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Structure constants of $\mathcal{U}(\mathfrak{sl}_2)$

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We derive a recursive formula for the structure constants for the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{K})$, where $\mathbb{K}$ is an algebraically closed field of characteristic zero.

1. Introduction

Given an algebra $\mathcal{A}$ over a field $\mathbb{K}$ with basis $\{v_k \mid k \in I\}$ indexed by some set I, the product of any two basis vectors can be expressed in terms of the basis; i.e., we can write $v_i v_j$ as $\sum_{k\in I} \gamma_{ijk} v_k$ for some scalars $\gamma_{ijk} \in \mathbb{K}$. The γ_{ijk} are referred to as structure constants and theoretically, we can discern properties of the algebra from identities satisfied, or not satisfied, by these scalars. For example, the algebra $\mathcal{A}$ will be commutative if and only if $\gamma_{ijk} = \gamma_{jik}$ for all i and j. From a practical standpoint, structure constants are of the most use for smaller algebras, i.e., finite-dimensional. For an infinite-dimensional algebra, computing the structure constants is a daunting task, especially since the structure constants with respect to one basis are likely to differ wildly from the structure constants with respect to another basis.

With that being said, the case we consider here is almost ideal for this type of problem. The Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ is ubiquitous in Lie theory, small (three-dimensional) and easily described through its structure constants. While its universal enveloping algebra, $\mathcal{U}(\mathfrak{sl}_2(\mathbb{K}))$ is infinite-dimensional, thanks to the very powerful Poincaré-Birkhoff-Witt theorem, it has a canonical basis generated by the basis of the underlying $\mathfrak{sl}_2(\mathbb{K})$. This makes computing the structure constants somewhat tractable. While the mathematics contained herein is not tremendously complicated, and relies heavily on induction, the result itself is quite interesting.

2. Statement of the problem

For the purpose of self-containment, we give several initial definitions and results, the latter without proof. We refer the interested reader to either [Erdmann and Wildon 2006] for a general introduction to Lie algebras and enveloping algebras or [Carter 2005; Humphreys 1972] for a more in-depth development of the theory.

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Definition 1. A *Lie algebra* is a vector space $\mathfrak{L}$ over a field $\mathbb{K}$ equipped with a bilinear product, called the bracket, $[\,\cdot\,,\cdot\,] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ satisfying

- (anticommutativity) $[x, x] = 0$ for all $x \in \mathfrak{L}$;
- (Jacobi identity) $[[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{L}$.

If $\mathbb{K}$ has characteristic other than 2, by applying bilinearity to $[x + y, x + y] = 0$ the immediate consequence is that $[x, y] = -[y, x]$; that is, we can reverse the order of the bracket at the cost of a negative.

Given an associative algebra $\mathcal{A}$, there is an associated Lie algebra $\mathcal{A}^-$ with the same underlying vector space equipped with the *commutator bracket*, $[x, y] := xy - yx$ for all $x, y \in \mathcal{A}^-$. For example, the associative algebra $\mathrm{Mat}_2(\mathbb{K})$ of 2×2-matrices can be equipped with the commutator bracket to yield the *general linear algebra*, $\mathfrak{gl}_2(\mathbb{K})$. Here, the bracket of two matrices A and B is just $[A, B] = AB - BA$, with AB and BA being the regular matrix product. Because AB rarely equals BA for arbitrary matrices, the bracket is usually nonzero. A crucial, albeit small, example of a Lie algebra is $\mathfrak{sl}_2(\mathbb{K})$, which arises as a Lie subalgebra of $\mathfrak{gl}_2(\mathbb{K})$.

Definition 2. The *special linear Lie algebra* of 2×2 matrices, $\mathfrak{sl}_2(\mathbb{K})$, is the Lie algebra consisting of all 2×2 matrices with trace zero equipped with the commutator bracket.

The field $\mathbb{K}$ does not play an important role in what follows, and consequently will be omitted; e.g., $\mathfrak{sl}_2$ instead of $\mathfrak{sl}_2(\mathbb{K})$. The Lie algebra $\mathfrak{sl}_2$ is three-dimensional with a natural basis consisting of the matrices

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The bracket of any of these with itself is zero by anticommutativity and the mixed brackets give the relations

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f. \tag{1}$$

A natural question to ask is whether a Lie algebra $\mathfrak{L}$ arises as the associated Lie algebra for some associative algebra, $\mathcal{A}$. That is, given $\mathfrak{L}$, does there exist an $\mathcal{A}$ such that $\mathfrak{L} = \mathcal{A}^-$? In certain cases the answer is yes (e.g., if $\mathfrak{L}$ is finite-dimensional), but in general the answer is no. That being said, we can often find an associative algebra $\mathcal{A}$ such that $\mathcal{L}$ is a subalgebra of $\mathcal{A}^-$; that is, $\mathfrak{L} \subseteq \mathcal{A}^-$. Such an algebra $\mathcal{A}$ is said to be an *enveloping algebra* of $\mathfrak{L}$. A Lie algebra can possess many distinct enveloping algebras, but the following theorem asserts the existence of a powerful and special one.

Theorem 3 (Poincaré–Birkhoff–Witt). *Let $\mathfrak{L}$ be a Lie algebra. Then there exists an associative algebra $\mathcal{U}(\mathfrak{L})$, called the **universal enveloping algebra**, satisfying*:

(1) *$\mathfrak{L}$ can be identified with a subalgebra of $\mathcal{U}(\mathfrak{L})$, where the bracket on $\mathfrak{L}$ is given by $[x, y] = xy - yx$.*

(2) *If $\varphi : \mathfrak{L} \to \mathcal{A}^-$ is any Lie algebra homomorphism, there is a unique associative algebra homomorphism*

$$\bar{\varphi} : \mathcal{U}(\mathfrak{L}) \to \mathcal{A}$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{\gamma} & \mathcal{U}(\mathfrak{L}) \\ & \searrow^{\varphi} & \downarrow \bar{\varphi} \\ & & \mathcal{A} \end{array}$$

Moreover, if $\mathfrak{L}$ has a basis $\{v_i \mid i \in I\}$, where I is equipped with a total order $<$, then $\mathcal{U}(\mathfrak{L})$ has a basis of all monomials of the form $v_{i_1}^{n_1} v_{i_2}^{n_2} \cdots v_{i_m}^{n_m}$, where $i_1 < i_2 < \cdots < i_m$ and $n_j \in \mathbb{N}$.

While a discussion of homomorphisms is beyond the scope of this paper, heuristically, what Theorem 3 is saying is that there is one enveloping algebra for $\mathfrak{L}$ that trumps all others — any enveloping algebra for a given $\mathfrak{L}$ can be derived from $\mathcal{U}(\mathfrak{L})$. In theory, if we know $\mathcal{U}(\mathfrak{L})$, we can determine any and all enveloping algebras of $\mathfrak{L}$.

In this paper, we are concerned with the universal enveloping algebra of $\mathfrak{sl}_2$. Since $\mathfrak{sl}_2$ is three-dimensional, we can order the canonical basis by $f < h < e$. Note that this ordering is arbitrary in the sense that there is nothing "larger" about e as compared to h and so forth. Really, it is just a reflection that f is lower triangular and we move "up" to h as a diagonal matrix and then farther "up" to e as upper triangular. With this convention set, Theorem 3 provides a basis of $\mathcal{U}(\mathfrak{sl}_2)$ consisting of all elements of the form $f^r h^s e^t$, where $r, s, t \in \mathbb{N}$.

For the sake of completeness, we give the formal definition of structure constants.

Definition 4. Let $\mathcal{A}$ be an algebra (not necessarily associative) with basis $\{v_k \mid k \in I\}$. The *structure constants* of $\mathcal{A}$ with respect to the basis $\{v_k \mid k \in I\}$ are scalars γ_{ijk}, where

$$v_i \cdot v_j = \sum_{k \in I} \gamma_{ijk} v_k,$$

where all but finitely many of the γ_{ijk} are zero.

With all this in place, we state the main problem. Note that while in Definition 4 the basis vectors are indexed by a single value, the PBW basis vectors $f^r h^s e^t$ are naturally indexed by the triple (r, s, t).

Main Problem. Given the Poincaré–Birkhoff–Witt basis $\{f^r h^s e^t \mid r, s, t \in \mathbb{N}\}$ of $\mathcal{U}(\mathfrak{sl}_2)$, what are the structure constants γ_{ijkl} such that

$$f^r h^s e^t \cdot f^u h^v e^w = \sum_{l \in L} \gamma_{ijkl} f^i h^j e^k \tag{2}$$

for some indexing set L?

Ideally, we would determine the constants explicitly. As we will see in Section 5, more realistically we develop a recursive formula for the γ_{ijkl} depending on the powers on the left-hand side of the equation.

3. Commutation identities: initial cases

To determine this, we need to use various commutation identities to reorder the components on the left-hand side of (2). In $\mathcal{U}(\mathfrak{sl}_2)$, the bracket identities from (1) give

$$[e, f] = ef - fe = h, \tag{3}$$

$$[h, e] = he - eh = 2e, \tag{4}$$

$$[h, f] = hf - fh = -2f. \tag{5}$$

These allow us to reorder the products of e, f, and h:

$$ef = fe + h, \tag{6}$$

$$eh = he - 2e, \tag{7}$$

$$hf = fh - 2f. \tag{8}$$

We make heavy use of mathematical induction in what follows and consequently begin by establishing several base cases.

Proposition 5. *Given a PBW basis element $f^r h^s e^t$, we have*

$$(f^r h^s e^t) f = f^{r+1}(h-2)^s e^t + t f^r h^{s+1} e^{t-1} - t(t-1) f^r h^s e^{t-1}. \tag{9}$$

Proof. To establish (9), we begin by determining expressions for $e^t f$ and $h^s f$ in terms of the PBW basis. Specifically, for the former we claim that

$$e^t f = f e^t + t h e^{t-1} - t(t-1) e^{t-1}. \tag{10}$$

If $t = 1$ in (10), it simplifies to (6) from earlier. Assuming the result for t, then

$$\begin{aligned}
(e^{t+1} f) = e(e^t f) &= e(f e^t + t h e^{t-1} - t(t-1) e^{t-1}) \\
&= (ef) e^t + t(eh) e^{t-1} - t(t-1) e^t \\
&= (fe + h) e^t + t(he - 2e) e^{t-1} - t(t-1) e^t \\
&= f e^{t+1} + h e^t + t h e^t - 2t e^t - t(t-1) e^t \\
&= f e^{t+1} + (t+1) h e^t - [2t + t(t-1)] e^t \\
&= f e^{t+1} + (t+1) h e^t - t(t+1) e^t.
\end{aligned}$$

Using a similar argument for $h^s f$, we claim

$$h^s f = f(h-2)^s. \tag{11}$$

If we take $s = 1$, it becomes (8) provided we distribute the f on the left. Assuming the result for s, we have

$$\begin{aligned}(h^{s+1} f) = h(h^s f) = h(f(h-2)^s) = (hf)(h-2)^s \\ = (fh - 2f)(h-2)^s = f(h-2)(h-2)^s = f(h-2)^{s+1}.\end{aligned}$$

Applying (10) and (11) to $(f^r h^s e^t) f$, we have

$$\begin{aligned}(f^r h^s)(e^t f) &= (f^r h^s)(f e^t + t h e^{t-1} - t(t-1)e^{t-1}) \\ &= f^r (h^s f) e^t + t f^r h^{s+1} e^{t-1} - t(t-1) f^r h^s e^{t-1} \\ &= f^r (f(h-2)^s) e^t + t f^r h^{s+1} e^{t-1} - t(t-1) f^r h^s e^{t-1} \\ &= f^{r+1} (h-2)^s e^t + t f^r h^{s+1} e^{t-1} - t(t-1) f^r h^s e^{t-1}.\end{aligned}$$ □

It should be noted that (11) is not technically in PBW format, although it is close. While $\mathcal{U}(\mathfrak{sl}_2)$ is highly noncommutative, elements of the field $\mathbb{K}$ commute with everything. In particular, the scalar 2 commutes with h and the term $(h-2)^s$ can be expanded using the binomial theorem. Doing so yields the product in (11) in terms of the PBW basis:

$$(f^r h^s e^t) f = \left(\sum_{k=0}^{s} \binom{s}{k} (-2)^k f^{r+1} h^{s-k} e^t\right) + t f^r h^{s+1} e^{t-1} - t(t-1) f^r h^s e^{t-1}. \quad (12)$$

Obviously the equation in Proposition 5 is more compact.

Next we turn to pushing an h through a generic PBW basis vector.

Proposition 6. *Given a PBW basis element $f^r h^s e^t$, we have*

$$(f^r h^s e^t) h = f^r h^{s+1} e^t - 2t f^r h^s e^t. \quad (13)$$

Proof. As with the preceding proof, we need to establish the identity

$$e^t h = h e^t - 2t e^t = (h - 2t) e^t. \quad (14)$$

If $t = 1$, this is (7). Assuming the result for t, we have

$$\begin{aligned}e^{t+1} h = e(e^t h) = e(h e^t - 2t e^t) = (eh) e^t - 2t e^{t+1} \\ = (he - 2e) e^t - 2t e^{t+1} = h e^{t+1} - 2(t+1) e^{t+1}.\end{aligned}$$

Applying this to (13), we have

$$\begin{aligned}(f^r h^s e^t) h &= f^r h^s (e^t h) = f^r h^s (h - 2t) e^t \\ &= f^r h^{s+1} e^t - 2t f^r h^s e^t.\end{aligned}$$ □

4. Commutation identities: secondary cases

Now, we turn our attention to the products $e^t h^v$ and $h^s f^u$.

Proposition 7. *The quantities $e^t h^v$ and $h^s f^u$ can be expressed as*

$$e^t h^v = (h-2t)^v e^t = \sum_{k=0}^{v} \binom{v}{k} (-2t)^k h^{v-k} e^t, \tag{15}$$

$$h^s f^u = f^u (h-2u)^s = \sum_{k=0}^{u} \binom{u}{k} (-2u)^k f^u h^{u-k}. \tag{16}$$

Proof. The far right-hand sides of both equations result from applying the binomial theorem to centers of their respective equations, so we focus on establishing the left-hand equalities. Moreover, the proof of the second is symmetric to the first and is omitted. For the first, we proceed by induction on v with t held constant. If $v=1$, we have $e^t h = (h-2t)e^t$, which was established in Proposition 6. Assuming the result for $e^t h^v$, we have

$$\begin{aligned} e^t h^{v+1} &= (e^t h^v) h = (h-2t)^v e^t h \\ &= (h-2t)^v (h-2t) e^t = (h-2t)^{v+1} e^t. \end{aligned}$$ □

Our next concern is how to express an element of the form $e^t f^u$ in terms of the PBW basis, with the case $u=1$ simply being (10). This is decidedly more complicated than the cases in Proposition 7. If we consider $e^t h^v$, for example, we are in essence repeatedly using (7): $eh = he - 2e$. We can think of this as reordering eh as a sum of elements of the form $h^i e$, where the power of e remains constant and the powers of h decrease. This is precisely what happens with $e^t h^v$ and symmetrically with $h^s f^u$.

On the other hand, when we wish to reorder $e^t f^u$, we need to repeatedly use (6): $ef = fe + h$. Here, the h-term is essentially new. Successive applications will generate more powers of h, which will interact in turn with the various powers of f. Consequently, more and more terms are spawned as we push successive powers of f past successive powers of e. Consequently, it becomes logistically impossible to express $e^t f^u$ explicitly in the PBW basis, but it is quite natural to be done recursively.

Theorem 8. *The element $e^t f^u$ can be expressed in the form*

$$\sum_{i=0}^{u} \sum_{l=0}^{i} \gamma_{u,i,l} f^{u-i} h^{i-l} e^{t-i}$$

for scalars $\gamma_{u,i,l}$. Moreover for fixed natural numbers a and b with $b \le a$, we have the following recursive formula for the coefficient of $f^{u+1-a} h^{a-b} e^{t-a}$ in terms of the coefficients $\gamma_{u,i,l}$:

$$\gamma_{u+1,a,b} = \left(\sum_{l+k=0}^{b} \binom{a-l}{k} (-2)^k \gamma_{u,a,l} + \gamma_{u,a-1,b}(t-a+1) \right) - \gamma_{u,a-1,b-1}(t-a)(t-a-1). \tag{17}$$

Proof. We first establish the claim that we can express $e^t f^u$ as a linear combination of terms of the form $f^{u-i}h^{i-l}e^{t-i}$ for various i and l, and not surprisingly we proceed by induction on u. We have established $u=1$ in (10). The general approach is to use the induction hypothesis and compute $e^t f^{u+1} = (e^t f^u) f$ and then do considerable rearrangement and reindexing along with applying Propositions 5 and 6. In general, $\gamma_{u,i,l}$ is the coefficient of $f^{u-i}h^{i-l}e^{t-i}$ with $i \le u$ and $l \le i$. In what follows, we assign a value of 0 to any scalar $\gamma_{u,i,l}$ where $i > u$ or $l > i$ since no such terms exist. Additionally, if $i < 0$ or $l < 0$, we make the same assignment:

$$\begin{aligned}
(e^t f^u) f &= \sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l} f^{u-i}h^{i-l}e^{t-i} f \\
&= \sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l} f^{u-i}h^{i-l}(f e^{t-i} + (t-i)he^{t-i-1} - (t-i)(t-i-1)e^{t-i-1}) \\
&= \sum_{i=1}^{u}\sum_{l=0}^{i} \gamma_{u,i,l} f^{u-i}(h^{i-l} f)e^{t-i} + \sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l}(t-i) f^{u-i}h^{i-l+1}e^{t-i-1} \\
&\quad - \sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l}(t-i)(t-i-1) f^{u-i}h^{i-l}e^{t-i-1} \\
&= \sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l} f^{u+1-i}(h-2)^{i-l}e^{t-i} + \sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l}(t-i) f^{u-i}h^{i-l+1}e^{t-i-1} \\
&\quad - \sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l}(t-i)(t-i-1) f^{u-i}h^{i-l}e^{t-i-1} \\
&= \sum_{i=0}^{u}\sum_{l=0}^{i}\sum_{k=0}^{i-l} \gamma_{u,i,l}\binom{i-l}{k}(-2)^k f^{u+1-i}h^{i-l-k}e^{t-i} \\
&\quad + \sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l}(t-i) f^{u-i}h^{i-l+1}e^{t-i-1} \\
&\qquad - \sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l}(t-i)(t-i-1) f^{u-i}h^{i-l}e^{t-i-1}.
\end{aligned}$$

Now we selectively reindex and rename indices as necessary. To begin, we separate the three distinct summations in the above:

$$\sum_{i=0}^{u}\sum_{l=0}^{i}\sum_{k=0}^{i-l} \gamma_{u,i,l}\binom{i-l}{k}(-2)^k f^{u+1-i}h^{i-l-k}e^{t-i}, \tag{18}$$

$$\sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l}(t-i) f^{u-i}h^{i-l+1}e^{t-i-1}, \tag{19}$$

$$\sum_{i=0}^{u}\sum_{l=0}^{i}\gamma_{u,i,l}(t-i)(t-i-1)f^{u-i}h^{i-l}e^{t-i-1}. \tag{20}$$

Concentrating on (19), we reindex by setting $j=i+1$, in which case we have

$$\sum_{j=1}^{u+1}\sum_{l=0}^{j-1}\gamma_{u,j-1,l}(t-j+1)f^{u+1-j}h^{j-l}e^{t-j}.$$

By our conventions on the constant $\gamma_{u,i,l}$ we may extend the second summation to $l=j$ without changing the value. Additionally, if $j=0$, we have $\gamma_{u,-1,l}$, which is also zero by convention. Thus we may rewrite (19) as

$$\sum_{j=0}^{u+1}\sum_{l=0}^{j}\gamma_{u,j-1,l}(t-j+1)f^{u+1-j}h^{j-l}e^{t-j},$$

which matches the required form.

Turning our attention to (20), we make the same reindexing with the substitution $j=i+1$, thereby yielding

$$\sum_{j=1}^{u+1}\sum_{l=0}^{j-1}\gamma_{u,j-1,l}(t-j+1)(t-j)f^{u+1-j}h^{j-1-l}e^{t-j}.$$

Replacing l by $s=l+1$ we have

$$\sum_{j=1}^{u+1}\sum_{s=1}^{j}\gamma_{u,j-1,s-1}(t-j+1)(t-j)f^{u+1-j}h^{j-s}e^{t-j}$$

and once again, if we extend the sums to $s=0$ and $j=0$ respectively, the resulting scalars are zero. Thus (20) becomes

$$\sum_{j=0}^{u+1}\sum_{s=0}^{j}\gamma_{u,j-1,s-1}(t-j+1)(t-j)f^{u+1-j}h^{j-s}e^{t-j}.$$

We rename the indices using the original index names in the second and third parts and consolidate to yield

$$\sum_{i=0}^{u+1}\sum_{l=0}^{j}\big(\gamma_{u,i-1,l}(t-i+1)-\gamma_{u,i-1,l-1}(t-i)(t-i-1)\big)f^{u+1-i}h^{i-l}e^{t-i}. \tag{21}$$

Finally turning our attention to (18), to achieve the desired form we must reindex the sum using the substitution $r=l+k$. Starting with binomial coefficent, note that

$$\binom{i-l}{k}=\binom{i-l}{i-l-k}=\binom{i-l}{i-r}$$

and that the power $(-2)^k$ is the same as $(-2)^{r-l}$. In terms of the summation bounds, if l ranges between 0 and i and k ranges between 0 and $i-l$, this is equivalent to l ranging between 0 and i and $r=l+k$ ranging between l and i. Thus (18) becomes

$$\sum_{i=0}^{u+1}\sum_{l=0}^{i}\sum_{r=l}^{i}\gamma_{u,i,l}\binom{i-l}{i-r}(-2)^{r-l}f^{u+1-i}h^{i-r}e^{t-i}. \tag{22}$$

Once again, we have extended the summation bounds using the conventions on $\gamma_{u,i,l}$. This gives the PBW basis elements in terms of the indices i and r for this part. Renaming l in (22) by s and r by l and consolidating with (21) we have that $e^t f^{u+1}$ can be expressed as

$$\begin{aligned} e^t f^{u+1} = &\sum_{i=0}^{u+1}\sum_{s=0}^{i}\sum_{l=s}^{i}\gamma_{u,i,s}\binom{i-s}{i-r}(-2)^{l-s}f^{u+1-i}h^{i-l}e^{t-i} \\ &+\sum_{i=0}^{u+1}\sum_{l=0}^{j}\big(\gamma_{u,i-1,l}(t-i+1)-\gamma_{u,i-1,l-1}(t-i)(t-i-1)\big)f^{u+1-i}h^{i-l}e^{t-i}. \end{aligned} \tag{23}$$

This gives $e^t f^{u+1}$ as a linear combination of terms of the form $f^{u+1-i}h^{i-l}e^{t-i}$ as required.

It is easier to establish the recursive formula if we use (18) in its original formulation. Fixing a and b with $b \le a$, the coefficient of $f^{u+a-1}h^{a-b}e^{t-a}$ in (18) is

$$\sum_{l+k=0}^{b}\binom{a-l}{k}(-2)^k\gamma_{u,a,l},$$

while the remaining term comes from setting $i=a$ and $l=b$ in (23) giving

$$\begin{aligned}\gamma_{u+1,a,b} = &\left(\sum_{l+k=0}^{b}\binom{a-l}{k}(-2)^k\gamma_{u,a,l}\right) \\ &+\gamma_{u,a-1,b}(t-a+1)-\gamma_{u,a-1,b-1}(t-a)(t-a-1),\end{aligned} \tag{24}$$

completing the proof. □

We can rewrite the sum in the last equation a little more explicitly as

$$\sum_{l=0}^{a}\sum_{k=b-l}^{a-l}\binom{a-l}{k}(-2)^k\gamma_{u,a,l}$$

if need be.

5. Main result

We are now ready to prove the main result, namely expressing $(f^r h^s e^t)(f^u h^v e^w)$ in terms of the PBW basis using the preceding results. Given that

$$e^t f^u = \sum_{i=0}^{u}\sum_{l=0}^{i}\gamma_{u,i,l}f^{u-i}h^{i-l}e^{t-i},$$

we have

$$\begin{aligned}
&(f^r h^s e^t)(f^u h^v e^w)\\
&= f^r h^s (e^t f^u) h^v e^w\\
&= f^r h^s \left(\sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l} f^{u-i} h^{i-l} e^{t-i}\right) h^v e^w\\
&= f^r h^s \left(\sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l} f^{u-i} h^{i-l} (e^{t-i} h^v)\right) e^w\\
&= f^r h^s \left(\sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l} f^{u-i} h^{i-l} (h-2(t-i))^v e^{t-i}\right) e^w\\
&= f^r \sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l} (h^s f^{u-i}) h^{i-l} (h-2(t-i))^v e^{w+t-i}\\
&= f^r \sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l} f^{u-i} (h-2(u-i))^s h^{i-l} (h-2(t-i))^v e^{w+t-i}\\
&= \sum_{i=0}^{u}\sum_{l=0}^{i} \gamma_{u,i,l} f^{r+u-i} (h-2(u-i))^s h^{i-l} (h-2(t-i))^v e^{w+t-i}\\
&= \sum_{i=0}^{u}\sum_{l=0}^{i}\sum_{j=0}^{s}\sum_{k=0}^{v} \gamma_{u,i,l} \binom{s}{j}\binom{v}{k} (-2)^{j+k} (u-i)^j (t-i)^k f^{r+u-i} h^{s+v+i-(j+k+l)} e^{t+w-i}.
\end{aligned}$$

The above, together with the recursion given by (24) allows the computation of the structure constants.

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