involve
a journal of mathematics

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msp

2020 vol. 13, no. 4

INVOLVE 13:4 (2020)
https://doi.org/10.2140/involve.2020.13.705

Some new Gompertz fractional difference equations

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(Communicated by Martin J. Bohner)

We introduce three new fractional Gompertz difference equations using the Riemann–Liouville discrete fractional calculus. These three models are based a nonfractional Gompertz difference equation, and they differ depending on whether a fractional operator replaces the difference operator, the integral operator defining the logarithm, or both simultaneously. An explicit solution to one of them is achieved with restricted parameters and recurrence relation solutions are derived for all three. Finally, we fit these models to data to compare them with a previously published discrete fractional Gompertz model and the continuous model.

1. Introduction

The Gompertz function dates back to the study of human mortality conducted by Benjamin Gompertz [1825]. Since then, it has been used in numerous ways, particularly as a population growth model; e.g., see [Alves et al. 2019; Easton 1999; Jane et al. 2020; Laird 1964; Pezzini et al. 2019; Winsor 1932]. The Gompertz differential equation is

$$y' = -ry \ln\left(\frac{y}{K}\right),$$

which has closed-form general solution as the iterated exponential $y(t) = Ke^{Ce^{-rt}}$.

In contrast to the continuous Gompertz model, we are concerned with a discrete theory. The fundamental operator of the "forward" difference calculus is the Δ operator that obeys $\Delta f(t) = f(t+1) - f(t)$. An enormous amount of research in discrete analogues to the theory of differential equations has been conducted for these operators, and there are many well-known popular introductions to it such as [Bohner and Peterson 2001; Elaydi 2005; Kelley and Peterson 2010]. There are numerous difference equations that have been called a "Gompertz difference equation" in the

MSC2020: primary 39A60; secondary 26A33, 92D25, 33E12.
Keywords: discrete fractional calculus, Gompertz, parameter estimation, data fitting.

literature, such as in [Akın et al. 2020; Nobile et al. 1982; Satoh 2000]. We now consider the Δ-Gompertz model from [Bohner 2005, (12)], defined by

$$\Delta y = (\ominus r)(a + \tilde{L}_y)y, \quad y(t_0) = y_0, \tag{1}$$

where r and a are constants and

$$\tilde{L}_y(t, t_0) = \sum_{k=t_0}^{t-1} \frac{y(k+1) - y(k)}{y(k)} \tag{2}$$

is the Δ-logarithm, originally defined in [Bohner 2005, (3)]. The model (1) originally used time-dependent r and a, but we will not consider that generality here.

We will generalize (1) to a model using fractional sum and difference operators. Fractional differential calculus dates back to the early days of the theory of calculus [Ross 1975]. The fundamental idea of fractional calculus is the extension of derivatives and integrals to noninteger order. The fractional discrete calculus dates back to [Kuttner 1957], with an explosion of interest in more recent times [Abdeljawad 2011; Atici and Eloe 2009; Dzieliński et al. 2010; Ferreira 2013; Miller and Ross 1989; Wu et al. 2015]; see in particular the recent monograph [Goodrich and Peterson 2015]. An analogue of the Gompertz differential equation using fractional differential operators has also been developed and studied [Solís-Pérez et al. 2019; Frunzo et al. 2019]. Fractional Gompertz difference equations are not a new idea; for example, see [Wang et al. 2014; Bolton et al. 2015; Atıcı and Şengül 2010], which use various fractional difference operators. We will take one model from the literature for comparison to the models we will propose: Atıcı et al. defined a fractional Δ-difference equation in [Atıcı et al. 2017, (3)] by

$$(c - b\log y(t))y(t) = \begin{cases} \Delta^\nu y^\sigma(t-\nu), & \nu \in (0, 1.5) \setminus \{1\}, \\ \Delta y(t), & \nu = 1, \end{cases} \tag{3}$$

where Δ^ν denotes the discrete Riemann–Liouville Δ-fractional difference operator. We emphasize that (3) uses the classical logarithm and not a logarithm resembling (2). This is the fundamental change we have made by choosing (1) as the basis for our models. The equation (3) has the following form, convenient for numerical computation [Atıcı et al. 2017, p. 323]:

$$y(t+1) = \begin{cases} (c - b\log(y(t)))y(t) - \sum_{s=0}^{t-1} \frac{\Gamma(t-\nu-s)}{\Gamma(t-s+1)\Gamma(-\nu)} y(s+1), & \nu \in (0, 1.5) \setminus \{1\}, \\ (c + 1 - b\log(y(t)))y(t), & \nu = 1. \end{cases} \tag{4}$$

The recurrence (4) was used to fit the model (3) to data collected on mice tumors.

2. Preliminaries and definitions

For $a, b \in \mathbb{R}$ with $a-b$ a positive integer, we define as in [Goodrich and Peterson 2015, p. 1], henceforth abbreviated [GP15], the sets

$$\mathbb{N}_a := \{a, a+1, a+2, a+3, \dots\}, \quad \mathbb{N}_a^b := \{a, a+1, a+2, a+3, \dots, b\}.$$

Throughout, we use the usual convention for empty products that $\prod_{j=a}^{b} f(j) = 1$ and for empty sums $\sum_{j=a}^{b} f(j) = 0$ whenever $b < a$. The backwards jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_{a-1}$ is given by [GP15, p. 150], where $\rho(t) = t-1$ for $t > 0$ and $\rho(0) = 0$. The backwards difference operator is [GP15, p. 150] $\nabla y(t) = y(t) - y(\rho(t))$. If $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $a \in \mathbb{N}_a$, and $b \in \mathbb{N}_{a+1}$, then the discrete ∇-integral of f is the summation defined by

$$\int_a^b f(t)\, \nabla t = \sum_{t=a+1}^{b} f(t)$$

[GP15, Definition 3.31].

The gamma function $\Gamma : (0, \infty) \to [1, \infty)$ is defined by

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x}\, \mathrm{d}x$$

[Artin 1964, (2.1)]. Integration by parts shows that the functional equation

$$\Gamma(t+1) = t\Gamma(t)$$

holds [Artin 1964, (2.2)], which extends the domain of Γ to $\mathbb{R}\backslash\{0, -1, -2, \dots\}$. For $r \in \mathbb{Z}^+$ and $t \in \mathbb{N}_0$, the rising factorial function is

$$t^{\bar{r}} = \frac{\Gamma(t+r)}{\Gamma(t)}$$

[GP15, Definition 3.3]. The gamma function is primarily used here to define the ∇-Taylor monomials, H_μ, which are defined for $\mu \neq -1, -2, -3, \dots$ and $t \in \mathbb{N}_a$ by [GP15, Definition 3.56],

$$H_\mu(t, a) = \frac{(t-a)^{\bar{\mu}}}{\Gamma(\mu+1)} = \frac{\Gamma(t-a+\mu)}{\Gamma(\mu+1)\Gamma(t-a)},$$

so if $\mu = 0$, then we observe $H_0(t, a) = 1$. In particular, we observe the following useful identity that we will occasionally use:

$$H_\mu(t, t-1) = \frac{\Gamma(t-(t-1)+\mu)}{\Gamma(\mu+1)\Gamma(t-(t-1))} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1)\Gamma(1)} = 1.$$

Given $p : \mathbb{N}_a \to \mathbb{R}$ such that $1 - p(t) \neq 0$ for all t, we define the $\boxminus$ operator by

$$(\boxminus p)(t) = \frac{-p(t)}{1-p(t)}$$

[GP15, (3.10)]. In particular, if $p(t) = r$ is a constant function, then

$$(\boxminus r)(t) = \frac{-r}{1-r}$$

is also a constant function. For $p : \mathbb{N}_{a+1} \to \mathbb{R}$ obeying $1-p(t) \neq 0$, the ∇-exponential function E_p is given by [GP15, (3.7)]

$$E_p(t,s) = \begin{cases} \prod_{\tau=s+1}^{t} 1/(1-p(\tau)), & t \in \mathbb{N}_s, \\ \prod_{\tau=t+1}^{s} [1-p(\tau)], & t \in \mathbb{N}_a^{s-1}. \end{cases} \tag{5}$$

The following result confirms that the ∇-exponential solves a first-order difference equation.

Theorem 1 [GP15, Theorem 3.6]. *Assume $p : \mathbb{N}_{a+1} \to \mathbb{R}$ such that $1 - p(t) \neq 0$, and let $t \in \mathbb{N}_{a+1}$. The ∇-exponential $y(t) = E_p(t,s)$ is the unique solution of the initial value problem*

$$\nabla y(t) = p(t)y(t), \quad y(s) = 1.$$

If $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$, and $t \in \mathbb{N}_a$, then the ν-th order ∇-fractional sum of f at t is defined by [GP15, Definition 3.58]

$$\nabla_a^{-\nu} f(t) = \int_a^t H_{\nu-1}(t,\rho(s)) f(s)\,\nabla s,$$

where we let $\nabla_a^{-0} f(t) = f(t)$. Let $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$, $t \in \mathbb{N}_{a+1}$, and $N \in \mathbb{N}_1$, where $N - 1 < \nu \leq N$. The ν-th order ∇-fractional difference of f at t is similarly defined by [GP15, (3.32)]

$$\nabla_a^{\nu} f(t) = \int_a^t H_{-\nu-1}(t,\rho(s)) f(s)\,\nabla s. \tag{6}$$

From the definition, it is easy to see that $\nabla_a^{\nu} f(a+1) = f(a+1)$. The following result shows how fractional differences and summations compose.

Theorem 2 [GP15, Theorem 3.109]. *If $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu, \mu > 0$, then*

$$\nabla_a^{\nu} \nabla_a^{-\mu} f(t) = \nabla_a^{\nu-\mu} f(t).$$

The fractional difference operator subtracts from the order of the ∇-Taylor monomials as $\nabla_{t_0}^{\nu} H_\mu(t,t_0) = H_{\mu-\nu}(t,t_0)$ [GP15, Theorem 3.93(ii)], and so for any constant C

$$\nabla_a^{\nu} C = C \nabla_a^{\nu} H_0(t,a) = C H_{-\nu}(t,a).$$

In order to solve fractional initial value problems in closed form, the ∇-convolution is needed. Let $f, g : \mathbb{N}_{a+1} \to \mathbb{R}$, and define the ∇-convolution by [GP15, Definition 3.77]

$$(f * g)(t) = \int_a^t f(t - \rho(\tau) + a) g(\tau) \nabla\tau.$$

The ∇-Mittag-Leffler function, $E_{p,\alpha,\beta}$, where $|p| < 1,\ \alpha > 0,\ \beta \in \mathbb{R}$, and $t \in \mathbb{N}_a$, is defined by [GP15, Definition 3.98]

$$E_{p,\alpha,\beta}(t, a) = \sum_{k=0}^{\infty} p^k H_{\alpha k+\beta}(t, a).$$

The following result is a discrete fractional analogue of the variation of constants formula for first-order initial value problems.

Theorem 3 [GP15, Theorem 3.104]. *If $f : \mathbb{N}_a \to \mathbb{R}$, $|c| < 1$, and $0 < \nu < 1$, then the unique solution of the fractional initial value problem*

$$\begin{cases} \nabla^{\nu}_{\rho(a)} x(t) + c x(t) = f(t), & t \in \mathbb{N}_{a+1}, \\ x(a) = A, & A \in \mathbb{R}, \end{cases}$$

is given by

$$x(t) = [E_{-c,\nu,\nu-1}(\,\cdot\,, \rho(a)) * f(\,\cdot\,)](t) + [A(c+1) - f(a)] E_{-c,\nu,\nu-1}(t, \rho(a)). \quad (7)$$

The recurrence (4) came from applying a Δ-analogue [Atıcı and Şengül 2010, p. 322] of the following lemma to (3).

Lemma 4. *If $f : \mathbb{N}_{s+1} \to \mathbb{R}$, then for $t \in \mathbb{N}_{s+1}$*

$$\nabla^{\nu}_{s} f(t) = f(t) + \sum_{k=s+1}^{t-1} \frac{\Gamma(t-k-\nu) f(k)}{\Gamma(-\nu)\Gamma(t-k+1)}.$$

Proof. Calculating directly from the definition,

$$\begin{aligned} \nabla^{\nu}_{s} f(t) &= \int_s^t H_{-\nu-1}(t, \rho(\tau)) f(\tau) \nabla\tau = \sum_{k=s+1}^{t} H_{-\nu-1}(t, \rho(k)) f(k) \\ &= H_{-\nu-1}(t, \rho(t)) f(t) + \sum_{k=s+1}^{t-1} H_{-\nu-1}(t, \rho(k)) f(k) \\ &= f(t) + \sum_{k=s+1}^{t-1} \frac{\Gamma(t-k-\nu) f(k)}{\Gamma(-\nu)\Gamma(t-k+1)}, \end{aligned}$$

completing the proof. □

We now define the various ∇-logarithms we will use. The natural ∇-analogue of (2) is

$$\mathrm{L}_y(t,t_0)=\int_{t_0}^{t}\frac{y^{\nabla}(\tau)}{y(\tau)}\,\nabla\tau,\qquad t\in\mathbb{N}_{t_0+1}.$$

We define the first fractional ∇_ν-logarithm $\mathrm{L}_{y,\nu}$ by

$$\mathrm{L}_{y,\nu}(t,t_0)=\left(\nabla_{t_0}^{-\nu}\frac{y^{\nabla}}{y}\right)(t),\qquad t\in\mathbb{N}_{t_0+1}. \tag{8}$$

we define the second fractional ∇_ν-logarithm $\ell_{y,\nu}$ by

$$\ell_{y,\nu}(t,t_0)=\int_{t_0}^{t}\frac{\nabla_{t_0}^{\nu}y(\tau)}{y(\tau)}\,\nabla\tau,\qquad t\in\mathbb{N}_{t_0+1}, \tag{9}$$

and the third ∇_ν-logarithm $\Lambda_{y,\nu}$ by

$$\Lambda_{y,\nu}(t,t_0)=\left(\nabla_{t_0}^{-\nu}\frac{\nabla_{t_0}^{\nu}y}{y}\right)(t),\qquad t\in\mathbb{N}_{t_0+1}. \tag{10}$$

3. Three Gompertz fractional ∇-difference equations

The natural ∇-analogue of (1) is

$$\nabla y=(\boxminus r)(a+\mathrm{L}_y)y,\qquad y(t_0)=y_0, \tag{11}$$

whose unique solution is easily found using the same method used in [Cuchta and Streipert 2020]. There are many ways in which a ∇-fractional analogue of (11) could be defined, depending on which operators in the equation are made fractional. We will consider three such analogues in this article. In light of (3), we will allow the order of our fractional derivative to lie in the interval (0, 1.5), where $\nu=1$ corresponds to the nonfractional ∇ difference. First, we preserve the nonfractional difference on y from (11), but we use the fractional ∇-logarithm (8) to obtain

$$\nabla y=(\boxminus r)y(a+\mathrm{L}_{y,\nu}),\qquad y(t_0)=y_0. \tag{12}$$

If instead the difference on y in (11) is a fractional difference, but the ∇-logarithm (9) is used, then we obtain

$$\nabla_{t_0}^{\nu}y(t)=(\boxminus r)y(a+\ell_{y,\nu}),\qquad y(t_0+1)=y_0. \tag{13}$$

Finally, if we make the difference on y in (11) a fractional difference and use the fractional ∇-logarithm (10), then we obtain

$$\nabla_{t_0}^{\nu}y(t)=(\boxminus r)y(a+\Lambda_{y,\nu}),\qquad y(t_0+1)=y_0. \tag{14}$$

Note that all these logarithms reduce to L_y as $\nu\to 1$. We now find a closed-form solution of (12) in terms of the ∇-Mittag-Leffler function.

Theorem 5. *If* $0 < \nu < 1$, $r < \frac{1}{2}$, *and, for* $t \in \mathbb{N}_{t_0}$,

$$\sum_{k=t_0+1}^{t} E_{\boxminus r,\nu,\nu-1}(t-k+1+t_0, t_0)H_{-\nu}(k, t_0) \neq -1 - \frac{1-r}{ar}, \tag{15}$$

then the initial value problem (12) *has the unique solution* $y(t) = y_0 E_p(t, t_0)$ *for* $t \in \mathbb{N}_{t_0}$, *where*

$$p(t) = (\boxminus r)a + (\boxminus r)\big(E_{\boxminus r,\nu,\nu-1}(\,\cdot\,, t_0) * aH_{-\nu}(\,\cdot\,, t_0)\big)(t).$$

Proof. If $z(t) = a + \mathrm{L}_{y,\nu}(t, t_0)$, then

$$(\nabla_{t_0}^{\nu} z)(t) = aH_{-\nu}(t, t_0) + \frac{\nabla y(t)}{y(t)}. \tag{16}$$

Now compute

$$\begin{aligned} a + \mathrm{L}_{y,\nu}(t_0+1, t_0) &= a + \left(\nabla_{t_0}^{-\nu} \frac{\nabla y}{y}\right)(t_0+1) \\ &= a + \int_{t_0}^{t_0+1} H_{\nu-1}(t_0+1, \rho(s)) \frac{\nabla y(s)}{y(s)} \nabla s \\ &= a + H_{\nu-1}(t_0+1, t_0) \frac{\nabla y(t_0+1)}{y(t_0+1)} = a + 1 - \frac{y_0}{y(t_0+1)}. \end{aligned}$$

We expand the difference equation (12) at $t = t_0 + 1$ to get

$$y(t_0+1) - y_0 = -\frac{r}{1-r} y(t_0+1)\left(a + 1 - \frac{y_0}{y(t_0+1)}\right),$$

and hence

$$y(t_0+1) = \frac{y_0}{1+ra}.$$

Immediately, we obtain $z(t_0+1) = a + \mathrm{L}_{y,\nu}(t_0+1, t_0) = a(1-r)$. Using the definition of z, (16), and dividing the difference equation (12) by y, the following initial value problem for z is derived:

$$\nabla_{t_0}^{\nu} z + (-\boxminus r)z = aH_{-\nu}(t, t_0), \quad z(t_0+1) = a(1-r). \tag{17}$$

Theorem 3 implies that (17) has the unique solution

$$z(t) = a + \big(E_{\boxminus r,\nu,\nu-1}(\,\cdot\,, t_0) * aH_{-\nu}(\,\cdot\,, t_0)\big)(t).$$

Therefore, we have the initial value problem $\nabla y = (\boxminus r)zy, \quad y(t_0) = y_0$. We need to ensure that the function $h : \mathbb{N}_{t_0} \to \mathbb{R}$ given by $h(t) = (\boxminus r)z(t)$ obeys $h(t) \neq 1$. Expanding out the definition of h reveals

$$1 \neq \left(\frac{-r}{1-r}\right)\left(a + ak \sum_{k=t_0}^{t} E_{\boxminus r,\nu,\nu-1}(t-k+1+t_0, t_0)H_{-\nu}(k, t_0)\right),$$

and by algebraic rearrangement, this becomes the condition (15). Thus, we may appeal to Theorem 1 to complete the proof. □

Theorem 5 provides an impractical solution for numerical computation for a variety of reasons. In particular, the ∇-Mittag-Leffler function requires the evaluation of an infinite series, the order is restricted to $0 < \nu < 1$, and condition (15) is not trivial to verify. So, we now derive a recurrence relation for numerical computation of the solution of (12).

Theorem 6. *The initial value problem (12) is algebraically equivalent to the recurrence*

$$y(t) = y_0 \prod_{k=t_0+1}^{t} \frac{1}{1-(\boxminus r)(a+z(k))}, \qquad t \in \mathbb{N}_{t_0},$$

where

$$z(t) = -ra - (1-r) \sum_{k=t_0+1}^{t-1} \frac{\Gamma(t-k-\nu)z(k)}{\Gamma(-\nu)\Gamma(t-k+1)}, \qquad t \in \mathbb{N}_{t_0+1}. \tag{18}$$

Proof. If $z(t) = \mathrm{L}_{y,\nu}(t,t_0)$, then

$$\nabla_{t_0}^{\nu} z(t) = \frac{y^{\nabla}(t)}{y(t)} = (\boxminus r)(a+z(t)).$$

Apply Lemma 4 to the left-hand side to get

$$z(t) + \sum_{k=t_0+1}^{t-1} \frac{\Gamma(t-k-\nu)z(k)}{\Gamma(-\nu)\Gamma(t-k+1)} = (\boxminus r)(a+z(t)). \tag{19}$$

We obtain (18) by rearranging (19) and solving for $z(t)$. Substituting $z(t)$ back into (12), we observe

$$\nabla y(t) = (\boxminus r)(a+z(t))y(t), \qquad y(t_0) = y_0.$$

By Theorem 1, we have $y(t) = y_0 E_{(\boxminus r)(a+z(t))}(t,t_0)$, and applying (5) completes the proof. □

Theorem 7. *The initial value problem (13) is algebraically equivalent to the following recurrence for $t > t_0+1$:*

$$y(t) = \frac{-\sum_{k=t_0+1}^{t-1} H_{-\nu-1}(t,k-1)y(k)}{1-r+r\Big(a+1+\sum_{k=t_0+1}^{t-1} \frac{1}{y(k)} \sum_{j=t_0+1}^{k} H_{-\nu-1}(k,j-1)y(j)\Big)}, \qquad t \in \mathbb{N}_{t_0+2}.$$

Proof. Expand the fractional derivative and the logarithm in (13) to obtain

$$\int_{t_0}^{t} H_{-\nu-1}(t,\rho(s))y(s)\,\nabla s = (\boxminus r)y(t)\left(a + \int_{t_0}^{t} \frac{\nabla_{t_0}^{\nu} y(\tau)}{y(\tau)}\,\nabla\tau\right).$$

Separating the final term of the ∇-integrals and rearranging algebraically yields

$$
\begin{aligned}
y(t) &= \frac{((\boxminus r) - 1) \int_{t_0}^{t-1} H_{-\nu-1}(t, \rho(s)) y(s) \, \nabla s}{1 - (\boxminus r)\big(a + 1 + \int_{t_0}^{t-1} \frac{\nabla_{t_0}^{\nu} y(\tau)}{y(\tau)} \, \nabla \tau\big)} \\
&= \frac{((\boxminus r) - 1) \int_{t_0}^{t-1} H_{-\nu-1}(t, \rho(s)) y(s) \, \nabla s}{1 - (\boxminus r)\big(a + 1 + \int_{t_0}^{t-1} \frac{1}{y(\tau)} \int_{t_0}^{\tau} H_{-\nu-1}(\tau, \rho(s)) y(s) \, \nabla s \, \nabla \tau\big)},
\end{aligned}
$$

and writing the ∇-integrals as summations, replacing $(\boxminus r)$ with $-r/(1-r)$, and routine algebraic simplification completes the proof. □

Theorem 8. *The initial value problem (14) is algebraically equivalent to the recurrence*

$$
y(t) = \frac{-\sum_{k=t_0+1}^{t-1} H_{-\nu-1}(t, k-1) y(k)}{1 - r + r\big(a + 1 + \sum_{k=t_0+1}^{t-1} \frac{H_{\nu-1}(t,k-1)}{y(k)} \sum_{j=t_0+1}^{k} H_{-\nu-1}(k, j-1) y(j)\big)}, \quad t \in \mathbb{N}_{t_0+2}. \tag{20}
$$

Proof. Expand (14) using definitions to get

$$
\int_{t_0}^{t} H_{-\nu-1}(t, \rho(\tau)) y(\tau) \, \nabla \tau = (\boxminus r) y(t) \bigg(a + \int_{t_0}^{t} H_{\nu-1}(t, \rho(\tau)) \frac{\nabla_{t_0}^{\nu} y(\tau)}{y(\tau)} \, \nabla \tau \bigg).
$$

Separate the final terms from the ∇-integrals and rearrange algebraically to obtain

$$
\begin{aligned}
y(t) &= \frac{((\boxminus r) - 1) \int_{t_0}^{t-1} H_{-\nu-1}(t, \rho(\tau)) y(\tau) \, \nabla \tau}{1 - (\boxminus r) a - (\boxminus r) - (\boxminus r) \int_{t_0}^{t-1} H_{\nu-1}(t, \rho(\tau)) \frac{\nabla_{t_0}^{\nu} y(\tau)}{y(\tau)} \, \nabla \tau} \\
&= \frac{((\boxminus r) - 1) \int_{t_0}^{t-1} H_{-\nu-1}(t, \rho(\tau)) y(\tau) \, \nabla \tau}{1 - (\boxminus r)\big(a + 1 + \int_{t_0}^{t-1} \frac{H_{\nu-1}(t, \rho(\tau))}{y(\tau)} \int_{t_0}^{\tau} H_{-\nu-1}(\tau, \rho(s)) y(s) \, \nabla s \, \nabla \tau\big)},
\end{aligned}
$$

and writing the ∇-integrals as summations and applying the definition of $(\boxminus r)$ completes the proof. □

4. Curve fitting and comparison of models

We will now present curve fittings of our three models (12), (13), and (14), the model (3), and the continuous model to data that had previously fit with (continuous) Gompertz curves. In order to test the fit of our models, we have programmed[1] each model into the Python programming language and used the `curve_fit` function from `scipy.optimize`. According to its documentation, the `curve_fit` function ultimately implements the Levenberg–Marquardt algorithm for solving nonlinear least squares problems. To implement the solutions of our models, we used the

[1] https://github.com/tomcuchta/cuchtafinchamdiscretefractionalgompertz

	ν	a	r	y_0	RSS	SE
(12)	$5.58\cdot10^{-1}$	$-2.38\cdot10^{-1}$	$2.33\cdot10^{0}$	$1.78\cdot10^{2}$	$8.49\cdot10^{6}$	$2.43\cdot10^{2}$
(13)	$1.13\cdot10^{0}$	$-1.52\cdot10^{0}$	$4.89\cdot10^{-1}$	$2.07\cdot10^{2}$	$8.61\cdot10^{6}$	$2.44\cdot10^{2}$
(14)	$8.77\cdot10^{-1}$	$-1.73\cdot10^{0}$	$4.87\cdot10^{-1}$	$2.08\cdot10^{2}$	$8.60\cdot10^{6}$	$2.44\cdot10^{2}$
	ν	c	b	y_0	RSS	SE
(3)	$5.88\cdot10^{-1}$	$4.46\cdot10^{0}$	$6.79\cdot10^{-1}$	$7.45\cdot10^{1}$	$8.61\cdot10^{6}$	$2.44\cdot10^{2}$
	L_∞	G	t_0		RSS	SE
ctn	$5.85\cdot10^{2}$	$3.81\cdot10^{-1}$	$1.79\cdot10^{0}$		$8.56\cdot10^{6}$	$2.43\cdot10^{2}$

Table 1. Results of curve fittings to data from [Hilling et al. 2016b] and their goodness of fit. The continuous curve was given there by the three-parameter formula $L_\infty e^{-e^{-G(t-t_0)}}$.

recurrences in Theorems 6, 7, and 8 directly. The initial condition argument t_0 was always chosen so that the data point with smallest independent variable agreed with the minimum of the domain of the solution. The effect this has is to horizontally shift the solutions so that they all "begin" at the same value of the independent variable in a given data set. Because those recurrence relations are self-referential at every iteration, they are computationally inefficient without optimizing the code in some way. So, we used the technique of memoization, meaning that we stored all newly generated values for a solution in a dictionary data structure to reference when computing the next value of the solution.

We present the parameters found, and as done in [Atıcı et al. 2017], we computed the residual sum of squared errors (RSS) and the standard error (SE). The RSS is defined by

$$\mathrm{RSS} = \sum_{i=1}^{n} (y_i - Y(t_i))^2,$$

where n denotes the number of data points, (t_i, y_i) denotes the data points themselves, and $Y(t_i)$ denotes the prediction of the fitted model at t_i, and SE is given by

$$\mathrm{SE} = \sqrt{\frac{\mathrm{RSS}}{n-k}},$$

where n denotes the total number of data points and k is the number of parameters of the model.

The article [Hilling et al. 2016b] concerns the relationship between the age (in years) and length (in millimeters) of 155 *ictalurus punctatus* ("channel catfish") caught in Cheat Lake in West Virginia. The lengths of the fish were measured, and a technique to determine their age was performed. Their data set is publicly accessible

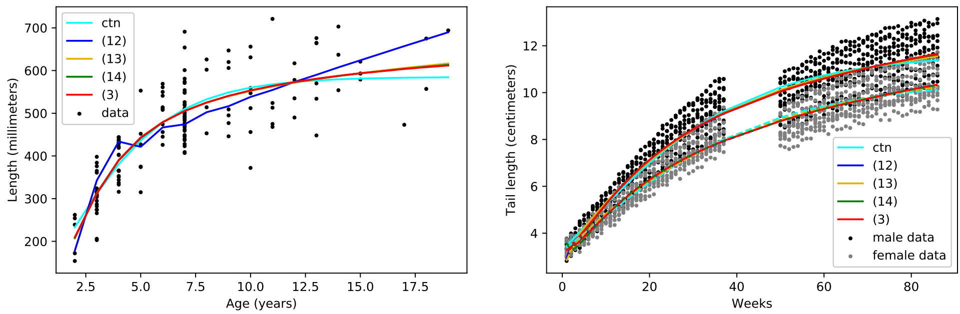

Figure 1. Left: Optimal models for the data from [Hilling et al. 2016b]. Right: Optimal models for the data from [Yang et al. 2019b]. Male (solid curve) and female (dashed curve) tail-length data. The curve for (12) is difficult to see because it is below the other curves.

male tail-length data						
	ν	a	r	y_0	RSS	SE
(12)	$8.92\cdot10^{-1}$	$-8.34\cdot10^{-1}$	$9.55\cdot10^{-2}$	$2.94\cdot10^{0}$	$2.72\cdot10^{4}$	$5.33\cdot10^{0}$
(13)	$1.13\cdot10^{0}$	$-1.53\cdot10^{0}$	$5.79\cdot10^{-2}$	$2.85\cdot10^{0}$	$2.72\cdot10^{4}$	$5.33\cdot10^{0}$
(14)	$8.25\cdot10^{-1}$	$-1.87\cdot10^{0}$	$1.39\cdot10^{-1}$	$3.56\cdot10^{0}$	$2.72\cdot10^{4}$	$5.33\cdot10^{0}$
	ν	c	b	y_0	RSS	SE
(3)	$2.30\cdot10^{-2}$	$1.15\cdot10^{0}$	$1.02\cdot10^{-1}$	$3.07\cdot10^{0}$	$2.72\cdot10^{4}$	$5.33\cdot10^{0}$
	A	B	K		RSS	SE
ctn	$1.17\cdot10^{1}$	$1.27\cdot10^{0}$	$4.52\cdot10^{-2}$		$2.72\cdot10^{4}$	$5.33\cdot10^{0}$

female tail-length data						
	ν	a	r	y_0	RSS	SE
(12)	$8.60\cdot10^{-1}$	$-6.76\cdot10^{-1}$	$1.03\cdot10^{-1}$	$2.89\cdot10^{0}$	$1.54\cdot10^{4}$	$4.35\cdot10^{0}$
(13)	$1.12\cdot10^{0}$	$-1.51\cdot10^{0}$	$4.62\cdot10^{-2}$	$2.84\cdot10^{0}$	$1.54\cdot10^{4}$	$4.35\cdot10^{0}$
(14)	$7.56\cdot10^{-1}$	$-1.72\cdot10^{0}$	$1.82\cdot10^{-1}$	$3.56\cdot10^{0}$	$1.54\cdot10^{4}$	$4.36\cdot10^{0}$
	ν	c	b	y_0	RSS	SE
(3)	$3.58\cdot10^{-2}$	$1.14\cdot10^{0}$	$1.23\cdot10^{-1}$	$3.35\cdot10^{0}$	$1.54\cdot10^{4}$	$4.35\cdot10^{0}$
	A	B	K		RSS	SE
ctn	$1.05\cdot10^{1}$	$1.18\cdot10^{0}$	$3.95\cdot10^{-2}$		$1.53\cdot10^{4}$	$4.35\cdot10^{0}$

Table 2. Results of curve fittings to data from [Yang et al. 2019b] and their goodness of fit. The continuous curve was given there by the three-parameter formula $Ae^{-Be^{-Kt}}$.

[Hilling et al. 2016a] and contains columns for the age and length (both positive integers), which we used without modification. Their analysis of the various growth curves concluded that the von Bertalanffy growth (proportional to $1 - e^{-k(t-t_0)}$, commonly used in length-based methods for fish population estimates [Pauly and Morgan 1987]) was the model of best fit among those considered. The results of our work are summarized in Table 1 and visualized in Figure 1, left. We see that the five models have essentially identical RSS, indicating that they all fit equally well. The models (12), (14), and (3) have optimal ν parameter between 0.55 and 0.87, while the optimal ν for (13) is larger than 1. The fit of (12) is not monotone, while the other four models are visually quite similar.

In [Yang et al. 2019b], the logistic, Gompertz, and von Bertalanffy growth curves were fit to data collected about *plestiodon elegans* (a type of skink) pertaining to some of their physical traits — specifically, the tail length of thirteen male and eleven female individuals were measured once a week for 85 weeks. Their data set is publicly accessible [Yang et al. 2019a] and contains columns for the animal identification number, week number (both positive integers), and tail length in centimeters (accurate to two decimal places), which we used without modification. The study concluded that the continuous Gompertz model was the best fit for both sexes for tail length data. We have fit the models to male and female tail-length data, and our results are summarized in Table 2, and visualized in Figure 1, right. We observe that the RSS and SE are nearly identical for all curve fittings, indicating that the models fit similarly well. The models (12) and (14) have ν-value between 0.74 and 0.89, (13) always has a ν value larger than 1, and the model (3) always has small ν.

5. Conclusion

We have investigated three new natural fractional analogues of the Gompertz ∇-difference equation. We have found a closed form solution of one of them in terms of the ∇-Mittag-Leffler function, but with necessarily restricted parameters. We derived recurrence relations for all three. We have fit the models to data sets and compared the fit to the existing fractional Gompertz model by Atıcı et al. It appears that both of the data sets were fit nearly equally well by all four models according to RSS and SE, and so using biological considerations or more advanced statistical techniques would be necessary to decide which model is best. We observed that the four discrete fractional models under consideration did not perform better than their continuous counterparts in terms of RSS or SE.

There are numerous directions of future research involving these models. Finding simple sufficient conditions for the regressivity of the solution of the first new fractional model is of great interest, not only for the fractional model but also for the original nonfractional model. Many qualitative properties remain unexplored, including analysis of stability and oscillation of their solutions. One interesting fractional generalization that we did not consider is taking the orders of the fractional difference and fractional sum in the third new fractional logarithm to be different. Such a model would gain an extra parameter, and so it may fit better in terms of RSS and SE. When comparing models with different numbers of parameters, a more robust statistical analysis should be conducted, e.g., the Akaike information criterion method.

Our choice of fractional derivative had some consequences that complicated some aspects of the work. Most importantly, in the proof of Theorem 5, the fractional derivative of $z(t) = a + \mathrm{L}_{y,\nu}(t, t_0)$ was computed yielding

$$(\nabla_{t_0}^{\nu} z)(t) = aH_{-\nu}(t, t_0) + \frac{\nabla y(t)}{y(t)}.$$

Since we used the Riemann–Liouville fractional difference, the derivative of the constant a was not zero, and we consequently had to use a convolution via Theorem 3 to express the closed-form solution of the first new model. This raises the general question of defining fractional Gompertz equations using the Caputo fractional differences (and others), where the fractional derivative of a constant is zero. Such models may be easier to analyze analytically, but may have other unforeseen downsides.

Acknowledgments

This research was made possible by NASA West Virginia Space Grant Consortium, Training Grant #NNX15AI01H. We also thank the referees for their valuable recommendations that improved the quality of this manuscript.

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Received: 2020-04-29 Revised: 2020-08-06 Accepted: 2020-08-10

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2020 vol. 13 no. 4