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Existence of multiple solutions to a discrete boundary value problem with mixed periodic boundary conditions

Kimberly Howard, Long Wang and Min Wang

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A second-order discrete boundary value problem with mixed periodic boundary conditions is studied. Sufficient conditions on the multiplicity of solutions in a weak sense are obtained by using the critical point theory. An example is given to demonstrate the applications of our results as well.

1. Introduction

We consider a nonlinear boundary value problem (BVP) consisting of a second-order difference equation

$$-\Delta^2 u(t-1) = f(t, u(t)), \quad t \in [2, N]_{\mathbb{Z}}, \tag{1-1}$$

and a pair of mixed periodic boundary conditions (BCs)

$$u(0) = -u(N), \quad \Delta u(0) = \Delta u(N), \tag{1-2}$$

where

- $N \geq 3$ is a positive integer and $[a, b]_{\mathbb{Z}}$ denotes the discrete interval $\{a, a+1, \dots, b\}$ for any integers a and b with $a \leq b$;
- Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$ and $\Delta^2 u(t) = \Delta(\Delta u(t))$; and
- $f : [2, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is continuous in its second variable.

By a solution of BVP (1-1), (1-2), we mean a function $u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R}$ that satisfies (1-1) and (1-2).

Remark 1.1. It is notable that in this paper, the solution is only required to satisfy (1-1) on $[2, N]_{\mathbb{Z}}$ instead of $[1, N]_{\mathbb{Z}}$. We call it *a solution in a weak sense*. Assume (1-1) is defined on $[1, N]_{\mathbb{Z}}$. By applying the standard variational technique in the

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traditional way, let

$$\widetilde{H} = \{u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R} \mid u \text{ satisfies the BC (1-2)}\},$$

$$\tilde{J}(u) = \frac{1}{2}\sum_{t=1}^{N}(\Delta u(t-1))^2 - \sum_{t=1}^{N}\int_0^{u(t)} f(t,s)\,ds.$$

If $u \in \widetilde{H}$ is a critical point of $\tilde{J}(u)$, then u must satisfy $\Delta u(0) = \Delta u(N) = 0$ and $u(0) = -u(N)$, which become a special case of the antiperiodic BCs studied in the literature; see for example [Lyons and Neugebauer 2015a; Kuang and Yang 2018]. We are not interested in such simple BCs. This is the motivation to consider the solution in a weak sense. The reader is referred to [Kong and Wang ≥ 2020, Remark 2.2] for more discussions on the solution in a weak sense.

BVPs have been an active research area for decades. Both continuous and discrete BVPs subject to various BCs have been investigated by numerous scholars. The reader is referred to [Davis et al. 2016; Feng et al. 2018; Garcia and Neugebauer 2019; Graef et al. 2008; 2013; 2017; 2018; Henderson and Luca 2018; 2019; Kuang and Yang 2018; Lyons and Neugebauer 2015a; 2015b; Liang and Weng 2007; Liu et al. 2018; Kong et al. 2017] for recent advances in BVPs. Although BVPs with either periodic BCs or antiperiodic BCs have been extensively studied, to the best of our knowledge, there are relatively few works on the problems with mixed periodic BCs, i.e., a combination of periodic and antiperiodic BCs. For continuous BVPs, Garcia and Neugebauer [2019] studied the existence of solutions for the BVPs consisting of the differential equation

$$u'' = f(t, u, u'), \quad 0 < t < 1,$$

and two sets of mixed periodic BCs

$$u(0) + u(1) = 0, \quad u'(0) - u'(1) = 0,$$

or

$$u(0) - u(1) = 0, \quad u'(0) + u'(1) = 0.$$

For discrete BVPs, Kong and Wang [2020] considered the existence of solutions (in a weak sense) for the BVP

$$\begin{aligned} -\Delta^2 u(t-1) &= f(u(t)), \quad t \in [2, N]_{\mathbb{Z}}, \\ u(0) = -u(N), &\quad \Delta u(0) = \Delta u(N), \end{aligned} \tag{1-3}$$

by the mountain pass lemma. In [Kong and Wang ≥ 2020], the existence of solutions (in a weak sense) for the BVP

$$\begin{aligned} -\Delta(r(t-1)\Delta u(t-1)) &= f(t, u(t)), \quad t \in [2, N]_{\mathbb{Z}}, \\ u(0) = u(N), &\quad r(0)\Delta u(0) = -r(N)\Delta u(N), \end{aligned}$$

was studied by Clark's theorem. Note that as commented in [Kong and Wang ≥ 2020], there was a typo in (1.1) in [Kong and Wang 2020]. The domain was mistakenly written as $t \in [1, N]_{\mathbb{Z}}$ and should be replaced by $t \in [2, N]_{\mathbb{Z}}$ as seen in (1-3) above.

We will extend [Kong and Wang 2020] by considering a more general problem. It is easy to see that BVP (1-1), (1-2) covers BVP (1-3) as a special case. By using the linking theorem, a new set of sufficient conditions for the existence of solutions will be derived. Our work will supplement the existing results and contribute to the application of the variational method to the BVPs with mixed periodic BCs.

This paper is organized as follows: after this introduction, the preliminary results are given in Section 2. Section 3 contains the main results. An example is given in Section 3 as well.

2. Preliminaries

We present some definitions and lemmas needed for the proof of our main result.

Definition 2.1. Assume U is a Banach space. We say that a functional $J \in C^1(U, \mathbb{R})$ satisfies the Palais–Smale (PS) condition if every sequence $\{u_n\} \subset U$, such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. The sequence $\{u_n\}$ is called a PS sequence.

The following lemma, which is the well-known linking theorem, plays a crucial role in the proof of our main results; see for example [Rabinowitz 1986, Theorem 5.3].

Lemma 2.2. *Let U be a real Banach space, $U = U_1 \oplus U_2$, where U_1 is finite-dimensional. Suppose that $J \in C^1(U, \mathbb{R})$ satisfies the PS condition and the following*:

(J1) *There are positive constants c and ρ such that $J|_{\partial B_\rho(0)\cap U_2} \geq c$.*

(J2) *There are $\mu \in \partial B_1(0) \cap U_2$ and a positive constant $\hat{c} \geq \rho$ such that $J|_{\partial\Omega} \leq 0$, where*

$$\Omega = (\bar{B}_{\hat{c}}(0) \cap U_1) \oplus \{s\mu \mid 0 < s < \hat{c}\}.$$

Then J possesses a critical value $c_0 \geq c$, where

$$c_0 = \inf_{d\in\Gamma} \sup_{u\in\Omega} J(d(u)) \tag{2-1}$$

and $\Gamma = \{d \in C(\bar{\Omega}, U) \mid d|_{\partial\Omega} = I\}$, where I denotes the identity operator.

In the rest of the paper, we will use the Banach space U defined by

$$U = \{u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R} \mid u(0) = -u(N),\ \Delta u(0) = \Delta u(N),\ u(1) = 0\}.$$

By [Kong and Wang 2020, Remark 2.1], U is an $(N-1)$-dimensional Banach space with the norm defined by

$$\|u\| = \left(\sum_{t=1}^{N} u^2(t)\right)^{\frac{1}{2}}.$$

Let $\tilde{f} : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{F} : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\tilde{f}(t, x) = \begin{cases} 0, & t = 1, \\ f(t, x), & t \in [2, N-1]_{\mathbb{Z}}, \\ f(t, x) + 2x, & t = N, \end{cases} \tag{2-2}$$

$$\tilde{F}(t, x) = \int_0^x \tilde{f}(t, s)\, ds. \tag{2-3}$$

Clearly, both $\tilde{f}(t, x)$ and $\tilde{F}(t, x)$ are continuous in x. Define $J : U \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \sum_{t=1}^{N} (\Delta u(t-1))^2 - \sum_{t=1}^{N} \tilde{F}(t, u(t)). \tag{2-4}$$

A critical point of J is a point u at which J', the Fréchet derivative of J, vanishes, and the value of J at u is then called a critical value of J; see for example [Rabinowitz 1986]. The following lemma links the critical point of J and the solution of BVP (1-1), (1-2).

Lemma 2.3. *The function $u \in U$ is a critical point of J if and only if u is a solution of BVP (1-1), (1-2) in a weak sense.*

Proof. For any $u \in U$, from (2-2)–(2-4), we obtain

$$J(u) = \frac{1}{2} \sum_{t=1}^{N} (\Delta u(t-1))^2 - \sum_{t=2}^{N} \int_0^{u(t)} f(t, s)\, ds - 2 \int_0^{u(N)} s\, ds.$$

It is easy to see that J is continuously differentiable. For any $v \in U$, we have that $J'(u)$ at $u \in U$ is given by

$$\langle J'(u), v \rangle = \sum_{t=1}^{N} \Delta u(t-1) \Delta v(t-1) - \sum_{t=2}^{N} f(t, u(t)) v(t) - 2u(N) v(N).$$

Using the summation by parts formula, we have

$$\begin{aligned} \sum_{t=1}^{N} \Delta u(t-1) \Delta v(t-1) &= \Delta u(N) v(N) - \Delta u(0) v(0) - \sum_{t=1}^{N} \Delta^2 u(t-1) v(t) \\ &= 2\Delta u(N) v(N) - \sum_{t=1}^{N} \Delta^2 u(t-1) v(t). \end{aligned}$$

Then it follows that

$$\begin{aligned}\langle J'(u), v\rangle &= 2\Delta u(N)v(N)-\sum_{t=1}^{N}\Delta^2 u(t-1)v(t)-\sum_{t=2}^{N} f(t,u(t))v(t)-2u(N)v(N)\\ &= -2\Delta u(0)v(0)-\sum_{t=1}^{N}\Delta^2 u(t-1)v(t)-\sum_{t=2}^{N} f(t,u(t))v(t)-2u(0)v(0)\\ &= -\sum_{t=1}^{N}\Delta^2 u(t-1)v(t)-\sum_{t=2}^{N} f(t,u(t))v(t)\\ &= -\sum_{t=2}^{N}[\Delta^2 u(t-1)+f(t,u(t))]v(t).\end{aligned}$$

Therefore $\langle J'(u), v\rangle = 0$ if and only if (1-1) holds. ☐

Now we consider an equivalent form of J. Let

$$A=\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 & 1\\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0\\ 0 & -1 & 2 & -1 & 0 & \cdots & 0\\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0\\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1\\ 1 & 0 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix}_{N\times N}, \quad N\geq 3. \tag{2-5}$$

Then it is easy to verify that for any $u\in U$,

$$J(u)=\tfrac{1}{2}u^T Au-\sum_{t=1}^{N}\tilde{F}(t,u(t)), \tag{2-6}$$

where $(\,\cdot\,)^T$ denotes the transpose and $u=(0,u(2),\dots,u(N))^T$. Note that by [Kong and Wang 2020, Remark 2.2], A has N positive eigenvalues $0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_N$ such that for any $u\in\mathbb{R}^N$, we have $\lambda_1\|u\|^2\leq u^T Au\leq\lambda_N\|u\|^2$.

3. Main results

The following theorem is our first main result.

Theorem 3.1. *Let $\lambda_1,\dots,\lambda_N$ be the eigenvalues of A defined by (2-5). Assume the function f from (1-1) satisfies $f(t,0)=0$, $t\in[2,N]_{\mathbb{Z}}$. Additionally, the function $\tilde{F}(t,x)$ defined by (2-3) satisfies the following assumptions*:

(F1) *There exist two constants $\delta_1>0$ and $a_1\in\big(0,\tfrac{1}{2}\lambda_1\big)$ such that $\tilde{F}(t,x)\leq a_1x^2$ for $|x|\leq\delta_1$ and $t\in[2,N]_{\mathbb{Z}}$.*

(F2) *There exist two constants* $a_2 \in \left(\frac{1}{2}\lambda_N, \infty\right)$ *and* $a_3 > 0$ *such that* $\tilde{F}(t, x) \geq a_2 x^2 - a_3$ *for all* $(t, x) \in [2, N]_{\mathbb{Z}} \times \mathbb{R}$.

Then BVP (1-1), (1-2) *admits at least three solutions, one being trivial, and two being nontrivial.*

The following lemma will be needed to prove Theorem 3.1.

Lemma 3.2. *Assume* (F2) *of Theorem 3.1 is satisfied. Then the functional* J *defined by* (2-4) (*or* (2-6)) *satisfies the PS condition.*

Proof. For any sequence $\{u_k\} \subseteq U$ with $J'(u_k) \to 0$ as $k \to \infty$ and $-B \leq J(u_k) \leq B$, where B is a positive constant, we claim $\{u_k\}$ is bounded. We will prove it by contradiction.

Assume $\{u_k\} \subseteq U$ is an unbounded sequence. Then there exists a subsequence $\{u_m\}$ of $\{u_k\}$ such that $\|u_m\| \to \infty$ and $J'(u_m) \to 0$ as $m \to \infty$, and $-B \leq J(u_m) \leq B$ for $m = 1, 2, \ldots$. From (F2) of Theorem 3.1 and (2-6), when m is large enough, we have

$$\begin{aligned} J(u_m) &= \tfrac{1}{2} u_m^T A u_m - \sum_{t=1}^{N} \tilde{F}(t, u_m(t)) \\ &\leq \tfrac{1}{2}\lambda_N \|u_m\|^2 - \sum_{t=2}^{N} \tilde{F}(t, u_m(t)) \leq \tfrac{1}{2}\lambda_N \|u_m\|^2 - \sum_{t=2}^{N} (a_2 (u_m(t))^2 - a_3) \\ &\leq \tfrac{1}{2}\lambda_N \|u_m\|^2 - a_2 \|u_m\|^2 + a_3 N = \left(\tfrac{1}{2}\lambda_N - a_2\right)\|u_m\|^2 + a_3 N. \end{aligned} \tag{3-1}$$

Hence $J(u_m) \to -\infty$ as $m \to \infty$, and this contradicts the fact that $\{J(u_m)\}$ is bounded. Thus, $\{u_k\}$ is bounded. Since U is a finite-dimensional space, we know $\{u_k\}$ has a convergent subsequence. Therefore, J satisfies the PS condition. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. It is obvious that $u \equiv 0$ is a solution of BVP (1-1), (1-2).

By (3-1), we can show $J(u) \to -\infty$ as $\|u\| \to \infty$ for any $u \in U$. Using the fact that $J(u)$ is bounded above in U, define $\bar{J} = \sup_{u \in U} J(u)$. Therefore, we have a sequence $\{u_k\}$ on U such that $\bar{J} = \lim_{k\to\infty} J(u_k)$. From the proof of Lemma 3.2, it follows that $\{u_k\}$ is bounded. Then $\{u_k\}$ has a convergent subsequence defined by $\{u_{k_n}\}$. Let $\bar{u} = \lim_{n\to\infty} u_{k_n}$, and it is easy to see that $\bar{u} \in U$ with $J(\bar{u}) = \bar{J}$. Therefore $\bar{u}$ is a critical point of J.

To apply Lemma 2.2, we need to separate U into two subspaces P and Q with $U = P \oplus Q$. Let $\{\eta_1, \ldots, \eta_N\} \subset \mathbb{R}^N$ be the orthonormal eigenvectors associated with $\{\lambda_1, \ldots, \lambda_N\}$. Define $P = \text{span}\{\eta_1, \ldots, \eta_M\}$ and $Q = \text{span}\{\eta_{M+1}, \ldots, \eta_N\}$, for some $M \in [2, N-1]_{\mathbb{Z}}$, such that $U = P \oplus Q$, $P \cap U \neq \{0\}$, and $Q \cap U \neq \{0\}$. Clearly, both P and Q are finite-dimensional spaces. Moreover, for any $u \in P$, there exist $b_1, \ldots, b_M \in \mathbb{R}$ such that $u = \sum_{i=1}^{M} b_i \eta_i$ and $\|u\|^2 = \sum_{i=1}^{M} b_i^2$; for any $u \in Q$, there exist $b_{M+1}, \ldots, b_N \in \mathbb{R}$ such that $u = \sum_{i=M+1}^{N} b_i \eta_i$ and $\|u\|^2 = \sum_{i=M+1}^{N} b_i^2$.

Since U is finite-dimensional, for any $u \in U$, there exist positive numbers κ_1 and κ_2 such that $\kappa_1 < \kappa_2$ and

$$\kappa_1 \max\{|u(1)|, |u(2)|, \ldots, |u(N)|\} \le \|u\| \le \kappa_2 \max\{|u(1)|, |u(2)|, \ldots, |u(N)|\}.$$

Observe for any $u \in Q$, such that $\|u\| = \kappa_1\delta_1$ and $\max\{|u(1)|, |u(2)|, \ldots, |u(N)|\} \le \delta_1$,

$$\begin{aligned} J(u) &= \tfrac{1}{2} u^T A u - \sum_{t=1}^{N} \tilde{F}(t, u(t)) \ge \tfrac{1}{2}\lambda_1 \|u\|^2 - \sum_{t=2}^{N} \tilde{F}(t, u(t)) \\ &\ge \tfrac{1}{2}\lambda_1 \|u\|^2 - \sum_{t=2}^{N} a_1 (u(t))^2 = \tfrac{1}{2}\lambda_1 \|u\|^2 - a_1 \|u\|^2 \\ &= \bigl(\tfrac{1}{2}\lambda_1 - a_1\bigr)(\kappa_1\delta_1)^2 > 0. \end{aligned} \tag{3-2}$$

Then assumption (J1) of Lemma 2.2 is satisfied. It is easy to see (3-2) also implies $\bar{J} > 0$. So $\bar{u}$ is a nontrivial solution of BVP (1-1), (1-2).

Next we verify (J2) of Lemma 2.2. Choose $\mu \in Q$ and a positive constant $\hat{c}$ such that $\|\mu\| = 1$ and

$$\hat{c} > \max\left\{\kappa_1\delta_1, \frac{2a_3 N}{2a_2 - \lambda_N}\right\}.$$

Let $\Omega = (B_{\hat{c}}(0) \cap P) \oplus \{s\mu \mid 0 < s < \hat{c}\}$. Note that for any $\nu \in P$

$$\begin{aligned} J(s\mu + \nu) &= \tfrac{1}{2}(s\mu + \nu)^T A (s\mu + \nu) - \sum_{t=1}^{N} \tilde{F}(t, s\mu(t) + \nu(t)) \\ &\le \tfrac{1}{2}\lambda_N s^2 + \tfrac{1}{2}\lambda_N \|\nu\|^2 - \sum_{t=2}^{N} \bigl(a_2 (s\mu(t) + \nu(t))^2 - a_3\bigr) \\ &\le \tfrac{1}{2}\lambda_N s^2 + \tfrac{1}{2}\lambda_N \|\nu\|^2 - a_2 s^2 - a_2 \|\nu\|^2 + a_3 N \\ &= \bigl(\tfrac{1}{2}\lambda_N - a_2\bigr) s^2 + \bigl(\tfrac{1}{2}\lambda_N - a_2\bigr)\|\nu\|^2 + a_3 N \\ &\le \bigl(\tfrac{1}{2}\lambda_N - a_2\bigr)\|\nu\|^2 + a_3 N. \end{aligned}$$

Therefore, $J|_{\partial\Omega} \le 0$. Hence (J2) holds.

Then by Lemma 2.2, we know $J(u)$ has a critical value $c_0 \ge \bigl(\tfrac{1}{2}\lambda_1 - a_1\bigr)(\kappa_1\delta_1)^2$. Let $\tilde{u} \in U$ be a critical point corresponding to c_0; i.e., $J(\tilde{u}) = c_0$. If $\bar{u} \ne \tilde{u}$, then by Lemma 2.3, $\bar{u}$ and $\tilde{u}$ are two different solutions of BVP (1-1), (1-2). If $\bar{u} = \tilde{u}$, by (2-1), $\sup_{u \in U} J(u) = \inf_{d \in \Gamma} \sup_{u \in \Omega} J(d(u))$. Choose distinct $d_1, d_2 \in \Gamma$ such that

$$\{d_1(u) \mid u \in \Omega \setminus \partial\Omega\} \cap \{d_2(u) \mid u \in \Omega \setminus \partial\Omega\} = \varnothing.$$

Then

$$\sup_{u\in\Omega} J(d_1(u)) = \sup_{u\in\Omega} J(d_2(u)) = \sup_{u\in U} J(u).$$

Hence there exist $u_1, u_2 \in \Omega$ such that $d_1(u_1)$ and $d_2(u_2)$ are two different critical points of J.

Therefore, we have shown BVP (1-1), (1-2) admits at least three solutions. □

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.3. *Assume the function f from (1-1) satisfies $f(t,0)=0$, $t \in [2,N]_{\mathbb{Z}}$, and $\tilde{F}(t,x)$ defined by (2-3) is of the form*

$$\tilde{F}(t,x) = a(t)|x|^{r(t)}, \quad a(t) > 0,\ r(t) > 2,\ t \in [2,N]_{\mathbb{Z}}.$$

Then BVP (1-1), (1-2) admits at least three solutions, one being trivial, and two being nontrivial.

We conclude the paper with the following example.

Example 3.4. Consider BVP (1-1), (1-2) with $N=4$ and f defined by

$$f(t,x) = \begin{cases} \frac{2}{5}x^3, & t = 2, 3, \\ \frac{2}{5}x^3 - 2x, & t = 4. \end{cases}$$

By (2-2) and (2-3), we know

$$\tilde{F}(t,x) = \tfrac{1}{10}x^4, \quad t = 2, 3, 4.$$

Then by Corollary 3.3, BVP (1-1), (1-2) admits at least three solutions.

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