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Peg solitaire is a classical one-person game that has been played in various countries on different types of boards. Numerous studies have focused on the solvability of the games on these traditional boards and more recently on mathematical graphs. In this paper, we go beyond traditional peg solitaire and explore the solvability on graphs with pegs of more than one color and arrive at results that differ from previous works on the subject. This paper focuses on classifying the solvability of peg solitaire in three colors on several different types of common mathematical graphs, including the path, complete bipartite, and star. We also consider the solvability of peg solitaire on the Cartesian products of graphs.

1. Introduction

Peg solitaire has been played in cultures across the world for over 300 years. Boards can come in a variety of shapes and sizes, each with a different number of pegs. The standard rules for a game of peg solitaire state that a board starts with a certain number of pegs and one hole placed anywhere on the board, usually the center. The player must have an ending state in which only one peg is remaining on the board by removing all other pegs achieved by jumping adjacent pegs over one another. There is a rich history of problems posed and solved on different types of boards [Beasley 1985] and recently there has been renewed interest in the subject [Bell 2007; 2008].

In the last few years Robert A. Beeler and his students have studied peg solitaire on graphs by modifying the rules as follows [Beeler et al. 2017; Beeler and Paul Hoilman 2011; 2012; Beeler and Walvoort 2015]: suppose there is an edge connecting vertices v_1 and v_2 and a second edge connecting vertices v_2 and v_3 with pegs in vertices v_2 and v_3 (see Figure 1). Then a player may jump the peg in v_3 over the peg in v_2 to obtain the result in Figure 1. Following [Beeler and Walvoort 2015], we denote such a jump by $v_3 \cdot \vec{v}_2 \cdot v_1$. Once the player jumps one peg over the other, as seen in Figure 1, the peg that was jumped over is then removed. Although

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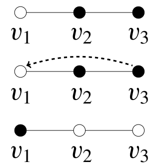

Figure 1. Example of a move in peg solitaire.

others have created and analyzed different variations of peg solitaire on graphs [Beeler and Rodriguez 2012; Engbers and Weber 2018; Engbers and Stocker 2015; Loeb and Wise 2015], the goal of this paper is to generalize the results in [Beeler and Paul Hoilman 2011] by adding an additional, "third," color to the game. We recognize that although the pegs in our version of the game come in two colors, it might be best to think of the "third" color as the white holes, since, as explained in Section 2, our version of the game is closely aligned with arithmetic in $\mathbb{Z}_3$. In Section 2 we present the rules of peg solitaire on graphs with pegs with different colors, in Section 3 we give results on different graphs with these new rules, and in Section 4 we discuss some open questions.

2. Three-color peg solitaire

As described in [Beeler and Paul Hoilman 2011], each game consists of a graph, $G = (V, E)$, with $|V| \geq 2$, that serves as the board, and a starting state S depicting the initial placement of the pegs. As in that paper, we also assume that all graphs are finite and undirected with no loops or multiple edges and that graphs are always connected. In this version of peg solitaire we include two different types of pegs and adjust the criteria for what moves can be executed. The starting state of each game includes the following components:

- a singleton set S_0 consisting of the vertex that has a peg of color 0, also called the initial hole in the board;
- a set S_1 which consists of vertices that have pegs of color 1; and
- a set S_2 which is made up of vertices that have pegs of color 2.

Thus a starting state S will be denoted as $S = (S_0, S_1, S_2)$

As in [Beeler and Paul Hoilman 2011] we note that a player can only execute a jump $x \cdot \vec{y} \cdot z$ if $xy, yz \in E$ and if z has color 0, while x and y do not. A difference in the rules is that when a player jumps a peg with the same color, the peg that has been jumped over then switches to the second color (see Figure 2). Moreover, when adjacent pegs have different colors, then the peg that jumps over the different colored peg creates a hole (see Figure 3).

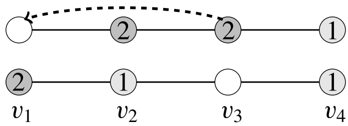

Figure 2. Example of moving a peg over a peg of the same color.

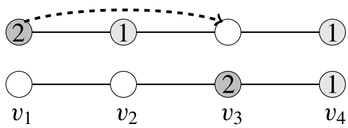

Figure 3. Example of moving a peg over a peg of a different color.

Moreover we note that we can translate the color rules in terms of modular arithmetic as follows: If we allow the darker gray color in Figure 2 to correspond to the number 2 and the lighter gray color to correspond to the number 1, then when a peg jumps over another peg into a hole, the middle peg is replaced with the result of an addition modulo 3. This generalizes the rules in [Beeler and Paul Hoilman 2011], since in that paper all pegs have color 1, and the result of any jump is $1+1=0 \bmod 2$, a hole. Using modular arithmetic we obtain the following notation for Figure 2:

$$\begin{array}{cccc} 0 & 2 & 2 & 1 \\ 2 & 1 & 0 & 1. \end{array}$$

In other words, when the peg in vertex v_3 jumped over the peg in vertex v_2 we add the values of the pegs in v_3 and v_2 modulo 3 and obtain the number 1 in vertex v_2. This corresponds to the move in Figure 2. The new notation would express the move in Figure 3 as

$$\begin{array}{cccc} 2 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1. \end{array}$$

Note that there is a game that is closely related to the one in the previous paragraph and which can be played in a similar manner, namely the game

$$0 \quad 1 \quad 1 \quad 2,$$

where each color in the original game is replaced with the opposite[1] color. Following the same two moves highlighted above we obtain the following configurations of the game:

$$\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2. \end{array}$$

[1]The opposite color refers to the fact that 1 and 2 are additive inverses in $\mathbb{Z}_3$.

Thus if Γ is a game on a graph G with starting state $S_1 = \{u_1, u_2, \dots, u_m\}$, $S_2 = \{v_1, v_2, \dots, v_n\}$, and $S_0 = \{v_o\}$, then the game Γ' given by $S_1' = \{v_1, v_2, \dots, v_n\}$, $S_2' = \{u_1, u_2, \dots, u_m\}$, and $S_0 = \{v_o\}$ is called the *opposite* of Γ. As in the example above, the properties of the addition table for $\mathbb{Z}_3$ imply that any move made on a game in three colors will be mirrored in the opposite game. This observation allows us to reduce later arguments in the sense that the actual colors being used in the game do not matter, only whether the peg that jumps is of the same or a different color than the one that is being jumped over.

Similar to the traditional game of peg solitaire, the goal of this version is to have one remaining peg, regardless of the color, on the board at the end of the game. The game is in its terminal state when one of two things happen. Firstly, the game is in its terminal state if the goal has been reached and there is only one peg remaining on the board. In the second option, the game is in its terminal state if the board has reached a point where there are no remaining moves to solve the game. For example, there may be no allowable moves remaining (see Proposition 3.4). Otherwise, there may be allowable moves creating a repeating loop in which no additional pegs are removed (see Theorem 3.7).

We define the *terminal state* $T = (T_0, T_1, T_2)$ of each game as follows:

- a set T_0 containing the vertices with remaining pegs of color 0, or holes;
- a set T_1 that consists of the vertices with remaining pegs of color 1; and
- a set T_2 that consists of the vertices with remaining pegs of color 2.

Thus the goal of playing each game is to reach a terminal state T, where $|T_1 \cup T_2| = 1$. We say that a game on a graph, i.e., a graph G together with a starting state S, has been *won* if there is a sequence of moves leading to a terminal state with $|T_1 \cup T_2| = 1$. We say that a game on a graph is a *losing game* if it is not a winning game: no sequence of moves leads to a terminal state with $|T_1 \cup T_2| = 1$. This leads us to the following definitions, where we intuitively think that a game is solvable if we can win at least one game; and freely solvable if we can win every game.

Definition 2.1 (solvable). A graph $G = (V, E)$ is solvable if there exists a starting state $S = (S_0, S_1, S_2)$ that has an associated terminal state $T = (T_0, T_1, T_2)$ such that $|T_1 \cup T_2| = 1$.

Definition 2.2 (freely solvable). A graph $G = (V, E)$ is freely solvable if for every starting state $S = (S_0, S_1, S_2)$, there exists an associated terminal state $T = (T_0, T_1, T_2)$ such that $|T_1 \cup T_2| = 1$.

Definition 2.3 (k-solvable). A graph $G = (V, E)$ is k-solvable for $k \in \mathbb{N}$ when k is the minimal value such that there exists a starting state $S = (S_0, S_1, S_2)$, with associated terminal state $T = (T_0, T_1, T_2)$ such that $|T_1 \cup T_2| = k$, and all vertices in $T_1 \cup T_2$ are nonadjacent.

We finish this section by noting our first result following from the discussion thus far.

Lemma 2.4. *Let G be a graph and let Γ be a game on G. Then Γ is a winning game if and only if its opposite is a wining game.*

In the next section we determine the solvability of many of the same graphs considered in [Beeler and Paul Hoilman 2011].

3. Results

3A. ***Games on path and cyclic graphs.*** We begin by showing that path graphs with n vertices, $n \geq 3$, are solvable but not freely solvable. We remark that this result provides a distinction between the results occurring in two [Beeler and Paul Hoilman 2011] versus three colors. We further note that while, at this point, we do not have an example of a graph that is solvable in two colors yet not solvable in three colors, there are individual games where this is the case, such as in P_4. See Theorem 2.3 in [Beeler and Paul Hoilman 2011] for these examples.

A path graph P_n has n vertices and $n-1$ edges connecting each of the vertices along a line. The cycle graph C_n is the same as the path graph with one additional edge connecting the first and last vertex. For more on path graphs and other basic graph theory terminology, refer to [West 2001]. In the proofs below we label the vertices as in Figure 4.

Theorem 3.1. *The path graph on n vertices, P_n, is solvable for $n \geq 2$. Moreover, P_n is not freely solvable for $n \geq 3$.*

Proof. We begin by noting that we can win every game on the path graph P_2; thus P_2 is freely solvable. We now show that P_n is solvable for all $n \geq 3$, proceeding by induction on the number of vertices. Suppose that we can win the game, Γ, with starting state $S_0 = \{v_1\}$, $S_1 = \{v_2, \ldots, v_n\}$, and $S_2 = \varnothing$; i.e., we can win the game

$$0 \ 1 \ 1 \ 1 \ 1 \ 1 \ \cdots \ 1.$$

Consider a similar game in P_{n+1}, denoted by Γ', with starting state $S'_0 = \{v_1\}$, $S'_1 = \{v_2, \ldots, v_n, v_{n+1}\}$, and $S'_2 = \varnothing$,

$$0 \ 1 \ 1 \ 1 \ 1 \ 1 \ \cdots \ 1 \ 1.$$

Then by moving $v_3 \cdot \vec{v}_2 \cdot v_1$ and then back $v_1 \cdot \vec{v}_2 \cdot v_3$, we see that we now have holes in vertices v_1 and v_2 and pegs of the color 1 in vertices $v_3, \ldots, v_{n+1}$. Now if

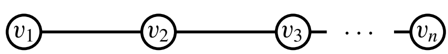

Figure 4. A path graph with n vertices.

we consider the vertices $v_2, \dots, v_{n+1}$ on their own, we see that their configuration matches the starting state of the game Γ. We can play the rest of the game Γ' as we would the game Γ and win the game. Thus P_n is solvable for all n.

We can show that P_n is not freely solvable for $n \geq 3$ in a similar manner. We begin by noting that we cannot win the game 1 0 2 in P_3. Now suppose $n = 2k+1$ for some integer k, and consider the game Γ in P_n with starting state given by $S_0 = \{v_1\},\ S_1 = \{v_2, v_4, \dots, v_{2k}\}$, and $S_2 = \{v_3, v_5, \dots, v_{2k+1}\}$, i.e.,

$$0\ 1\ 2\ 1\ 2\ 1\ \cdots\ 1\ 2.$$

Note that we only have one move, $v_3 \cdot \vec{v}_2 \cdot v_1$, at our disposal and we are left with

$$2\ 0\ 0\ 1\ 2\ 1\ \cdots\ 1\ 2.$$

We are again left with one possible move, $v_5 \cdot \vec{v}_4 \cdot v_3$, and now obtain

$$2\ 0\ 2\ 0\ 0\ 1\ \cdots\ 1\ 2.$$

We continue in this manner, always forced to make one move, $v_{2i+1} \cdot \vec{v}_{2i} \cdot v_{2i-1}$, as i goes from 3 to k so that our terminal state becomes $T_0 = \{v_2, v_4, \dots, v_{2k}, v_{2k+1}\}$, $T_1 = \varnothing$, and $T_2 = \{v_1, v_3, \dots, v_{2k-1}\}$, i.e.,

$$2\ 0\ 2\ 0\ 2\ 0\ 2\ \cdots\ 2\ 0\ 0.$$

Thus $|T_1 \cup T_2| \neq 1$ and since this is the only way to play this particular game we conclude that P_n is not freely solvable for n odd.

Now consider a game in $P_{2k},\ k \geq 2$, with the starting state $S_0 = \{v_1\},\ S_1 = \{v_2, v_4, \dots, v_{2k}\},\ S_2 = \{v_3, v_5, \dots, v_{2k-1}\}$, given by

$$0\ 1\ 2\ 1\ 2\ 1\ \cdots\ 2\ 1.$$

Then just as before we note that we are forced to make one move each time we try to play, namely, $v_{2i+1} \cdot \vec{v}_{2i} \cdot v_{2i-1}$, as i goes from 1 to $k-1$. Our terminal state is similar to the one above: $T_0 = \{v_2, v_4, \dots, v_{2k-2}, v_{2k-1}\},\ T_1 = \{v_{2k}\}$, and $T_2 = \{v_1, v_3, \dots, v_{2k-3}\}$, illustrated as

$$2\ 0\ 2\ 0\ 2\ 0\ 2\ \cdots\ 2\ 0\ 1.$$

Thus, $|T_1 \cup T_2| \neq 1$ and so P_n is not freely solvable when n is even. □

The following result follows immediately from Theorem 3.1 and the fact that P_n is a spanning subgraph of C_n.

Corollary 3.2. *The cyclic graph on n vertices, C_n, is solvable for $n \geq 2$.*

The following lemma will be useful in a couple of proofs, both in this paper and a subsequent paper, so we state it here. Moreover, this lemma introduces an interesting phenomenon with cycle graphs alluded to below.

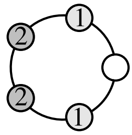

Figure 5. A game that cannot be won in C_5.

Lemma 3.3. *The cycle graph on three vertices*, C_3, *is freely solvable.*

Proof. Let v_1, v_2, and v_3 be the three vertices of the graph C_3 such that there are edges between v_1 and v_2, v_2 and v_3, and v_3 and v_1. Without loss of generality, suppose that $S_0 = \{v_1\}$. Then we have two cases to consider.

- Case 1: The two pegs on the graph, in vertices v_2 and v_3, are different colors. Then by $v_3 \cdot \vec{v}_2 \cdot v_1$ we win this game.
- Case 2: The two pegs on the graph, in vertices v_2 and v_3, are the same color. Then by $v_3 \cdot \vec{v}_2 \cdot v_1$ we obtain a game equivalent to the game in Case 1. □

At this time we cannot determine if C_n is freely solvable for all $n \geq 2$, but playing each game in $C_2, C_3, C_4, \ldots, C_{11}$, with the aid of a computer program [Sopena 2019] we have determined that each of these graphs are freely solvable with the exception of C_5. The game in Figure 5 is a game that cannot be won in C_5, which we prove below.

Proposition 3.4. *The game* Γ *with starting state* $S_0 = \{v_1\}$, $S_1 = \{v_2, v_5\}$, *and* $S_2 = \{v_3, v_4\}$, *or any equivalent game*, *cannot be won.*

Proof. Note that our first move is either $v_3 \cdot \vec{v}_2 \cdot v_1$ or $v_4 \cdot \vec{v}_5 \cdot v_1$. Regardless, we create a situation where there is a vertex (either v_2 or v_5) that has a peg of one color, while the vertices adjacent to it have pegs of a different color. Moreover, the other two vertices have holes. Again, there is a choice of two moves but either one leads to a terminal state with two pegs. □

3B. ***Games on complete bipartite graphs.*** In this section we will prove a result that complements Theorem 2.7 of [Beeler and Paul Hoilman 2011], namely that the complete bipartite graph is freely solvable. The complete bipartite graph, $K_{m,n}$, has as its vertex set the union of disjoint sets of vertices $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. The set of edges E connects all vertices from X to all vertices in Y, but none from X to other vertices in X and none from Y to other vertices in Y.

Theorem 3.5. *The complete bipartite graph* $K_{m,n}$ *is freely solvable for all* $m, n > 1$.

Proof. Without loss of generality, let $S_0 = \{x_1\}$. The first step is to remove the peg in y_1 by $x_2 \cdot \vec{y}_1 \cdot x_1$. If the pegs in x_2 and y_1 are of the same color, we add a second move of jumping back $x_1 \cdot \vec{y}_1 \cdot x_2$. After relabeling, we now have holes in x_1 and y_1.

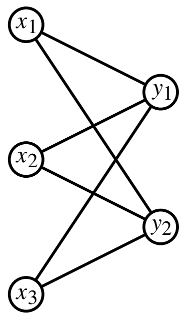

Figure 6. An example of a complete bipartite graph, $K_{3,2}$.

We repeat this action $n-2$ more times to remove all but one peg in Y until we have holes in exactly the vertices $y_1, \ldots, y_{n-1}$ and x_1.

Now we will remove the peg in x_2 by jumping $y_n \cdot \vec{x}_2 \cdot y_{n-1}$. If the pegs in x_2 and y_n are of the same color, we add a second move of jumping $y_{n-1} \cdot \vec{x}_2 \cdot y_n$. After relabeling, we now have holes in x_1, x_2, and $y_1, \ldots, y_{n-1}$. We repeat this action $m-2$ more times to remove all pegs in X until we have holes in all vertices except y_n. □

Corollary 3.6. *The complete graph, K_n, is freely solvable for $n > 1$.*

Proof. When $n = 2$, the graph is P_2, which is clearly freely solvable. When $n = 3$, the graph is C_3, which is freely solvable by Lemma 3.3. For $n > 3$, the complete bipartite graph is a spanning subgraph of the complete graph, so the result follows from Theorem 3.5. □

Theorem 3.7. *The star graph $K_{1,n}$ is $(n-1)$-solvable for $n > 2$.*

Proof. Note that for $n = 1$, the graph is P_2 and so is freely solvable, and for $n = 2$ the graph is P_3, which is also solvable. We remind the reader that the graph $K_{1,n}$ has vertex set $\{x_1, y_1, y_2, \ldots, y_n\}$ with an edge between x_1 and y_i for all $1 \le i \le n$, and no other edges.

If $S_0 = \{x_1\}$, then there are no possible moves. Otherwise, note that a peg in x_1 is the only one that can possibly be removed from the graph. Suppose without loss of generality that $S_0 = \{y_n\}$ and that the first jump is $y_1 \cdot \vec{x}_1 \cdot y_n$. If the color of y_1 is the same as x_1, then the peg in x_1 has not been removed. Therefore, we should assume that the color of y_1 is different than x_1, in which case the move creates a hole in x_1 and a hole in y_1. There are no possible moves after this. The terminal state has $|T_1 \cup T_2| = n - 1$. □

3C. ***Cartesian products of graphs.*** In this section we show that if we have a graph G that is solvable, and H is any graph, then the Cartesian product of the two graphs, G and H, is also solvable. If $V(G)$ is the set of vertices a graph G and $V(H)$ is the set of vertices of graph H, then the set of vertices of the Cartesian

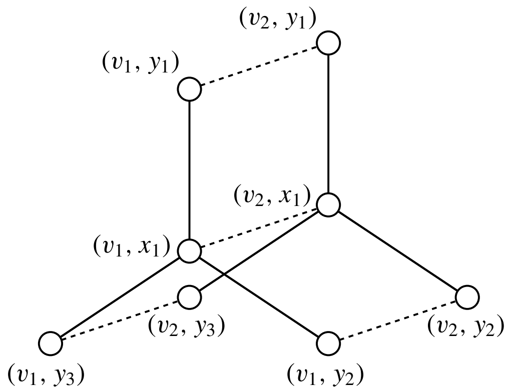

Figure 7. An example of the Cartesian product of P_2 and $K_{1,3}$.

product $G \square H$ is given by $V(G \square H) = \{(u, v) \mid u \in V(G),\ v \in V(H)\}$. Moreover, as in [Beeler and Paul Hoilman 2011], we define G_u to be the copy of G induced by $u \in V(H)$. Note that Figure 7 shows us the Cartesian product $P_2 \square K_{1,3}$ and that there are new edges introduced between both copies of $K_{1,3}$.

We begin by making a remark about a strategy that we have used before (see proof of Theorem 3.1) and will be used heavily hereafter.

Remark 3.8 (there and back strategy). Let x, y and z be three vertices with edges xy and yz. Suppose that a hole exists in either vertex x or z and that the other two vertices contain pegs of the same color. Without loss of generality suppose the hole is in the vertex z. Then we remove the peg in vertex y by $x \cdot \vec{y} \cdot z$ followed by $z \cdot \vec{y} \cdot x$. We shall denote this "there and back move" by $z \cdot \overleftrightarrow{y} \cdot x$.

Theorem 3.9. *Let G be a solvable graph and let T be a tree. Then $G \square T$ is solvable.*

Proof. Let $V(G)$ be the vertices of G and $V(T)$ be the vertices of T. Furthermore, let u_r be the root of T, suppose that the height of T is n, and let U_i be the set of all descendants of u_r at height i for $i \in \{1, \dots, n\}$. Consider a winning game, Γ, in G with starting state (S_0, S_1, S_2) and associated winning terminal state (T_0, T_1, T_2). Define the following starting state (S_0', S_1', S_2') for $G \square T$:

- $S_0' = \{(v_0, u_r)\}$ for $S_0 = \{v_0\}$.
- S_1' contains all of the vertices (v, u) such that $v \in S_1$ and $u = u_r$ or $u \in U_i$ for $i \in \{1, 2, \dots, n\}$.
- S_2' contains all of the vertices (v, u) such that $v \in S_2$ and $u = u_r$ or $u \in U_i$ for $i \in \{1, 2, \dots, n\}$.

Note that the description above has placed a copy of the winning game Γ in each copy of G. We now fill in the vertices that are copies of the hole in vertex (v_0, u_r).

We choose and fix a vertex in $V(G)$ that is adjacent to v_0, call it v', and note the color of the peg in v'. Without loss of generality suppose that v' is in S_1. Let $(v_0, u) \in S_1'$ for all $u \in U_i$ with $i \in \{1, \dots, n\}$. In other words, all of the vertices in $\{v_0\} \square T$ that are not (v_0, u_r) have a peg of the same color as the peg in v'.

The proof proceeds in three steps, the first of which removes pegs that are in vertices (v_0, u) with $u \in U_i$ with $i \in \{1, \dots, n\}$ so that we have holes in all of the vertices in $\{v_0\} \square T$. The second step wins the game Γ now found in each copy of T. The final step eliminates the remaining pegs except for one.

We begin the first step by inductively using Remark 3.8. We take the pegs in vertices $(v', u) \in G \square U_1$ to remove all pegs in vertices $(v_0, u) \in G \square U_1$. In other words, we are removing the pegs in vertices $(v_0, u) \in G \square U_1$ by $(v', u) \cdot \overleftrightarrow{(v_0, u)} \cdot (v_0, u_r)$ for every $u \in U_1$. We now inductively use the vertices $(v', u) \in G \square U_2$ to remove all pegs in vertices $(v_0, u) \in G \square U_2$ via $(v', u) \cdot \overleftrightarrow{(v_0, u)} \cdot (v_0, u_r)$ and proceed until all of the vertices $(v_0, u) \in G \square T$ have holes. Thus there is a copy of Γ in each copy of G given by $G \square \{u\}$ for $u \in T$.

The second step follows easily as it involves winning the game Γ on each copy of G on T. This leads to a peg on one vertex for each copy of G on T, say (t, u) for some $t \in G$ and for every $u \in T$. Again we proceed inductively but this time from the "bottom up". We begin by using the pegs in vertices (t, u) for all $u \in U_{n-1}$ to remove the pegs in vertices (t, w) for all $w \in U_n$ by $(t, u) \cdot \overleftrightarrow{(t, w)} \cdot (v, w)$ for some $v \in G$ adjacent to $t \in G$. We then repeat this step using the pegs in vertices (t, u) for all $u \in U_{n-2}$ to remove the pegs in (t, w) for all $w \in U_{n-1}$ in a similar manner until we reach the top of the tree. That is, after this process the last peg is in vertex (t, u_r). □

Corollary 3.10. *Let G be a solvable graph and let H be any graph. Then $G \square H$ is solvable.*

Proof. Every graph has a spanning tree T. Now we use the spanning tree of H and Theorem 3.9. □

We leave the proof of the following corollary to the reader.

Corollary 3.11. *Let G be a solvable graph and T a tree. Then $G \square T^n$ is solvable for all $n \geq 1$.*

4. Discussion

We conclude this paper with some open questions related to this project. Following [Beeler et al. 2017; Beeler and Walvoort 2015] it is natural to ask about solvability of peg solitaire in three colors on caterpillars and trees. It would be interesting to explore peg solitaire in three colors on more traditional boards as well. As noted in Section 3A, it remains a natural open question, building on the characterization of free solvability of cycle graphs in two colors given by [Beeler and Paul Hoilman 2011], as to whether the cycle graph C_n is freely solvable in three colors. We would also propose creating a three-color variant of fool's solitaire to compare with the results in [Beeler and Rodriguez 2012].

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