □

Finally, the notion of a fixed point in Definition 4 will be important in determining for which values of x_0 the function $P(x_0)$ will have intervals of constancy, as seen in Figures 1 and 2. Since the fixed points of f_k result in the output being equal to the input, if $x_{j-1} = p_k$, then $x_j = p_k$ as long as the same function has been chosen for the next iteration. Thus if we had only one function f_k in Definition 6, then for $x_{j-1} > p_k$ we would have $x_j > x_{j-1}$ and thus $P(x_0) = 1$ (because the x_j sequence relies on linear functions with integer coefficients, so if it is increasing it must be increasing without bound); similarly for $x_{j-1} < p_k$ we would have $x_j < x_{j-1}$ and thus $P(x_0) = 0$. Of course, our situation is more complicated, since we will randomly choose from among n different functions at each x_{j-1}.

Example 8. Consider the same functions from Example 2:

$$f_1(x) = 20x - 1, \quad f_2(x) = 6x - 3, \quad f_3(x) = 3x - 2.$$

We calculate their fixed points. Since the general form of the fixed points for the functions in Definition 3 is $p_k = b_k/(a_k - 1)$, we have $p_1 = \frac{1}{19}$, $p_2 = \frac{3}{5}$, and $p_3 = \frac{8}{9}$. We also note that in Figure 2, the step at height 0 ends at the fixed point $p_1 = \frac{1}{19}$, and the step at height 1 begins at $p_3 = \frac{8}{9}$. As we shall see in Theorems 11 and 12, this is not a coincidence.

Figure 4 illustrates how the values of p_k are related to $f_k^{-1}(0)$ and $f_k^{-1}(1)$ for three functions. Since each f_k is an increasing function and $f_k^{-1}(p_k) = p_k$, we see that $f_k^{-1}(0) < p_k < f_k^{-1}(1)$.

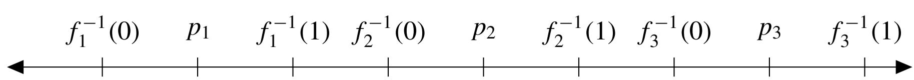


Figure 4. The ordering of the preimages of 0 and 1 and fixed points for three functions.

Probabilities for three functions

In this section, we completely describe the function $P(x_0)$ when $n=3$ in Definition 3. We first prove that the function $P(x_0)$ is nondecreasing. This proof does not rely on the number of functions and is valid for any set of n functions.

Theorem 9. *The function $P(x_0)$ given in Definition 6 is nondecreasing. That is, if $y_0 < z_0$, then $P(y_0) \le P(z_0)$.*

Proof. By Lemma 7, we only need to consider the case where $0 < y_0 < z_0 < 1$. Consider a particular sequence of functions $\{f_{k_j}\}$ for a fixed sequence $\{k_j\}$ with each $k_j \in \{1, 2, \dots, n\}$. Let the sequence given in Definition 6 be identified as $\{y_j\}$ if the seed value is y_0 and $\{z_j\}$ if the seed value is z_0. If $\lim_{j\to\infty}\{y_j\} = \infty$, then the same is true for $\lim_{j\to\infty}\{z_j\}$ by Lemma 7. If $\lim_{j\to\infty}\{y_j\} \ne \infty$, we only need to consider the case where $\lim_{j\to\infty}\{z_j\} \ne \infty$ (because if $\lim_{j\to\infty}\{z_j\} = \infty$, then $\lim_{j\to\infty}\{y_j\} \le \lim_{j\to\infty}\{z_j\}$). In the case where both limits are finite, we know via Definition 3 that $f_k(y_0) < f_k(z_0)$ for all f_k. Thus in this case, we have that $\lim_{j\to\infty}\{y_j\} \le \lim_{j\to\infty}\{z_j\}$, because limits preserve inequalities.

For any one fixed sequence of functions $\{f_{k_j}\}$, we have that $\lim_{j\to\infty}\{y_j\} \le \lim_{j\to\infty}\{z_j\}$. This is true for all sequences of functions; thus $P(y_0) \le P(z_0)$. □

The following theorem provides the foundation for the results in Theorem 13.

Theorem 10. *Consider three linear functions f_k defined by Definition 3 and $P(x_0)$ defined by Definition 6:*

(a) *For $x_0 \in \mathbb{R}$, $P(x_0) = \frac{1}{3}$ if and only if $x_0 \in I_1 = [f_1^{-1}(p_3), f_2^{-1}(p_1)]$.*

(b) *For $x_0 \in \mathbb{R}$, $P(x_0) = \frac{2}{3}$ if and only if $x_0 \in I_2 = [f_2^{-1}(p_3), f_3^{-1}(p_1)]$.*

We will prove part (a), as the proof of part (b) is similar.

Proof. Let $x_0 \in \mathbb{R}$. First, suppose $x_0 \in I_1 = [f_1^{-1}(p_3), f_2^{-1}(p_1)]$. We consider three cases.

<u>Case 1</u>: $x_1 = f_1(x_0)$. Consider the lower bound of I_1, which is $x_0 \ge f_1^{-1}(p_3)$. Thus,

$$x_1 = f_1(x_0) \ge f_1(f_1^{-1}(p_3)) = p_3.$$

Since all elements in $f_1(I_1)$ are greater than or equal to p_3, consider the sequence defined by $x_j = f_3(x_{j-1})$. When $x_{j-1} > p_3$, we have $x_j = f_3(x_{j-1}) > x_{j-1}$, and the sequence increases without bound due to the restrictions on a_k and b_k. By Lemma 7, if the sequence is defined by $x_j = f_{k_j}(x_{j-1})$, with $x_j \in \{1, 2, 3\}$, this sequence also increases without bound and $P(x_0) = 1$. In the particular case where $x_0 = f_1^{-1}(p_3)$ and $x_1 = f_1(x_0) = p_3$, the only sequence that does not diverge to infinity is the one where $x_j = f_3(x_{j-1})$ for all $j \ge 2$. But if at any point one element of the sequence is $x_j = f_2(x_{j-1})$ or $x_j = f_1(x_{j-1})$, then $x_j > p_3$ and the sequence will diverge to infinity. But the sequence where $x_1 = p_3$ and $x_j = f_3(x_{j-1})$ for all $j \geq 2$ occurs with 0 probability, so $P(x_0) = 1$ if $x_0 = f_1^{-1}(p_3)$ and $x_1 = f_1(x_0)$.

Case 2: $x_1 = f_2(x_0)$. Consider the upper bound of I_1, that is, $x_0 \leq f_2^{-1}(p_1)$. We have

$$f_2(x_0) \leq f_2(f_2^{-1}(p_1)) = p_1.$$

Since all elements of the interval $f_2(I_1)$ are less than or equal to p_1, we know that if $x_2 = f_1(x_1)$, the sequence $\{x_j\}$ cannot diverge to infinity. By Lemma 5, we know that $f_2(x_1)$ and $f_3(x_1)$ are negative, so no sequence of functions will cause $\{x_j\}$ to diverge to infinity. Thus $P(x_0) = 0$ for this case.

Case 3: $x_1 = f_3(x_0)$. Since the probability of the sequence $\{x_j\}$ diverging to infinity is 0 for any element of I_1 if $x_1 = f_2(x_0)$, the same is true for $x_1 = f_3(x_0)$ by Lemma 7. Thus $P(x_0) = 0$ in this case.

Since each of the cases has a probability of $\frac{1}{3}$ of occurring, the resultant probability is $\frac{1}{3}(1 + 0 + 0) = \frac{1}{3}$. We conclude that if $x_0 \in I_1 = [f_1^{-1}(p_3), f_2^{-1}(p_1)]$ then $P(x_0) = \frac{1}{3}$.

To show that if $P(x_0) = \frac{1}{3}$ then $x_0 \in I_1$, we instead show the contrapositive. To that end, we consider $x_0 \notin I_1$, so $x_0 < f_1^{-1}(p_3)$ or $x_0 > f_2^{-1}(p_1)$.

Case 1: $0 < x_0 < f_1^{-1}(p_3)$. For $0 < x_0 < f_1^{-1}(p_3)$, we know that if $x_1 = f_2(x_0)$ or $x_1 = f_3(x_0)$, then the limit of the sequence $\{x_j\}$ is finite, by Cases 2 and 3 above, and $P(x_0) = 0$ for those two cases. Now if $x_1 = f_1(x_0)$, we have that $f_1(0) < x_1 < f_1(f_1^{-1}(p_3)) = p_3$.

If the rest of the sequence is defined by $x_j = f_3(x_{j-1})$ for $j \geq 2$, then the sequence would be bounded above by p_3. Thus $P(x_0) < 1$ in this case. Putting together the three different scenarios for x_1, we see that $P(x_0) < \frac{1}{3}(1+0+0)$, so $P(x_0) < \frac{1}{3}$.

Case 2: $f_2^{-1}(p_1) < x_0 \leq 1$. We first recall that by Definitions 3 and 4 and Lemma 5, we have

$$f_1^{-1}(0) \leq p_1 \leq f_1^{-1}(1) < f_2^{-1}(0) \leq p_2 \leq f_2^{-1}(1) < f_3^{-1}(0) \leq p_3 \leq f_3^{-1}(1), \quad (2)$$

as illustrated in Figure 4.

If $f_2^{-1}(p_1) < x_0 \leq 1$, we consider the three different possibilities for x_1. If $x_1 = f_1(x_0)$, then

$$x_1 > f_1(f_2^{-1}(p_1)) \geq f_1(f_2^{-1}(0)) > f_1(f_1^{-1}(1)) = 1,$$

so by Lemma 7, $\{x_j\} \to \infty$. If $x_1 = f_2(x_0) > f_2(f_2^{-1}(p_1)) = p_1$, then $x_1 > p_1$. If $x_2 = f_1(x_1)$, then $x_2 > f_1(p_1) = p_1$ and the function is increasing; hence the sequence would diverge to ∞ if we continued to apply f_1. Thus $P(x_0)$ is strictly positive. So we have determined that for one of the cases $P(x_0) = 1$ and for one of the cases $P(x_0) > 0$. Regardless of the outcome of the third case (where $x_1 = f_3(x_0)$), we know that $P(x_0) > \frac{1}{3}(1+0)$. Therefore if $x_0 > f_2^{-1}(p_1)$ then $P(x_0) > \frac{1}{3}$. We have shown the contrapositive is true, so if $P(x_0) = \frac{1}{3}$, then $x_0 \in I_1$, as desired.

The proof of part (b) of the theorem uses similar reasoning to that in part (a). □

The following two theorems combine to prove for which x_0-values $P(x_0)$ is either 0 or 1.

Theorem 11. *Consider three linear functions f_k defined by Definition 3 and $P(x_0)$ defined by Definition 6. For $x_0 \in \mathbb{R}$, $P(x_0) = 0$ if and only if $x_0 \leq p_1$.*

Proof. First, assume that $x_0 \leq p_1$. We note that by (2),

$$f_2(x_0) \leq f_2(p_1) \leq f_2(f_1^{-1}(1)) < f_2(f_2^{-1}(0)) = 0.$$

The same would be true if we replaced f_2 with f_3, so by Lemma 7 we have $P(x_0) = 0$ in these cases. So we only need to consider what happens if $x_1 = f_1(x_0)$. Since f_1 is an increasing function, we have $f_1(x_0) \leq f_1(p_1) = p_1$. Thus all terms in the sequence $\{x_j\}$ are bounded above by p_1, so $P(x_0) = 0$.

To show that if $P(x_0) = 0$ then $x_0 \leq p_1$, we instead show the contrapositive. Suppose that $x_0 > p_1$. The sequence $\{f_1(x_{j-1})\}$ will have the property that $x_j > x_{j-1}$ for all j. Since $f_1(x)$ is a linear function with positive integer slope, the fact that $\{x_j\}$ is increasing and positive means that it is increasing without bound, so $P(x_0) > 0$. Thus if $P(x_0) = 0$, we know that $x_0 \leq p_1$. □

Theorem 12. *Consider three linear functions f_k defined by Definition 3 and $P(x_0)$ defined by Definition 6. For $x_0 \in \mathbb{R}$, $P(x_0) = 1$ if and only if $x_0 \geq p_3$.*

Proof. We first assume that $x_0 \geq p_3$. Consider the sequence $\{x_j\}$ defined by $x_j = f_3(x_{j-1})$ for all $j \geq 1$. Since $f_3(x_0) \geq f_3(p_3) = p_3$, we know that each term of the sequence is greater than p_3, except in the situation where $x_0 = p_3$. Furthermore, by the restrictions of the coefficients of f_3, the sequence $\{x_j\}$ increases without bound. By Lemma 7, any sequence with initial value x_0 will increase without bound since $\{f_3(x_j)\}$ is unbounded so $P(x_0) = 1$. If $x_0 = p_3$, the sequence $\{f_3(x_{j-1})\}$ will remain as p_3, but if any one of the $f_{k_j}(x_{j-1})$ of the sequence is f_2 or f_1, the sequence will once again increase without bound. Since the probability of the sequence of functions having all $k_j = 3$ is 0, $P(x_0) = 1$ when $x_0 = p_3$.

Now we prove the other direction of the biconditional by proving the contrapositive. Suppose that $x_0 < p_3$ and write $x_0 = p_3 - \varepsilon$, where $\varepsilon > 0$. We consider the sequence of $\{x_j\}$ where the first several terms are found by applying f_3 to the previous term. Observe that

$$x_1 = f_3(x_0) = a_3(p_3 - \varepsilon) - b_3 = p_3 - a_3\varepsilon,$$
$$x_2 = a_3(p_3 - a_3\varepsilon) - b_3 = p_3 - a_3^2\varepsilon.$$

The general term is $x_k = p_3 - a_3^k \varepsilon$, which will be negative for a finite value of k. Such a sequence will be bounded above, and so will not diverge to infinity. These bounded sequences occur with nonzero probability; thus $P(x_0) < 1$. □

Now that we have established for which intervals the probability $P(x_0) \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, we provide a theorem which demonstrates how to recursively find intervals for where $P(x_0) = \gamma/3^k$ for all nonnegative integer choices of γ and k.

Theorem 13. *Consider three linear functions f_k defined by Definition 3 and $P(x_0)$ defined by Definition 6. Assume*

$$J = \left\{ x : P(x_0) = \frac{\gamma}{3^k} \right\},$$

where $\gamma \in \mathbb{N} \cup \{0\}$ and $3 \nmid \gamma$. Then

$$\left\{ y_0 : P(y_0) = \frac{\alpha}{3} + \frac{\gamma}{3^{k+1}} \right\} = \begin{cases} f_1^{-1}(J), & \alpha = 0, \\ f_2^{-1}(J), & \alpha = 1, \\ f_3^{-1}(J), & \alpha = 2. \end{cases}$$

Example 14. Suppose we wanted to find which x_0-values give rise to $P(x_0) = \frac{52}{81}$. We write

$$\begin{aligned} \tfrac{52}{81} &= \tfrac{1(27)}{81} + \tfrac{2(9)}{81} + \tfrac{2(3)}{81} + \tfrac{1(1)}{81} = \tfrac{1}{3} + \tfrac{25}{81} \\ &= \tfrac{1}{3} + \tfrac{1}{3}\left(\tfrac{25}{27}\right) = \tfrac{1}{3} + \tfrac{1}{3}\left(\tfrac{2}{3} + \tfrac{7}{27}\right) \\ &= \tfrac{1}{3} + \tfrac{1}{3}\left(\tfrac{2}{3} + \tfrac{1}{3}\left(\tfrac{2}{3} + \tfrac{1}{3} \cdot \tfrac{1}{3}\right)\right). \end{aligned}$$

Then $P(x_0) = \frac{52}{81}$ if and only if $x_0 \in f_2^{-1}(f_3^{-1}(f_3^{-1}(I_1)))$, where we have built the interval recursively from I_1 via Theorem 13.

Proof. We assume that J is as stated in Theorem 13. We need to show

$$\left\{ y_0 : P(y_0) = \frac{\alpha}{3} + \frac{\gamma}{3^{k+1}} \right\} = \begin{cases} f_1^{-1}(J), & \alpha = 0, \\ f_2^{-1}(J), & \alpha = 1, \\ f_3^{-1}(J), & \alpha = 2. \end{cases}$$

We start with the set containment of

$$\left\{ y_0 : P(y_0) = \frac{\alpha}{3} + \frac{\gamma}{3^{k+1}} \right\} \supseteq \begin{cases} f_1^{-1}(J), & \alpha = 0, \\ f_2^{-1}(J), & \alpha = 1, \\ f_3^{-1}(J), & \alpha = 2. \end{cases}$$

We proceed with proof by cases.

<u>Case 1</u>: $x_0 \in f_1^{-1}(J)$. Since $x_0 \in f_1^{-1}(J)$ and $J \subseteq [0, 1]$, we have

$$x_0 \in f_1^{-1}(J) \subseteq f_1^{-1}([0, 1]) \subseteq [0, 1].$$

Now $x_1 = f_\beta(x_0)$ for $\beta \in \{1, 2, 3\}$. We examine each β-value in subcases.

Subcase 1.1: $x_1 = f_1(x_0)$. Since $x_0 \in f_1^{-1}(J)$, we apply f_1 to both sides to get $x_1 = f_1(x_0) \in f_1(f_1^{-1}(J)) = J$. Since $x_1 \in J$, we have $P(x_1) = P(y)$ for any $y \in J$; thus $P(x_1) = \gamma/3^k$.

Subcase 1.2: $x_1 = f_2(x_0)$. Since x_0 is bounded above by $f_1^{-1}(1)$, we have $x_1 = f_2(x_0) < f_2(f_1^{-1}(1))$. By Lemma 5, $f_1^{-1}(1) < f_2^{-1}(0)$, so we conclude that

$$x_1 = f_2(x_0) < f_2(f_1^{-1}(1)) < f_2(f_2^{-1}(0)) = 0.$$

Since $x_1 < 0$, we conclude by Lemma 7 that the probability of the sequence $\{x_j\}$ diverging to ∞ is 0.

Subcase 1.3: $x_1 = f_3(x_0)$. As in Subcase 1.2, we conclude that

$$x_1 = f_3(x_0) < f_3(f_1^{-1}(1)) < f_3(f_3^{-1}(0)) = 0,$$

and the probability that $\{x_j\} \to \infty$ is 0.

We determined the probability that $\{x_j\} \to \infty$ for each of the three possible values of x_1, and we know each of these values has a probability of $\frac{1}{3}$ of being chosen. The value of $P(x_0)$ when $x_0 \in f_1^{-1}(J)$ is

$$P(x_0) = \frac{1}{3}\left(\frac{\gamma}{3^k} + 0 + 0\right) = \frac{\gamma}{3^{k+1}}.$$

Case 2: $x_0 \in f_2^{-1}(J)$. Once again, we have $x_0 \in f_2^{-1}(J) \subseteq f_2^{-1}([0, 1]) \subseteq [0, 1]$. Again, x_1 could come from applying any of the three functions to x_0, so we proceed with examining each subcase individually.

Subcase 2.1: $x_1 = f_1(x_0)$. Since $x_0 \in f_2^{-1}(J)$, and J is bounded below by 0, we know that in this case $x_0 > f_2^{-1}(0)$. Hence, by Lemma 5, $f_2^{-1}(0) > f_1^{-1}(1)$; thus $x_0 > f_1^{-1}(1)$. Therefore, $x_1 = f_1(x_0) > 1$ and we conclude the probability that $\{x_j\} \to \infty$ is 1.

Subcase 2.2: $x_1 = f_2(x_0)$. Using the same reasoning from Subcase 1.1, we see $x_1 = f_2(x_0) \in f_2(f_2^{-1}(J)) = J$. We conclude $P(x_1) = \gamma/3^k$.

Subcase 2.3: $x_1 = f_3(x_0)$. As demonstrated in Subcases 1.2 and 1.3, here $P(x_1) = 0$.

Combining the three subcases, we have

$$P(x_0) = \frac{1}{3}\left(1 + \frac{\gamma}{3^k} + 0\right) = \frac{1}{3} + \frac{\gamma}{3^{k+1}}.$$

Case 3: $x_0 \in f_3^{-1}(J)$. We use the methods of Cases 1 and 2 and get $P(x_1) = 1$ when $x_1 = f_1(x_0)$ and $x_1 = f_2(x_0)$, and $P(x_1) = \gamma/3^k$ when $x_1 = f_3(x_0)$. Thus

$$P(x_0) = \frac{1}{3}\left(1 + 1 + \frac{\gamma}{3^k}\right) = \frac{2}{3} + \frac{\gamma}{3^{k+1}}$$

when $x_0 \in f_3^{-1}(J)$.

Now we prove that

$$\left\{ y_0 : P(y_0) = \frac{\alpha}{3} + \frac{\gamma}{3^{k+1}} \right\} \subseteq \begin{cases} f_1^{-1}(J), & \alpha = 0, \\ f_2^{-1}(J), & \alpha = 1, \\ f_3^{-1}(J), & \alpha = 2. \end{cases}$$

We proceed with proof by contrapositive, which is if

$$x \notin f_1^{-1}(J) \cup f_2^{-1}(J) \cup f_3^{-1}(J)$$

then

$$P(x_0) \neq \frac{\alpha}{3} + \frac{\gamma}{3^{k+1}}. \tag{3}$$

We define $J = [r, s] \subseteq [0, 1]$, where $r, s \in \mathbb{R}$ and $0 \leq r < s \leq 1$. The assumption that $x \notin \bigcup_{i=1}^{3} f_i^{-1}(J)$ is equivalent to $x_0 \in (-\infty, f_1^{-1}(r))$ or $x_0 \in (f_1^{-1}(s), f_2^{-1}(r))$ or $x_0 \in (f_2^{-1}(s), f_3^{-1}(r))$ or $x_0 \in (f_3^{-1}(s), \infty)$. We examine each piece of this statement individually.

Case 1: $x_0 \in (-\infty, f_1^{-1}(r))$. We examine $x_1 = f_\beta(x_0)$ for $\beta \in \{1, 2, 3\}$ by subcases.

Subcase 1.1: $x_1 = f_1(x_0)$. We have

$$x_1 = f_1(x_0) < f_1(f_1^{-1}(r)) = r.$$

However, $J = [r, s]$ so the fact that $f_1(x_0) < r$ indicates that $P(x_0) < P(y)$, where $y \in J$ so $P(x_0) < \gamma/3^k$. As the probability is a nondecreasing function, we have for any $x_0 \in (-\infty, f_1^{-1}(r))$ that $0 \leq P(x_0) < \gamma/3^k$.

Subcase 1.2: $x_1 = f_2(x_0)$. Similarly to Subcase 1.1, we have

$$x_1 = f_2(x_0) < f_2(f_1^{-1}(r)) < f_2(f_2^{-1}(0)) = 0.$$

Thus, $x_1 < 0$ and we conclude $P(x_0) = 0$ for the upper bound of this interval. Since our probability function is a nondecreasing function, we conclude that $P(x_0) = 0$ for any x_0 in this interval by Lemma 7.

Subcase 1.3: $x_1 = f_3(x_0)$. The argument for this subcase is similar to that of Subcase 1.2, for

$$x_1 = f_3(x_0) < f_3(f_1^{-1}(r)) < f_3(f_3^{-1}(0)) = 0,$$

so $P(x_0) = 0$.

Now since each case represents a possible x_1 that all have an equal probability of $\frac{1}{3}$, we can multiply each case's probability by $\frac{1}{3}$ and sum them to get $P(x_0)$ for $x_0 \in (-\infty, f_1^{-1}(r))$. This gives

$$\frac{1}{3}(0+0+0) \leq P(x_0) < \frac{1}{3}\left(\frac{\gamma}{3^k} + 0 + 0\right) \quad \text{or} \quad 0 \leq P(x_0) < \frac{\gamma}{3^{k+1}}.$$

Thus for $x_0 \in (-\infty, f_1^{-1}(r))$, we have shown (3).

Case 2: $x_0 \in (f_1^{-1}(s), f_2^{-1}(r))$. We will examine $x_1 = f_\beta(x_0)$ for $\beta \in \{1, 2, 3\}$ by subcases.

Subcase 2.1: $x_1 = f_1(x_0)$. We examine the lower bound of our interval $f_1^{-1}(s) < x_0$. Since $x_1 = f_1(x_0)$, we apply the function f_1 to both sides of our inequality to get

$$x_1 = f_1(x_0) > f_1(f_1^{-1}(s)) = s.$$

However, $J = [r, s]$ so $f_1(x_0) > s$ indicates for any $y \in J$ that $P(x_0) > P(y) = \gamma/3^k$. As our probability function is nondecreasing, we have $\gamma/3^k < P(x_0) \leq 1$.

Subcase 2.2: $x_1 = f_2(x_0)$. We examine the upper bound of our interval $x_0 < f_2^{-1}(r)$. Again, we apply f_2 to both sides of the inequality to get

$$f_2(x_0) < f_2(f_2^{-1}(r)) = r.$$

Since $J = [r, s]$ and $f_2(x_0) < r$, we know for any $y \in J$, $P(x_0) < P(y) = \gamma/3^k$.

Thus, for any $x_0 \in (f_1^{-1}(s), f_2^{-1}(r))$, we have $0 \leq P(x_0) < \gamma/3^k$.

Subcase 2.3: $x_1 = f_3(x_0)$. We have $x_0 < f_2^{-1}(r)$, and we apply f_3 to both sides of the inequality to get $x_1 = f_3(x_0) < f_3(f_2^{-1}(r)) < f_3(f_3^{-1}(0)) = 0$. Thus $P(x_0) = 0$ by Lemma 7.

As in Case 1, we multiply each of the subcases by $\frac{1}{3}$ and sum them to see

$$\frac{1}{3}\left(\frac{\gamma}{3^k} + 0 + 0\right) < P(x_0) < \frac{1}{3}\left(\frac{\gamma}{3^k} + 1 + 0\right) \quad \text{or} \quad \frac{\gamma}{3^{k+1}} < P(x_0) < \frac{1}{3} + \frac{\gamma}{3^{k+1}}.$$

Thus for $x_0 \in (f_1^{-1}(s), f_2^{-1}(r))$, we have shown (3).

Case 3: $x_0 \in (f_2^{-1}(s), f_3^{-1}(r))$. The proof reduces to the subcases examined in Case 2, therefore again, we have

$$\frac{1}{3}\left(1 + \frac{\gamma}{3^k} + 0\right) < P(x_0) < \frac{1}{3}\left(1 + 1 + \frac{\gamma}{3^k}\right) \quad \text{or} \quad \frac{1}{3} + \frac{\gamma}{3^{k+1}} < P(x_0) < \frac{2}{3} + \frac{\gamma}{3^{k+1}}.$$

Thus for $x_0 \in (f_2^{-1}(s), f_3^{-1}(r))$, we have shown (3).

Case 4: $x_0 \in (f_3^{-1}(s), \infty)$. As before, we examine $x_1 = f_\beta(x_0)$ for $\beta \in \{1, 2, 3\}$ by subcases.

Subcase 4.1: $x_1 = f_1(x_0)$. We examine the lower bound of the interval: $x_0 > f_3^{-1}(s)$. Since $f_3^{-1}(s) > p_1$, we know $x_0 > p_1$; therefore $P(x_0) = 1$ for the lower bound.

Subcase 4.2: $x_1 = f_2(x_0)$. By the same reasoning as Subcase 4.1, since $f_3^{-1}(s) > p_2$, we know $P(x_0) = 1$ for the lower bound.

Subcase 4.3: $x_1 = f_3(x_0)$. We examine the lower bound of the interval: $x_0 > f_3^{-1}(s)$. Since $x_1 = f_3(x_0)$, we apply f_3 to both sides of the inequality to get $x_1 = f_3(x_0) > f_3(f_3^{-1}(s)) = s$. However, $J = [r, s]$ so the fact that $f_3(x_0) > s$

indicates for any $y \in J$ that $P(x_0) > P(y)$, which means $\gamma/3^k < P(x_0)$. Thus for this x_1, for $x_0 \in (f_3^{-1}(s), \infty)$, we have $\gamma/3^k < P(x_0)$.

Now since each case represents a possible x_1 that all have an equal probability of $\frac{1}{3}$,

$$\frac{1}{3}\left(1+1+\frac{\gamma}{3^k}\right) < P(x_0) \quad \text{or} \quad \frac{2}{3}+\frac{\gamma}{3^{k+1}} < P(x_0).$$

Thus for $x_0 \in (f_1^{-1}(s), \infty)$, we have shown (3).

Thus we have shown (3) for each of the four cases of $x_0 \notin \bigcup_{i=1}^{3} f_i^{-1}(J)$. □

We end this section with a corollary to Theorem 13 that helps describe the intervals of constant height, as seen in Figure 2.

Corollary 15. *Consider three linear functions f_k defined by Definition 3 and $P(x_0)$ defined by Definition 6. For any value of $\gamma/3^k$, with $\gamma, k \in \mathbb{N}$ and $\gamma \leq 3^k$, there is some interval of x_0-values such that $P(x_0) = \gamma/3^k$ for all x_0 in that interval.*

Probabilities for *n* functions

The situation for n functions is quite similar to that of three functions. We begin this section by first generalizing Theorem 10 and then generalizing Theorem 13.

Theorem 16. *Consider the linear functions f_k defined by Definition 3 and $P(x_0)$ as in Definition 6, and let n be the number of functions. For $x_0 \in \mathbb{R}$, $P(x_0) = \beta/n$ if and only if $x_0 \in I_\beta = [f_\beta^{-1}(p_n), f_{\beta+1}^{-1}(p_1)]$, where $\beta \in \{1, 2, \ldots, n-1\}$.*

Proof. Let $\beta \in \{1, 2, \ldots, n-1\}$ and let n be the number of functions. We assume $x_0 \in I_\beta = [f_\beta^{-1}(p_n), f_{\beta+1}^{-1}(p_1)]$. We examine $x_1 = f_\lambda(x_0)$ for $\lambda \in \{1, 2, \ldots, n\}$.

Case 1: $\lambda = \beta$. Consider the lower bound of I_β, which is $x_0 \geq f_\beta^{-1}(p_n)$. Since $\lambda = \beta$, we have $f_\lambda(x_0) \geq f_\beta(f_\beta^{-1}(p_n)) = p_n$. Since the lower bound of the interval is greater than or equal to p_n, we know that iteration through f_n will result in a sequence where $\{x_j\} \to \infty$. Therefore, due to Lemma 7, iteration using the set of all possible functions results in $\{x_j\} \to \infty$. Thus, $P(x_0) = 1$ for this case.

Case 2: $\lambda < \beta$. By Lemma 7, since $P(x_0) = 1$ for Case 1 when $x_1 = f_\lambda(x_0)$, we know that $P(x_0) = 1$ for this case as well.

Case 3: $\lambda > \beta$. Consider the upper bound of I_β, which is $x_0 \leq f_{\beta+1}^{-1}(p_1)$. The smallest value for λ in this case is $\beta+1$. Let $\lambda = \beta+1$. We apply f_λ to the upper bound to get $f_\lambda(x_0) \leq f_{\beta+1}(f_{\beta+1}^{-1}(p_1)) = p_1$. Thus $x_1 \leq p_1$ and $\{x_j\}$ is bounded above by p_1. By Lemma 7, since $P(x_0) = 0$ for $\lambda = \beta+1$, which is the smallest value for λ in this case, $P(x_0) = 0$ for all $\lambda > \beta$. Therefore, for Case 3, we conclude that $P(x_0) = 0$.

Each function has a probability of $1/n$ of being chosen on each iteration; thus $P(x_0)$ for any $x_0 \in I_\beta$ is

$$P(x_0) = \frac{1}{n}(\underbrace{1+1+\cdots+1}_{\beta\text{-times}} + \underbrace{0+0+\cdots+0}_{n-\beta\text{-times}}) = \frac{\beta}{n}.$$

Assume $P(x_0) = \beta/n$. Since the probability is of the form $P(x_0) = \beta/n$, we know that only β choices of x_1 can cause $\{x_j\} \to \infty$. This means that $n-\beta$ choices for x_1 will lead to $\{x_j\} \not\to \infty$. Thus the lower bound for x_0 must be greater than $f_\beta^{-1}(p_n)$, to ensure that when $x_1 = f_\beta(x_0)$, we have $f_\beta(x_0) > p_n$. Since x_1 is greater than the fixed point of f_n, we know $\{x_j\} \to \infty$. By Lemma 7, since $x_1 = f_\beta(x_0)$ leads to $\{x_j\} \to \infty$, all choices of $x_1 = f_\lambda(x_0)$ where $\lambda < \beta$ will lead to $\{x_j\} \to \infty$. However, to ensure that $n-\beta$ choices for x_1 lead to the sequence $\{x_j\}$ not diverging to infinity, the upper bound for x_0 must be $f_{\beta+1}^{-1}(p_1)$. This is because for all $x_0 < f_{\beta+1}^{-1}(p_1)$, it is true that $f_{\beta+1}(f_{\beta+1}^{-1}(p_1)) = p_1$, and for any $\lambda > \beta+1$, we will have $f_\lambda(f_{\beta+1}^{-1}(p_1)) < 0$, so all choices of $f_\lambda(x_0)$ will lead to $\{x_j\} \not\to \infty$. Therefore, we conclude that if $P(x_0) = \beta/n$ then $x_0 \in I_\beta = [f_\beta^{-1}(p_n), f_{\beta+1}^{-1}(p_1)]$. □

Theorem 17. *Consider the linear functions f_k defined by Definition 3 and $P(x_0)$ as in Definition 6, and let n be the number of functions. Assume*

$$J = \left\{x_0 : P(x_0) = \frac{\gamma}{n^k}\right\},$$

where $\gamma \in \mathbb{N} \cup \{0\}$ and $n \nmid \gamma$. Then for $\alpha \in \{0, 1, \dots, n-1\}$ we have

$$\left\{y_0 : P(y_0) = \frac{\alpha}{n} + \frac{\gamma}{n^{k+1}}\right\} = f_{\alpha+1}^{-1}(J).$$

Example 18. Assume that we have n functions, and we want to know for which x_0-values $P(x_0) = v/n^k$, where v/n^k is fully reduced. The value of $k-1$ is the number of preimages of the n functions you take to find this $P(x_0)$. We can express v as $v = c_{k-1}n^{k-1} + c_{k-2}n^{k-2} + \cdots + c_0 n^0$, where $c_j \in \mathbb{Z} \cap [0, n]$. The coefficient c_j is the number of functions that cause $\{x_j\} \to \infty$ under iteration using j preimages. We next give a numerical example.

Example 19. Suppose we have five functions, and we wish to see for which x_0-values $P(x_0) = \frac{427}{625}$. We know that $625 = 5^4$. Thus, we would need to take three preimages using our five functions to find this $P(x_0)$. We rewrite $\frac{427}{625}$ as

$$\begin{aligned}\tfrac{427}{625} &= \tfrac{3(125)}{625} + \tfrac{2(25)}{625} + \tfrac{0(5)}{625} + \tfrac{2(1)}{625} = \tfrac{3}{5} + \tfrac{1}{5}\left(\tfrac{52}{125}\right)\\ &= \tfrac{3}{5} + \tfrac{1}{5}\left(\tfrac{2}{5} + \tfrac{2}{125}\right) = \tfrac{3}{5} + \tfrac{1}{5}\left(\tfrac{2}{5} + \tfrac{1}{5}\left(\tfrac{0}{5} + \tfrac{2}{25}\right)\right).\end{aligned}$$

Therefore, we have $P(x_0) = \frac{427}{625}$ if and only if $x_0 \in f_4^{-1}(f_3^{-1}(f_1^{-1}(I_2)))$. This is the same method for finding the interval of constant $P(x_0)$ as used in Example 14.

Proof. We assume

$$J = \left\{ x_0 : P(x_0) = \frac{\gamma}{n^k} \right\},$$

where $\gamma \in \{\mathbb{N} \cup 0\}$ and $n \nmid \gamma$. Let $\alpha \in \{0, 1, \dots, n-1\}$. Since we are proving set equality, we start with proving that

$$f_{\alpha+1}^{-1}(J) \subseteq \left\{ y_0 : P(y_0) = \frac{\alpha}{n} + \frac{\gamma}{n^{k+1}} \right\}.$$

Let $x_0 \in f_{\alpha+1}^{-1}(J)$. Since $x_1 = f_\beta(x_0)$ for any $\beta \in \{1, 2, \dots, n\}$, we proceed with proof by cases.

<u>Case 1</u>: $\beta < \alpha + 1$. Since $x_0 \in f_{\alpha+1}^{-1}(J)$, we know that $x_0 \geq f_{\alpha+1}^{-1}(0) > f_\beta^{-1}(1)$ by Lemma 5. Thus $x_1 > f_\beta(f_\beta^{-1}(1)) = 1$ and $\{x_j\} \to \infty$, so $P(x_0) = 1$.

<u>Case 2</u>: $\beta = \alpha + 1$. As $x_0 \in f_{\alpha+1}^{-1}(J)$, when $x_1 = f_\beta(x_0)$, we have $x_1 = f_\beta(x_0) \in f_\beta(f_{\alpha+1}^{-1}(J))$ but since $\alpha + 1 = \beta$, this is

$$x_1 = f_\beta(x_0) \in f_{\alpha+1}(f_{\alpha+1}^{-1}(J)) = J.$$

Thus, $P(x_0) = P(y)$ for any $y \in J$, so $P(x_0) = \gamma / n^k$.

<u>Case 3</u>: $\beta > \alpha + 1$. As $x_0 \in f_{\alpha+1}^{-1}(J)$ and $\beta > \alpha + 1$,

$$x_1 < f_\beta(f_{\alpha+1}^{-1}(1)) < f_\beta(f_\beta^{-1}(0)) = 0;$$

thus $\{x_j\}$ is bounded above by 0 and $P(x_0) = 0$ by Lemma 7. Since $\alpha \in \{0, 1 \dots, n-1\}$ and $\beta \in \{1, 2, \dots, n\}$, we know this case will occur $n - (\alpha + 1)$ times.

Since each function has the probability of $1/n$ of being chosen on the first iteration, we have

$$P(x_0) = \frac{1}{n} \Bigg(\underbrace{1 + 1 + \cdots + 1 + 1}_{\alpha-\text{times}} + \frac{\gamma}{n^k} + \underbrace{0 + 0 + \cdots + 0}_{n-\alpha-1\text{-times}} \Bigg) = \frac{\alpha}{n} + \frac{\gamma}{n^{k+1}}.$$

We have accounted for all n cases because $\alpha + 1 + (n - \alpha - 1) = n$.

We now show the set containment of

$$\left\{ y_0 : P(y_0) = \frac{\alpha}{n} + \frac{\gamma}{n^{k+1}} \right\} \subseteq f_{\alpha+1}^{-1}(J).$$

We proceed with proof by contrapositive; that is, if $x_0 \notin \bigcup_{\alpha=0}^{n-1} f_{\alpha+1}^{-1}(J)$ then

$$P(x_0) \neq \frac{\alpha}{n} + \frac{\gamma}{n^{k+1}}. \tag{4}$$

We examine all cases, using the convention that $J = [r, s]$.

<u>Case 1</u>: $x_0 \in (-\infty, f_1^{-1}(r))$. There are two subcases.

<u>Subcase 1.1</u>: $x_1 = f_1(x_0)$. Consider the upper bound of the interval, $x_0 < f_1^{-1}(r)$. When $x_0 = f_1(x_0)$, we get $x_0 = f_1(x_0) < f_1(f_1^{-1}(r)) = r$. However, since $J = [r, s]$

and $x_1 < r$, we know that $P(x_0) \leq P(y)$, where $y \in J$. Since the lower bound of $P(x_0)$ is 0, we have $0 \leq P(x_0) < \gamma/n^k$.

Subcase 1.2: $x_1 = f_\tau(x_0)$, where $\tau \in \mathbb{Z} \cap [2, n]$. We know $x_0 < f_1^{-1}(r)$; thus since $f_1^{-1}(r) < f_\tau^{-1}(0)$, we have $f_\tau(x_0) < f_\tau(f_\tau^{-1}(0)) = 0$. Therefore, by Lemma 7, we conclude $P(x_0) = 0$ for this subcase.

Since each function has a probability of $1/n$ of being chosen, the probability for Case 1 is

$$\frac{1}{n}(0+0) \leq P(x_0) < \frac{1}{n}\left(0 + \frac{\gamma}{n^k}\right) \quad \text{or} \quad 0 \leq P(x_0) < \frac{\gamma}{n^{k+1}}.$$

Hence, we have shown (4).

Case 2: $x_0 \in (f_\alpha^{-1}(s), f_{\alpha+1}^{-1}(r))$. We examine $x_1 = f_\beta(x_0)$. Note $\beta \in \{1, 2, \ldots, n\}$ and $\alpha \in \{0, 1, \ldots, n-1\}$. We proceed with proof by cases, letting $x_0 \in (f_\alpha^{-1}(s), f_{\alpha+1}^{-1}(r))$.

Subcase 2.1: $\beta < \alpha$. When we have $\beta < \alpha$, the x_0-value is greater than $f_\beta^{-1}(1)$, so $x_1 > f_\beta(f_\beta^{-1}(1)) = 1$. Therefore, by Lemma 7, we have $P(x_0) = 1$ for these $\alpha - 1$ cases.

Subcase 2.2: $\beta = \alpha$. Since $x_1 = f_\alpha(x_0) > f_\alpha(f_\alpha^{-1}(s)) = s$, and $J = [r, s]$, we know that $P(x_0) > \gamma/n^k$. Thus for this one instance of $\beta = \alpha$, we have $\gamma/n^k < P(x_0) \leq 1$.

Subcase 2.3: $\beta = \alpha + 1$. Let $x_1 = f_{\alpha+1}(x_0)$. Now

$$f_{\alpha+1}(x_0) < f_{\alpha+1}(f_{\alpha+1}^{-1}(r)) = r,$$

so $P(x_0) < \gamma/n^k$ for this case.

Subcase 2.4: $\beta > \alpha + 1$. Since $x_1 = f_\beta(x_0) < f_\beta(f_{\alpha+1}^{-1}(r))$ and $f_{\alpha+1}^{-1}(r) < f_\beta^{-1}(0)$ by Definition 3, we have $x_1 < 0$ and $P(x_0) = 0$. There are $n - (\alpha + 1) = n - \alpha - 1$ instances where $\beta > \alpha + 1$ would be true.

Since each function has a probability of $1/n$ of being chosen, $P(x_0)$ is bounded by

$$\frac{1}{n}\Bigg(\underbrace{1+\cdots+1}_{\alpha-1\text{-times}} + \frac{\gamma}{n^k} + \underbrace{0+\cdots+0}_{n-\alpha\text{-times}}\Bigg) < P(x_0) < \frac{1}{n}\Bigg(\underbrace{1+\cdots+1}_{\alpha\text{-times}} + \frac{\gamma}{n^k} + \underbrace{0+\cdots+0}_{n-\alpha-1\text{-times}}\Bigg),$$

which simplifies to

$$\frac{\alpha-1}{n} + \frac{\gamma}{n^{k+1}} < P(x_0) < \frac{\alpha}{n} + \frac{\gamma}{n^{k+1}}.$$

Therefore, for Case 2 we have shown (4).

Case 3: $x_0 \in (f_n^{-1}(s), \infty)$. There are two subcases, where we consider the lower bound of $x_0 > f_n^{-1}(s)$.

Subcase 3.1: $\beta < n$. We know $f_n^{-1}(s) > f_\beta^{-1}(1)$ by Lemma 5. Thus $P(f_n^{-1}(s)) = 1$, and $P(x_0)$ is nondecreasing, so we conclude $P(x_0) = 1$ in this subcase.

Subcase 3.2: $\beta = n$. We apply f_n to our x_0-value to get $f_n(x) > f_n(f_n^{-1}(s)) = s$. However, since $J = [r, s]$ and the lower bound of the interval is greater than the upper bound of J, we know that $P(x_0) > P(y)$ where $y \in J$. Thus $P(x_0) > \gamma/n^k$. Therefore, we conclude that $\gamma/n^k < P(x_0) \leq 1$.

From Subcase 3.1, we have shown that the probability of every function except for f_n iterating our x_0-value toward ∞ is 1. Since the probability of each function being chosen on the first iteration is $1/n$, we have

$$\frac{1}{n}\biggl(\underbrace{1+1+\cdots+1}_{n-1\text{-times}}+\frac{\gamma}{n^k}\biggr) < P(x_0) \leq \frac{1}{n}(\underbrace{1+1+\cdots+1+1}_{n\text{-times}}),$$

which simplifies to

$$\frac{n-1}{n} + \frac{\gamma}{n^{k+1}} < P(x_0) \leq 1.$$

Therefore, for Case 3 we have shown (4). □

Finally, we can apply Theorem 17 to see that the function $P(x_0)$ can have intervals of constant height for any value of γ/n^k.

Corollary 20. *Consider the linear functions f_k as in Definition 3 and $P(x_0)$ as in Definition 6, and let n be the number of functions. For any value of γ/n^k, with $\gamma, k \in \mathbb{N}$ and $\gamma \leq n^k$, there is some interval of x_0-values such that $P(x_0) = \gamma/n^k$ for all x_0 in that interval.*

Future directions

The functions studied here are sets of linear functions where we restrict the coefficients of $f_k(x) = a_k x - b_k$ such that a_k, b_k are whole numbers with $a_k > b_k$. There are numerous natural extensions to this work. One such extension would be to remove the whole number restrictions on a_k and b_k. Even further, one might explore the possibility of a_k, b_k being complex numbers. Additionally, we can expand this research into higher-order functions, such as taking a set of functions of the form $F = \{f_k(x) = a_k x^2 + b_k x + c_k\}$. The interesting thing to note with quadratic functions is that each function can have zero, one, or two fixed points. Lastly, another natural extension is to consider what behavior $P(x_0)$ would have if the functions $f_k(x)$ were not chosen with equal probability at each iteration.

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