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We first describe an observation based on an analysis of data regarding the outcomes of decisions in cases considered by the United States Supreme Court. Based on this observation, we propose a simple model aiming toward producing an objective notion of an *ideology index*. As an initial step in justifying this concept we produce explicit formulas for the highest-energy eigenvectors of reversible Markov chains with rank-2 transition matrices.

1. Introduction

The idea for this article came about when Zilli noticed a strong correlation between the entries of an eigenvector built intrinsically only from the number of disagreements between justices on the United States Supreme Court and an extrinsic index defined roughly as the percent conservative decisions throughout the justices' tenure, where each decision's direction is determined by a scheme developed by political scientists.

The instances of disagreements between justices on the United States Supreme Court (or any voting body wherein pairwise agreement or disagreement between members can be discerned from vote records) can be expressed as weights on edges of an undirected graph. That is, each justice is represented by a node, and the edge connecting two nodes has weight equal to the number of times the corresponding justices disagreed on the outcome of a case they both participated in. It was hypothesized that techniques of spectral decomposition applied to such a graph could allow for the quantification of each justice's ideology. The significance of this approach is that it does not require legal analysis of the ideological implications of voting in a particular way on each case, which one would intuit as being essential in studying judicial ideology. We apply this method to data from the Washington University Law School's Supreme Court Database [SCDB 2018] and compare the results to the percent conservative measure of ideology resulting from the curated, legal analysis of each case included in the database. The results correlate well (see

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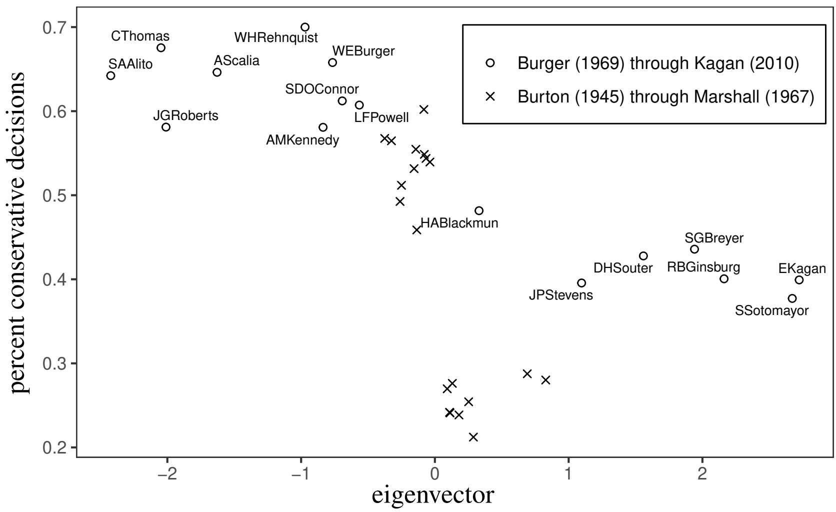

Figure 1. We construct a disagreement graph for decisions rendered from 1946 to 2018 (sourced from [SCDB 2018]) and derive the highest-energy eigenvector for the random walk on this graph. The plot shows the correlation between the eigenvector value and the percent conservative decisions (calculated from the expert classifications in the database) of each justice from Burton (appointed in 1945) through Kagan (2010): $R^2 = 0.349$. When considering only those justices from Burger (1969) through Kagan (represented by $\circ$ on the plot), the correlation is much stronger: $R^2 = 0.831$.

Figure 1), and we believe that this method may be applied to yield insight into bodies for which such extensive expert analysis does not exist.

Specifically, we consider a weighted graph where the nodes represent justices, and the weight on the edge between nodes x and y is a positive integer that represents the number of cases on which justices x and y disagreed. Such a weighted graph induces a reversible Markov chain. We propose the use of the (normalized) eigenvector $\boldsymbol{v}$ that corresponds to the highest energy as an objective measure of the ideological inclination of each of the justices, and we term it *ideology index*. The ideology index for the justice represented by node x will be the value $v(x)$ in the eigenvector $\boldsymbol{v}$.

There are two reasons that led us to consider weights given by disagreements rather than agreements. Firstly, instances of agreement between two judges include, along with cases which have ideological implications, the cases which were relatively uncontroversial (and would not be as indicative of judicial ideology). Conversely, the very existence of a disagreement between judges indicates that the case was contentious. That is, if one were to count the agreements between judges, there would be no objective way to discern which of those agreements were a result of the judges hearing a relatively noncontentious case and which of the agreements were along ideological grounds, whereas all disagreements are attributable to contention between the judges, which is more likely caused by differences in their ideology. Secondly, when we used the agreements data, the strongest correlation to ideology was with the eigenvector corresponding to the eighth-largest eigenvalue (for a 37×37 probability transition matrix). However, we were not able to find why this particular eigenvector would have the strongest correlation with ideology. On the other hand, when using disagreements, the strongest correlation was with the highest-energy eigenvector. In hindsight, this is something that could have been expected from considering the extreme case of bipartite graphs.

While developed independently, we believe that our model has similarities to existing *ideal point* models, such as that of [Martin and Quinn 2002]. In an ideal point model, one considers a justice's ideological position as a point in multidimensional Euclidean space, and the ideological implication of each decision direction on each case is also represented as a point in that space. In such models, the justice will vote in the direction whose point is closest to his or her own in Euclidean distance. Our model is simpler in many of its assumptions compared to Martin and Quinn's, specifically in that we do not consider a justice's ideal point as dynamic throughout his or her career. On the other hand, if one has to describe a justice's voting record with just one number, we consider the measure given by this model to be suitable and objective.

The structure of the paper is as follows. In Section 2 we collect a number of background results on the random walk associated to a reversible Markov chain and discuss the spectrum of the associated probability transition matrix. Section 3 contains the analysis of the highest-energy eigenvector for reversible Markov chains with rank-2 transition matrices. The explicit formulas obtained in the rank-2 case are then used to obtain estimates of the spectrum of a perturbed (higher-rank) "disagreement matrix". In Section 4 we conclude with a review of our results and an invitation to the continuation of this study.

2. Random walk preliminaries

In this section we introduce briefly the notion of random walk associated to a reversible Markov chain and some spectral properties; for details we refer to Section 1.5 in [Chung 1997]. An alternative point of view is that of random walks on electrical networks. For a thorough presentation of this approach we refer to [Doyle and Snell 1984] or Chapter 9 in [Levin et al. 2009].

A *network* is an undirected connected graph $G = (V, E)$, together with nonnegative weights $w(x, y)$ defined for all $x, y \in V$. The numbers $w(x, y)$ are *conductances*, which are assumed positive if (x, y) is an edge in E and zero otherwise. Note that we may allow *loops* (i.e., an edge from a vertex to itself) in which case $(x, x) \in E$ and $w(x, x)$ is positive. We assume the collection of weights forms a symmetric $N \times N$ matrix W, where $N = |V|$ is the number of vertices in the graph; i.e.,

$$\text{for any two vertices } x, y \in V, \quad w(x, y) = w(y, x).$$

Define the *degree* of a vertex $x \in V$ by

$$d(x) = \sum_{y \in V} w(x, y). \tag{2-1}$$

We denote by $\mathbf{1}$ the column vector with N entries all equal to 1.

The matrix P with entries

$$P(x, y) := \frac{w(x, y)}{d(x)} \text{ satisfies } P\mathbf{1} = \mathbf{1}, \text{ i.e.,} \sum_{y \in V} P(x, y) = 1 \text{ for every fixed } x. \tag{2-2}$$

Therefore, the vertices V are the states of a Markov chain with probability transition matrix P. An associated *random walk* is a sequence of random variables $\{X_i\}_{i \geq 0}$ satisfying for all $i \geq 0$ and all $x, y \in V$ that

$$\text{Prob}(X_{i+1} = y \mid X_i = x) = P(x, y).$$

Note that even though the matrix of weights W is symmetric, in general P is not a symmetric matrix.

On the set of nodes V we define a probability distribution $\boldsymbol{\pi}$ (think of it as a row vector) with entries given by

$$\pi(x) = \frac{d(x)}{\sum_y d(y)} = \frac{d(x)}{\operatorname{vol} G},$$

where $\operatorname{vol} G = \sum_y d(y)$ is the total degree (the sum of the degrees of all vertices in V). It is easy to check that the Markov chain thus defined is *reversible* with respect to the probability distribution $\boldsymbol{\pi}$. By definition, this means that for any $x, y \in V$ it holds that

$$\pi(x) P(x, y) = \pi(y) P(y, x). \tag{2-3}$$

A probability distribution $\boldsymbol{\sigma}$ on the set of states V of a Markov chain is called *stationary* if and only if

$$\boldsymbol{\sigma} P = \boldsymbol{\sigma},$$

i.e., $\boldsymbol{\sigma}$ is a left eigenvector of P with eigenvalue 1. A direct check using (2-3) shows that a reversible distribution $\boldsymbol{\pi}$ is also stationary.

A connected graph is *bipartite* if and only if V is the union of two nonempty disjoint sets V_1 and V_2 such that $w(x, y) = 0$ whenever $x, y \in V_1$ or $x, y \in V_2$. The case of bipartite graphs will be examined in what follows only as a limiting case.

Throughout most of this work the graph G will be assumed nonbipartite. On a nonbipartite connected graph any associated random walk is *ergodic*; i.e., there exists a unique stationary distribution $\boldsymbol{\sigma}$ on V such that for any initial distribution $\boldsymbol{\mu}$ it holds that

$$\lim_{s\to\infty} \boldsymbol{\mu} P^s = \boldsymbol{\sigma}.$$

One of the major questions in the theory of reversible Markov chains is characterizing the speed in the convergence

$$\lim_{s\to\infty} \boldsymbol{\mu} P^s = \boldsymbol{\pi}. \tag{2-4}$$

In answering this question, the more powerful techniques use the spectrum (the eigenvalues) and the eigenvectors of P.

The fact that $P\mathbf{1} = \mathbf{1}$ in (2-2) expresses the fact that $\mathbf{1}$ is an eigenvector of P corresponding to eigenvalue 1. Actually, when coupled with the fact that the entries of P are nonnegative, this relation means that every one of the rows of P is a probability distribution on V. By applying P to a column vector $\boldsymbol{u}$ each entry in the resulting vector $P\boldsymbol{u}$ is an average of the entries of $\boldsymbol{u}$, weighted with respect to the corresponding row of P. This averaging characteristic makes P into a contraction with respect to every convex norm on $\mathbb{R}^N$ (this follows from Jensen's inequality). For example, on $\mathbb{R}^N$ (which is identified with the space $L^2(V)$ of functions from V to $\mathbb{R}$) consider the norm

$$\|\boldsymbol{u}\|_2 = \left(\sum_x u^2(x)\right)^{\frac{1}{2}}.$$

Then for every $\boldsymbol{u} \in L^2(V)$ it holds that

$$\|P\boldsymbol{u}\|_2 \leq \|\boldsymbol{u}\|_2.$$

In particular, if $\boldsymbol{u}$ is an eigenvector of P with eigenvalue α we have

$$\|P\boldsymbol{u}\|_2 = \|\alpha\boldsymbol{u}\|_2 = |\alpha|\,\|\boldsymbol{u}\|_2 \leq \|\boldsymbol{u}\|_2.$$

Therefore, every eigenvalue of P has to satisfy $|\alpha| \leq 1$.

Another consequence of reversibility is the fact that all eigenvalues of P are real. This is due to the fact that relations (2-3) imply that P is similar to a symmetric matrix S. Indeed, let $\boldsymbol{d}$ denote the vector of degrees, with entries given by (2-1), and let D denote the diagonal matrix with $\boldsymbol{d}$ on the diagonal (and zero everywhere else). Then, it is clear that

$$S := D^{\frac{1}{2}} P D^{-\frac{1}{2}} = D^{-\frac{1}{2}} W D^{-\frac{1}{2}}$$

is a symmetric matrix. Since S and P have the same (real) eigenvalues, and because of the contractive property of P, $|\alpha| \leq 1$, it follows that all eigenvalues of P are contained in the interval $[-1, 1]$. Let

$$\Pi = \frac{1}{\operatorname{vol} G} D = \operatorname{diag}(\boldsymbol{\pi}).$$

From the spectral decomposition

$$S = \sum_{i=1}^{N} \alpha_i \phi_i \phi_i^{\mathsf{T}}, \tag{2-5}$$

with $\{\phi_i\}_i$ a set of orthonormal eigenvectors of the matrix S, we get that the eigenvectors of P are given by $\Pi^{-1/2}\phi_i$ and are orthonormal in L^2_π (see below for the definition of L^2_π).

We have already seen that $\alpha_1 = 1$ is an eigenvalue of P with eigenvector $\mathbf{1}$. It is a standard argument (again based on the averaging property of P) to show that when the graph G is connected, $\alpha_1 = 1$ is a simple eigenvalue. Therefore the remaining $N-1$ eigenvalues of P are strictly less than 1.

The second-largest eigenvalue α_2 of P is of great importance in estimating the convergence speed in (2-4). By analogy with the spectral theory on manifolds, one defines the matrix

$$L = \operatorname{Id} - P,$$

which is called the (normalized) *Laplacian* of the weighted graph G. The spectrum of L consists of eigenvalues $\lambda = 1 - \alpha \in [0, 2]$. We define the inner product space $L^2_\pi(V)$, weighted with respect to the stationary distribution $\boldsymbol{\pi}$, where the inner product is given by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_\pi = \sum_x \pi(x) u(x) v(x).$$

Proposition 1. *The equality*

$$\langle \boldsymbol{u}, L\boldsymbol{v} \rangle_\pi = \langle L\boldsymbol{u}, \boldsymbol{v} \rangle_\pi = \frac{1}{\operatorname{vol} G} \sum_{(x,y)\in E} w(x, y)(u(x) - u(y))(v(x) - v(y)) \tag{2-6}$$

holds.

Proof. If we denote by $\delta_{x,y}$ the Kronecker delta function, we have

$$\langle \boldsymbol{u}, L\boldsymbol{v} \rangle_\pi = \sum_{x\in V} \pi(x) u(x) (L\boldsymbol{v})(x) = \sum_{x,y\in V} \frac{d(x)}{\operatorname{vol} G} u(x)(\delta_{x,y} - P(x, y)) v(y),$$

therefore

$$\begin{aligned} \langle \boldsymbol{u}, L\boldsymbol{v} \rangle_\pi &= \frac{1}{\operatorname{vol} G} \left\{ \sum_{x\in V} d(x) u(x) v(x) - \sum_{x,y\in V} w(x, y) u(x) v(y) \right\} \\ &= \frac{1}{\operatorname{vol} G} \left\{ \sum_{x,y\in V} w(x, y) u(x)(v(x) - v(y)) \right\}. \end{aligned}$$

For every edge $(x, y) \in E$ with $x \neq y$ (recall that loops are allowed, but note that the terms in the sum for which $x = y$ are zero), terms involving x and y appear twice in the sum above, with x and y reversed. That is, for each edge (x, y) with $x \neq y$, there are two corresponding terms in the sum:

$$w(x, y)u(x)(v(x) - v(y)) \quad \text{and} \quad w(y, x)u(y)(v(y) - v(x)).$$

Since W is symmetric, $w(x, y) = w(y, x)$, and the sum of these terms is

$$w(x, y)(u(x) - u(y))(v(x) - v(y)).$$

Therefore, indexing the sum over edges, we have (2-6). □

For $\boldsymbol{u} \in \mathbb{R}^N$ we define its *energy* by

$$\mathcal{E}(\boldsymbol{u}) := \langle \boldsymbol{u}, L\boldsymbol{u} \rangle_\pi = \frac{1}{\operatorname{vol} G} \sum_{(x,y)\in E} w(x, y)(u(x) - u(y))^2. \tag{2-7}$$

The method of Lagrange multipliers gives that the eigenvalues of L are precisely the critical levels of $\mathcal{E}$ subject to the constraint $\langle \boldsymbol{u}, \boldsymbol{u} \rangle_\pi = 1$.

The minimum level of the energy $\mathcal{E}$ gives $\lambda_1 = 0$, and it is achieved by the constant eigenvector $\boldsymbol{u} = \mathbf{1}$. This corresponds to the largest eigenvalue of P, $\alpha_1 = 1 - \lambda_1 = 1$. The second-largest eigenvalue of P, $\alpha_2 = 1 - \lambda_2$, corresponds to the second smallest eigenvalue of L, given by

$$\lambda_2 = \inf_{\langle \boldsymbol{u}, \mathbf{1} \rangle_\pi = 0} \frac{\langle \boldsymbol{u}, L\boldsymbol{u} \rangle_\pi}{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_\pi}.$$

The difference

$$1 - \alpha_2 = \lambda_2$$

is called the *spectral gap* of the graph and it gives a measure of the speed of convergence in (2-4).

In this paper we will be concerned on the contrary with the smallest eigenvalue of P, which we denote by $\beta = \alpha_N$, and with its corresponding eigenvector. In terms of the eigenvalues of L, this corresponds to the largest eigenvalue, that is, the maximum of $\mathcal{E}$ restricted to the ellipsoid $\langle \boldsymbol{u}, \boldsymbol{u} \rangle_\pi = 1$. For this reason, we will call the eigenvector corresponding to β the *highest-energy eigenvector*. As we have seen above, β has to be a number in the interval $[-1, 1)$. For a connected graph, -1 is an eigenvalue if and only if G is bipartite. In our general setting G is not bipartite; therefore except for the one eigenvalue 1, the remaining eigenvalues (including β) will be in $(-1, 1)$.

3. Disagreement matrices

Assume we have a graph with a fixed number N of nodes (the justices), and each node x has a probability $p_x \in [0, 1]$ associated to it. In our proposed model, we assume there exists an abstract notion of, say, ideal conservator, which indicates how a pure ideological conservative would vote on every case. Then, p_x will represent the probability that justice x will vote with the ideal conservator. We will assume that for each justice, this probability is constant throughout their career. The probabilities p_x will be collected as entries of a vector $\boldsymbol{p}$ which will be fixed throughout. Our assumption that each p_x is constant throughout a justice's career is a major simplification in our model over earlier approaches, as for example in [Martin and Quinn 2002].

For each x, let $q_x := 1 - p_x$ also in $[0, 1]$. The *disagreement matrix* of this system is defined as the matrix B with entries

$$B_{xy} = \begin{cases} 0 & \text{if } x = y, \\ p_x q_y + p_y q_x & \text{if } x \neq y. \end{cases}$$

It has zero on the diagonal since a justice always votes in agreement with her/himself and the off-diagonal entries represent the probabilities of disagreement (justices x and y vote contrary to each other).

3.1. ***Rank-2 weight matrices.*** In this subsection we study the highest-energy eigenvector in a simpler (and less realistic) case when the diagonal entries in the disagreement matrix are not equal to zero. In $\mathbb{R}^N$, let $\boldsymbol{p}$ and $\boldsymbol{p}^\intercal$ denote the column and the row vector, respectively, with entries $p_x \in [0, 1]$, and similarly define the vectors $\boldsymbol{q}$ and $\boldsymbol{q}^\intercal$ with entries $q_x = 1 - p_x \in [0, 1]$. We discuss the spectrum of the probability transition matrix P_T associated with the rank-2 weight matrix $T := \boldsymbol{p}\boldsymbol{q}^\intercal + \boldsymbol{q}\boldsymbol{p}^\intercal$. We note that B is equal to T except for the diagonal entries, which are replaced by zero.

We begin with the following:

Proposition 2. *In the generic case, i.e., $\boldsymbol{p}$ is not a constant vector, we have that $\boldsymbol{p}$ and $\boldsymbol{q}$ are linearly independent vectors. In particular, T is a matrix of rank 2.*

Proof. Assume that for real numbers a and b we have $a\boldsymbol{p} + b\boldsymbol{q} = \boldsymbol{0}$. Then

$$\boldsymbol{0} = a\boldsymbol{p} + b(\boldsymbol{1} - \boldsymbol{p}) = (a - b)\boldsymbol{p} + b\boldsymbol{1}, \quad \text{i.e.,} \quad (b - a)\boldsymbol{p} = b\boldsymbol{1}.$$

Since $\boldsymbol{p}$ is nonconstant, from the last equality we must have $b = a = 0$; therefore the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are linearly independent.

Next, we show that T has rank 2. Let $\boldsymbol{p}^\perp$ denote the $(N-1)$-dimensional space of vectors in $\mathbb{R}^N$ which are orthogonal to $\boldsymbol{p}$, and similarly define $\boldsymbol{q}^\perp$. Since $\boldsymbol{p}$ and

$\boldsymbol{q}$ are linearly independent vectors we have

$$\dim(\boldsymbol{p}^{\perp} \cap \boldsymbol{q}^{\perp}) = N - 2.$$

Since

$$\boldsymbol{p}^{\perp} \cap \boldsymbol{q}^{\perp} \subset \ker T,$$

we have that

$$\dim(\ker T) \geq N - 2. \tag{3-1}$$

On the other hand, the vectors $T\boldsymbol{p}$ and $T\boldsymbol{q}$ are linearly independent. Indeed, $aT\boldsymbol{p} + bT\boldsymbol{q} = \boldsymbol{0}$ implies

$$\boldsymbol{0} = a(\boldsymbol{p}^{\mathsf{T}}\boldsymbol{q})\boldsymbol{p} + a|\boldsymbol{p}|^2\boldsymbol{q} + b|\boldsymbol{q}|^2\boldsymbol{p} + b(\boldsymbol{p}^{\mathsf{T}}\boldsymbol{q})\boldsymbol{q},$$

and since $\boldsymbol{q}$ and $\boldsymbol{p}$ are linearly independent we must have

$$a|\boldsymbol{p}|^2 + b(\boldsymbol{p}^{\mathsf{T}}\boldsymbol{q}) = 0 \quad \text{and} \quad a(\boldsymbol{p}^{\mathsf{T}}\boldsymbol{q}) + b|\boldsymbol{q}|^2 = 0.$$

The determinant of this system is

$$\begin{vmatrix} |\boldsymbol{p}|^2 & \boldsymbol{p}^{\mathsf{T}}\boldsymbol{q} \\ \boldsymbol{p}^{\mathsf{T}}\boldsymbol{q} & |\boldsymbol{q}|^2 \end{vmatrix} = |\boldsymbol{p}|^2|\boldsymbol{q}|^2 - (\boldsymbol{p}^{\mathsf{T}}\boldsymbol{q})^2 > 0$$

because the value of the determinant equals the square of the area of the parallelogram spanned by $\boldsymbol{p}$ and $\boldsymbol{q}$. Therefore, again we must have $b = a = 0$. Since $T\boldsymbol{p}$ and $T\boldsymbol{q}$ are linearly independent, it means

$$\dim(\operatorname{range} T) \geq 2. \tag{3-2}$$

From the rank-nullity theorem we have

$$\dim(\ker T) + \dim(\operatorname{range} T) = N.$$

From this, it follows that both inequalities (3-1) and (3-2) are in fact equalities; i.e., T has rank 2. □

Define

$$s_p = \sum_x p_x \quad \text{and} \quad s_q = \sum_x q_x, \quad \text{both in } [0, N], \text{ with } s_p + s_q = N.$$

Since we work with the weight matrix $T = \boldsymbol{p}\boldsymbol{q}^{\mathsf{T}} + \boldsymbol{q}\boldsymbol{p}^{\mathsf{T}}$, the degree of node x is the sum of entries on row x of T,

$$d(x) = \sum_y T_{xy} = p_x\Big(\sum_y q_y\Big) + q_x\Big(\sum_y p_y\Big) = p_x s_q + q_x s_p.$$

Note that we can write

$$d(x) = (N - s_p)p_x + s_p(1 - p_x),$$

and since both numbers $N - s_p$ and s_p are strictly positive, we have that $d(x) > 0$ for every x. We collect the degrees of all nodes in the vector $\boldsymbol{d}$, therefore

$$\boldsymbol{d} = s_q \boldsymbol{p} + s_p \boldsymbol{q},$$

and denote by $D := \operatorname{diag}(\boldsymbol{d})$ the diagonal matrix with entries $d(x)$ on the diagonal. Define the probability transition matrix

$$P_T := D^{-1} T.$$

We have the following:

Theorem 3. *In the generic case, i.e., $\boldsymbol{p}$ is not a constant vector, the matrix P_T has*

(1) *the largest eigenvalue is equal to* 1, *it is simple, and has corresponding eigenvector* $\mathbf{1}$;

(2) *eigenvalue* 0 *with multiplicity $N - 2$ and corresponding eigenspace consisting of all vectors orthogonal to both $\boldsymbol{p}$ and $\boldsymbol{q}$, i.e.,*

$$\ker P_T = \ker T = \boldsymbol{p}^{\perp} \cap \boldsymbol{q}^{\perp};$$

(3) *simple eigenvalue $\beta \in [-1, 0)$, with eigenvector $\boldsymbol{v}$ given explicitly in the formulas* (3-5) *and* (3-11) *below.*

Remark 4. As a consequence of the formulas (3-5) and (3-11) we will note that the eigenvector $\boldsymbol{v}$ is monotonic in $\boldsymbol{p}$. This means that as we order the vertices x of the graph in increasing order of $\boldsymbol{p}$, the values in the entries of $\boldsymbol{v}$ will also be either in increasing or decreasing order.

Another consequence of these formulas are estimates (explicitly in terms of $\boldsymbol{p}$ or $\boldsymbol{d}$) for the maximum values $\|\boldsymbol{v}\|_\infty$ given in (3-6) and (3-12).

Proof of Theorem 3. Part (1) follows from the general theory presented in Section 2.

Part (2) is immediate from the fact that the matrix D^{-1} is diagonal with no zeros on the diagonal, together with the fact from Proposition 2 that $\operatorname{rank} T = 2$.

For part (3) we will distinguish the following two cases:

Case 1: $s_p = N/2$. Since $\boldsymbol{q} = \mathbf{1} - \boldsymbol{p}$ we have

$$s_q = N - s_p = \frac{N}{2}, \quad \boldsymbol{d} = s_q \boldsymbol{p} + s_p \boldsymbol{q} = \frac{N}{2}\mathbf{1} \quad \text{and} \quad D = \frac{N}{2}\operatorname{Id}.$$

Also,

$$|\boldsymbol{q}|^2 = \sum_x q_x^2 = \sum_x (1 - p_x)^2 = \sum_x (1 - 2p_x + p_x^2) = N - 2s_p + \sum_x p_x^2 = |\boldsymbol{p}|^2,$$

and

$$\boldsymbol{p}^{\mathsf{T}} \boldsymbol{q} = \boldsymbol{p}^{\mathsf{T}} (\mathbf{1} - \boldsymbol{p}) = s_p - |\boldsymbol{p}|^2 = \frac{N}{2} - |\boldsymbol{p}|^2.$$

Consider the vector

$$\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{q} = 2\boldsymbol{p} - \mathbf{1}.$$

Then

$$T\boldsymbol{u} = (\boldsymbol{p}\boldsymbol{q}^\top + \boldsymbol{q}\boldsymbol{p}^\top)(\boldsymbol{p}-\boldsymbol{q}) = (\boldsymbol{p}^\top\boldsymbol{q})(\boldsymbol{p}-\boldsymbol{q}) + |\boldsymbol{p}|^2\boldsymbol{q} - |\boldsymbol{q}|^2\boldsymbol{p} = \Big(\frac{N}{2} - 2|\boldsymbol{p}|^2\Big)(\boldsymbol{p}-\boldsymbol{q}).$$

Therefore

$$T\boldsymbol{u} = \left(1 - \frac{4|\boldsymbol{p}|^2}{N}\right)\frac{N}{2}\boldsymbol{u} = \left(1 - \frac{4|\boldsymbol{p}|^2}{N}\right)D\boldsymbol{u}. \tag{3-3}$$

Note that because $s_p = N/2$, by the Cauchy–Schwarz inequality we have

$$\frac{N^2}{4} = \left(\sum_x p_x\right)^2 \le N \sum_x p_x^2 = N|\boldsymbol{p}|^2,$$

i.e.,

$$1 \le \frac{4|\boldsymbol{p}|^2}{N}.$$

In fact, the inequality above is strict as the only possibility for equality is when $\boldsymbol{p}$ is constant.

Since $P_T = D^{-1}T$, by multiplying (3-3) on the left by D^{-1}, we get that $P_T\boldsymbol{u} = \beta\boldsymbol{u}$ with

$$\beta = 1 - \frac{4|\boldsymbol{p}|^2}{N} < 0. \tag{3-4}$$

The stationary distribution in this case is the uniform distribution, i.e., $\pi(x) = 1/N$. It is common to normalize the eigenvectors in the L^2_π norm. That is, if $\boldsymbol{v} = c\boldsymbol{u}$, we require

$$1 = \|\boldsymbol{v}\|^2_\pi = \frac{c^2}{N}|\boldsymbol{u}|^2 = \frac{c^2}{N}(|\boldsymbol{p}|^2 + |\boldsymbol{q}|^2 - 2\boldsymbol{p}^\top\boldsymbol{q}) = c^2\left(\frac{4|\boldsymbol{p}|^2}{N} - 1\right).$$

Therefore

$$\boldsymbol{v} = c(\boldsymbol{p} - \boldsymbol{q}), \quad \text{with } c = \left(\frac{4|\boldsymbol{p}|^2}{N} - 1\right)^{-\frac{1}{2}}. \tag{3-5}$$

We note that, because $\boldsymbol{v} = c(\boldsymbol{p} - \boldsymbol{q}) = c(2\boldsymbol{p} - 1)$ with $c > 0$, we have that $\boldsymbol{v}$ is monotonically increasing in $\boldsymbol{p}$; that is, nodes x with larger p_x also have larger v_x.

Because $-\mathbf{1} \le 2\boldsymbol{p} - 1 \le \mathbf{1}$, from (3-5) we get the pointwise estimate for the entries of $\boldsymbol{v}$,

$$\|\boldsymbol{v}\|_\infty = c\max\{(2p_{\max} - 1), (1 - 2p_{\min})\} \le \frac{1}{\sqrt{4|\boldsymbol{p}|^2/N - 1}}. \tag{3-6}$$

Case 2: $s_p \ne N/2$. In this case,

$$\boldsymbol{d} = s_q\boldsymbol{p} + s_p\boldsymbol{q} = s_p\mathbf{1} + (N - 2s_p)\boldsymbol{p}$$

is not a constant vector anymore, and together $\mathbf{1}$ and $\boldsymbol{d}$ form a basis for the range of T. Indeed, from Proposition 2 we know that rank $T = 2$. In fact, the range of T is equal to the span of the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$. That $\boldsymbol{p}$ is in the range of T can be seen by applying T to the component of $\boldsymbol{q}$ orthogonal to $\boldsymbol{p}$

$$\operatorname{orth}_{\boldsymbol{p}} \boldsymbol{q} := \boldsymbol{q} - \frac{\boldsymbol{p}^\mathsf{T}\boldsymbol{q}}{|\boldsymbol{p}|^2}\boldsymbol{p}.$$

Then

$$T \operatorname{orth}_{\boldsymbol{p}} \boldsymbol{q} = \left(|\boldsymbol{q}|^2 - \frac{(\boldsymbol{p}^\mathsf{T}\boldsymbol{q})^2}{|\boldsymbol{p}|^2}\right)\boldsymbol{p}.$$

Since the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are not proportional, the Cauchy–Schwarz inequality

$$(\boldsymbol{p}^\mathsf{T}\boldsymbol{q})^2 \le |\boldsymbol{p}|^2|\boldsymbol{q}|^2$$

is strict and therefore the coefficient of $\boldsymbol{p}$ above is strictly positive. By a similar argument, $\boldsymbol{q}$ is in the range of T. Since the vectors

$$\mathbf{1} = \boldsymbol{p} + \boldsymbol{q} \quad \text{and} \quad \boldsymbol{d} = T\mathbf{1}$$

are in the range of T and are linearly independent, they form a basis for the range of T.

We introduce the notation

$$A(\boldsymbol{d}) = \frac{\sum_x d(x)}{N} \quad \text{and} \quad H(\boldsymbol{d}) = \frac{N}{\sum_x 1/d(x)}$$

for the arithmetic and harmonic means, respectively, of the entries of the vector of degrees $\boldsymbol{d}$. Consider the vector

$$\boldsymbol{u} = \mathbf{1} - A(\boldsymbol{d})D^{-1}\mathbf{1}. \tag{3-7}$$

We show that for a $\beta \in [-1, 0)$ specified below in (3-8), we have $P_T\boldsymbol{u} = \beta\boldsymbol{u}$.

First we note that because $\boldsymbol{d}$ is not a constant vector, we have

$$D\boldsymbol{u} = \boldsymbol{d} - A(\boldsymbol{d})\mathbf{1} \neq \mathbf{0}.$$

From this, together with

$$\mathbf{1}^\mathsf{T} D\boldsymbol{u} = 0,$$

we deduce that $\mathbf{1}$ and $D\boldsymbol{u}$ are nonzero, perpendicular, vectors in the range of T.

Since the range of T is 2-dimensional, if we show that $T\boldsymbol{u}$ is also perpendicular to $\mathbf{1}$, it follows that for some real β,

$$T\boldsymbol{u} = \beta D\boldsymbol{u}, \quad \text{i.e.,} \quad P_T\boldsymbol{u} = \beta\boldsymbol{u}.$$

We show that $\mathbf{1}^\mathsf{T} T\boldsymbol{u} = 0$. Indeed, we have

$$\mathbf{1}^\mathsf{T} T\boldsymbol{u} = (T\mathbf{1})^\mathsf{T}(\mathbf{1} - A(\boldsymbol{d})D^{-1}\mathbf{1}) = \boldsymbol{d}^\mathsf{T}(\mathbf{1} - A(\boldsymbol{d})D^{-1}\mathbf{1}) = NA(\boldsymbol{d}) - A(\boldsymbol{d})\boldsymbol{d}^\mathsf{T}D^{-1}\mathbf{1} = 0.$$

The only thing left to check is that $-1 \leq \beta < 0$. We do this by calculating the trace of P_T. On one hand, since the trace of a square matrix equals the sum of its eigenvalues, and $N-2$ of the eigenvalues of P_T are zero, we have $\operatorname{Tr} P_T = 1+\beta$, and therefore

$$\beta = \operatorname{Tr} P_T - 1. \tag{3-8}$$

On the other hand,

$$\begin{aligned}\operatorname{Tr} P_T &= \sum_x \frac{2p_x q_x}{d(x)} = \sum_x \frac{2p_x q_x}{s_q p_x + s_p q_x} \\ &= \sum_x \frac{2}{s_q/q_x + s_p/p_x} \leq \sum_x \frac{q_x/s_q + p_x/s_p}{2} = 1.\end{aligned}$$

The inequality above follows from the term-by-term inequality between the harmonic and arithmetic means of two positive numbers

$$\frac{2}{s_q/q_x + s_p/p_x} \leq \frac{q_x/s_q + p_x/s_p}{2},$$

with equality if and only if

$$\frac{q_x}{s_q} = \frac{p_x}{s_p}.$$

Since the equality cannot hold for all x (otherwise $\boldsymbol{p}$ would be constant), we obtain that $0 < \operatorname{Tr} P_T < 1$, that is, $-1 < \beta < 0$. In the calculation above, we implicitly assumed that for all x we have $p_x \in (0, 1)$. If p_x is equal to either 0 or 1, then the term $2p_x q_x/d(x)$ corresponding to x in $\operatorname{Tr} P_T$ is zero, and thus strictly smaller than

$$\frac{q_x/s_q + p_x/s_p}{2}.$$

As we shall see in the next subsection, if for every x we have that p_x is equal to either 0 or 1, then $\operatorname{Tr} P_T = 0$ and therefore $\beta = -1$. This is the case of bipartite graphs.

Alternatively, one may estimate β from $T\boldsymbol{u} = \beta D\boldsymbol{u}$ by multiplying by $\boldsymbol{u}^\top$ on the left. We get

$$\beta = \frac{\boldsymbol{u}^\top T\boldsymbol{u}}{\boldsymbol{u}^\top D\boldsymbol{u}}. \tag{3-9}$$

The denominator in (3-9) is readily calculated as

$$\boldsymbol{u}^\top D\boldsymbol{u} = (\mathbf{1}^\top - A(\boldsymbol{d})\mathbf{1}^\top D^{-1})D\boldsymbol{u} = \mathbf{1}^\top D\boldsymbol{u} - A(\boldsymbol{d})\mathbf{1}^\top\boldsymbol{u}.$$

Since $\mathbf{1}^\top D\boldsymbol{u} = 0$, we get

$$\boldsymbol{u}^\top D\boldsymbol{u} = -A(\boldsymbol{d})\biggl(N - A(\boldsymbol{d})\sum_x \frac{1}{d(x)}\biggr) = -A(\boldsymbol{d})\biggl(N - N\frac{A(\boldsymbol{d})}{H(\boldsymbol{d})}\biggr),$$

which means

$$\boldsymbol{u}^{\mathsf{T}} D\boldsymbol{u} = NA(\boldsymbol{d})\left(\frac{A(\boldsymbol{d})}{H(\boldsymbol{d})} - 1\right) > 0. \tag{3-10}$$

The fact that the quantity in parentheses above is positive follows from the inequality between harmonic and arithmetic means. The calculation that shows that $\boldsymbol{u}^{\mathsf{T}} T\boldsymbol{u} < 0$, however, can be quite tedious.

The equilibrium measure in this case is not uniform anymore. It will be given by

$$\pi(x) = \frac{d(x)}{\sum_y d(y)} = \frac{d(x)}{NA(\boldsymbol{d})}.$$

Again, we would like the eigenvector corresponding to eigenvalue β to be normalized in the L^2 norm weighted by $\boldsymbol{\pi}$. That is, if $\boldsymbol{v} = c\boldsymbol{u}$, we require

$$1 = \|\boldsymbol{v}\|_\pi^2 = \frac{c^2}{NA(\boldsymbol{d})}\boldsymbol{u}^{\mathsf{T}} D\boldsymbol{u} = \frac{c^2}{NA(\boldsymbol{d})} NA(\boldsymbol{d})\left(\frac{A(\boldsymbol{d})}{H(\boldsymbol{d})} - 1\right) = c^2\left(\frac{A(\boldsymbol{d})}{H(\boldsymbol{d})} - 1\right).$$

Therefore the normalized eigenvector corresponding to eigenvalue β is

$$\boldsymbol{v} = c(\mathbf{1} - A(\boldsymbol{d})D^{-1}\mathbf{1}), \quad \text{with } c = \left(\frac{A(\boldsymbol{d})}{H(\boldsymbol{d})} - 1\right)^{-\frac{1}{2}}. \tag{3-11}$$

Since we are in the case $s_p \neq N/2$, we have that

$$\boldsymbol{d} = s_q\boldsymbol{p} + s_p\boldsymbol{q} = (s_q - s_p)\boldsymbol{p} + s_p\mathbf{1}$$

is monotonic in $\boldsymbol{p}$ and therefore so is the vector $D^{-1}\mathbf{1}$. From the formula (3-11) we conclude that $\boldsymbol{v}$ is monotonic in $\boldsymbol{p}$. Also, observe that $\boldsymbol{v}$ changes sign precisely between the values of $\boldsymbol{d}$ which are above and below $A(\boldsymbol{d})$.

From the monotonicity of $\boldsymbol{v}$ we conclude that

$$\max_x |v_x| = \|\boldsymbol{v}\|_\infty \quad \text{for } x \text{ such that either } p_x = p_{\min} \text{ or } p_x = p_{\max}.$$

Precisely, we have

$$\|\boldsymbol{v}\|_\infty = \frac{\max\{(1 - A(\boldsymbol{d})/d_{\max}), (A(\boldsymbol{d})/d_{\min} - 1)\}}{\sqrt{A(\boldsymbol{d})/H(\boldsymbol{d}) - 1}}, \tag{3-12}$$

completing the proof. □

In the next subsection we discuss briefly the case when every p_x is either 0 or 1.

3.2. ***The case $p_x \in \{0, 1\}$.*** In our setting with $p_x \in [0, 1]$, because every term in $\operatorname{Tr} P_T$ is nonnegative, we only get $\operatorname{Tr} P_T = 0$ when all the p_x are categorical, i.e.,

$p_x = 0$ or $p_x = 1$. Assume m of p_x are equal to 1 and $n = N - m$ are equal to 0; then the matrix T has the form

$$T = \left[\begin{array}{c|c} \mathbf{0}_{m\times m} & \mathbf{1}_{m\times n} \\ \hline \mathbf{1}_{n\times m} & \mathbf{0}_{n\times n} \end{array}\right],$$

where the diagonal zero blocks are of size $m \times m$ and $n \times n$. The degrees are

$$d_x = n \quad \text{when } p_x = 1 \qquad \text{and} \qquad d_x = m \quad \text{when } p_x = 0.$$

The graph associated to the Markov chain with probability transition matrix $P_T = D^{-1}T$ is bipartite and P_T has the smallest eigenvalue $\beta = -1$. The corresponding eigenvector $\boldsymbol{v}$ has

$$v_x = 1 \quad \text{when } p_x = 1 \qquad \text{and} \qquad v_x = -1 \quad \text{when } p_x = 0.$$

Indeed, both formulas given in (3-5) and in (3-11) reduce to $\beta = -1$ and to the eigenvector $\boldsymbol{v}$ above:

If $m = n = N/2$ we have $|\boldsymbol{p}|^2 = N/2$ so that (3-5) yields $c = 1$.

If $m < n$, then

$$A(\boldsymbol{d}) = \frac{2mn}{N}, \qquad H(\boldsymbol{d}) = \frac{Nmn}{m^2 + n^2},$$

and from (3-11) we get

$$\boldsymbol{u} = \frac{n-m}{N}\begin{bmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{bmatrix} \quad \text{and} \quad c = \frac{N}{n-m},$$

and thus $\boldsymbol{v} = c\boldsymbol{u}$ as above.

3.3. ***The eigenvalues of the transition matrix P_B.*** A more realistic model would consider a disagreement matrix B equal to T everywhere except on the diagonal, where the entries are made equal to 0 (a justice never votes contrary to their own vote). As before, we define the degrees vector $\boldsymbol{d}_B$ with entries

$$d_B(x) = \sum_y B_{xy} = \sum_{y \neq x} T_{xy} = p_x s_q + q_x s_p - 2p_x q_x.$$

Let $P_B = D_B^{-1}B$ be the probability transition matrix associated to B, where D_B is the diagonal matrix with $\boldsymbol{d}$ on the diagonal. The study of the spectrum and the eigenvectors of P_B is itself an interesting problem.

Observe that for $N = 2$ we have

$$P_B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and its eigenvalues are 1 and -1 with eigenvectors given respectively by

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

In general, the matrices B, and consequently P_B, no longer have rank 2. However, numerical simulations lead (at least in the case when N is relatively large and the values of $\boldsymbol{p}$ are well distributed in [0, 1]) to the following

Conjecture 5. *Besides the eigenvalue $\lambda = 1$, the remaining $N - 1$ eigenvalues are negative with $N-2$ of them close to 0, while the smallest is less than β (the smallest eigenvalue of P_T).*

Conjecture 6. *The highest-energy eigenvector of P_B seems to remain monotonic in $\boldsymbol{p}$ and it is close to the corresponding eigenvector of P_T.*

We should warn the reader that our use of the word "conjecture" in this paper is not in the sense it is used traditionally in mathematics, but more with the meaning of "guess".

While the conjecture about the sign of the eigenvalues of P_B is true, as we prove in the Theorem 8 next, the conjecture about the monotonicity in $\boldsymbol{p}$ of the highest-energy eigenvector of P_B fails. This was again observed numerically for small values of N ($N = 3$ and $N = 4$) and it naturally leads to the following:

Question 7. *What conditions on $\boldsymbol{p}$ ensure the monotonicity (in $\boldsymbol{p}$) of the highest-energy eigenvector of P_B?*

We now prove the following:

Theorem 8. *The probability transition matrix P_B has the largest eigenvalue 1, while the remaining $N - 1$ eigenvalues are negative, with the smallest satisfying $\beta_B \le \beta_T$, with equality if and only if $p_x \in \{0, 1\}$ for all x (hence the bipartite case from Section 3.2).*

Proof. Our method of proof is to relate the eigenvalues of P_B to those of P_T in Section 3.1. For this we denote by $\boldsymbol{e}$ the vector with entries $e(x) = 2p_x q_x$, and by $E := \operatorname{diag}(\boldsymbol{e})$, the diagonal matrix with $\boldsymbol{e}$ on the diagonal. Note that

$$T = B + E \quad \text{and} \quad \boldsymbol{d}_T = \boldsymbol{d}_B + \boldsymbol{e}.$$

Let $\boldsymbol{u}$ be a nonconstant eigenvector of P_B corresponding to an eigenvalue $\alpha < 1$. We show that the eigenvalue of $\mathrm{Id} - P_B$ given by $\lambda = 1 - \alpha$ is greater than 1 and therefore $\alpha < 0$. For this we estimate the energy $\mathcal{E}_B(\boldsymbol{u})$ given by (2-7) with $w(x, y) = p_x q_y + p_y q_x$.

From the spectral decomposition (2-5) we have that $\boldsymbol{u}$ is perpendicular to $\mathbf{1}$ with respect to $\boldsymbol{\pi}$, and therefore with respect to the weight $\boldsymbol{d}_B$; i.e.,

$$\langle \boldsymbol{u}, \mathbf{1}\rangle_{d_B} = \langle \boldsymbol{u}, \boldsymbol{d}_B\rangle = \sum_{x\in V} d_B(x)u(x) = 0.$$

Consider the (unique) decomposition

$$\boldsymbol{u} = a\mathbf{1} + \boldsymbol{v},$$

where a is a real number, and $\boldsymbol{v}$ is perpendicular to $\mathbf{1}$ with respect to $\boldsymbol{d}_T$. Since $\boldsymbol{u}$ is nonconstant, the vector $\boldsymbol{v}$ is nonzero. The coefficient of $\mathbf{1}$ in the decomposition of $\boldsymbol{u}$ above is the scalar projection of $\boldsymbol{u}$ on $\mathbf{1}$ with respect to the inner product weighted by $\boldsymbol{d}_T$, and it can be calculated as

$$a = \frac{\langle \boldsymbol{u}, \boldsymbol{e}\rangle}{\operatorname{vol}(G_T)} = \frac{\langle \boldsymbol{u}, \boldsymbol{e}\rangle}{2s_p s_q}. \tag{3-13}$$

Indeed, from the fact that $\boldsymbol{d}_T = \boldsymbol{d}_B + \boldsymbol{e}$, we obtain

$$\langle \boldsymbol{u}, \boldsymbol{d}_T\rangle = \underbrace{\langle \boldsymbol{u}, \boldsymbol{d}_B\rangle}_{=0} + \langle \boldsymbol{u}, \boldsymbol{e}\rangle, \qquad \text{therefore} \quad \langle a\mathbf{1}, \boldsymbol{d}_T\rangle + \underbrace{\langle \boldsymbol{v}, \boldsymbol{d}_T\rangle}_{=0} = \langle \boldsymbol{u}, \boldsymbol{e}\rangle.$$

Since $\langle \mathbf{1}, \boldsymbol{d}_T\rangle = \operatorname{vol}(G_T) = 2s_p s_q$, formula (3-13) follows.

Observe that because $\boldsymbol{v}$ is perpendicular to $\mathbf{1}$ with respect to the weight $\boldsymbol{d}_T$ then $\boldsymbol{v}$ is in the sum of the eigenspaces of T with nonpositive eigenvalues. Therefore the energy $\mathcal{E}_T(\boldsymbol{v})$ is bounded from below by $\|\boldsymbol{v}\|^2_{\pi_T} = \langle \boldsymbol{v}, \boldsymbol{v}\rangle_{\pi_T}$. This means,

$$\operatorname{vol}(G_T)\mathcal{E}_T(\boldsymbol{v}) \ge \langle \boldsymbol{v}, \boldsymbol{v}\rangle_{d_T} = \sum_{x\in V} d_T(x)v^2(x) = \sum_{x\in V} d_B(x)v^2(x) + \sum_{x\in V} e(x)v^2(x).$$

As $v(x) = (u(x) - a)$ for every x, we have

$$\begin{aligned}\sum_{x\in V} d_B(x)v^2(x) &= \sum_{x\in V} d_B(x)(u(x)-a)^2\\ &= \sum_{x\in V} d_B(x)u^2(x) + a^2 \operatorname{vol}(G_B) - 2a \underbrace{\sum_{x\in V} d_B(x)u(x)}_{=0}.\end{aligned}$$

Since $\mathcal{E}_T(\boldsymbol{u}) = \mathcal{E}_T(\boldsymbol{v})$, we obtain

$$\operatorname{vol}(G_B)\mathcal{E}_B(\boldsymbol{u}) = \operatorname{vol}(G_T)\mathcal{E}_T(\boldsymbol{v}) \ge \sum_{x\in V} d_B(x)u^2(x) + a^2 \operatorname{vol}(G_B) + \sum_{x\in V} e(x)v^2(x).$$

Therefore

$$\mathcal{E}_B(\boldsymbol{u}) \ge \|\boldsymbol{u}\|^2_{\pi_B} + a^2 + \frac{1}{\operatorname{vol}(G_B)} \sum_{x\in V} e(x)v^2(x),$$

and so

$$\mathcal{E}_B(\boldsymbol{u}) = (1-\alpha)\|\boldsymbol{u}\|^2_{\pi_B} \geq \|\boldsymbol{u}\|^2_{\pi_B}, \quad \text{i.e.,} \quad \alpha \leq 0.$$

Observe that from (3-13) we have $a^2 > 0$ unless $\boldsymbol{u} \perp \boldsymbol{e}$. Also, under the assumption that $p_x \in (0, 1)$ for all x, we have $\sum_{x\in V} e(x)v^2(x) > 0$. Thus, the only cases of equality $\mathcal{E}_B(\boldsymbol{u}) = \|\boldsymbol{u}\|^2_{\pi_B}$ happen when $a = 0$, i.e., $\boldsymbol{u} = \boldsymbol{v}$, and $u(x) = 0$ for any x with $p_x \in (0, 1)$.

To estimate the highest-energy eigenvalue β_B of P_N, we use the variational characterization

$$(1-\beta_B) = \max_{\boldsymbol{u}\neq \boldsymbol{0}} \frac{\mathcal{E}_B(\boldsymbol{u})}{\|\boldsymbol{u}\|^2_{\pi_B}}.$$

Let $\boldsymbol{v}$ be the eigenvector corresponding to the highest-energy eigenvalue β_T of P_T and consider a test vector $\boldsymbol{u} = a\mathbf{1} + \boldsymbol{v}$, where a is picked such that $\boldsymbol{u}$ is perpendicular to $\mathbf{1}$ with respect to the weight $\boldsymbol{d}_B$. Then,

$$(1-\beta_B) \geq \frac{\mathcal{E}_B(\boldsymbol{u})}{\|\boldsymbol{u}\|^2_{\pi_B}} = \operatorname{vol}(G_T)\frac{\mathcal{E}_T(\boldsymbol{u})}{\sum_{x\in V} d_B(x)u^2(x)} = (1-\beta_T)\frac{\sum_{x\in V} d_T(x)v^2(x)}{\sum_{x\in V} d_B(x)u^2(x)}.$$

As before,

$$\begin{aligned}\sum_{x\in V} d_T(x)v^2(x) &= \sum_{x\in V} d_B(x)v^2(x) + \sum_{x\in V} e(x)v^2(x)\\ &= \operatorname{vol}(G_B)a^2 + \sum_{x\in V} d_B(x)u^2(x) + \sum_{x\in V} e(x)v^2(x) \geq \sum_{x\in V} d_B(x)u^2(x).\end{aligned}$$

Therefore $(1-\beta_B) \geq (1-\beta_T)$ with strict inequality if $p_x \in (0, 1)$ for all x. □

4. Conclusions and future directions

There are two aspects associated with the subject of this paper. One is the modeling aspect together with the proposal of a new measure, the ideology index, in the concrete case of the United States Supreme Court, and the other aspect consists of the theoretical investigations of further simplifications of the model.

Regarding the model, we begin with a data set that records only the number of cases on which any given pair of justices voted opposite to each other. From this matrix we create a network to which we associate a reversible Markov chain. The states of the Markov chain are the nodes of the network, and in our model each node corresponds to a Supreme Court justice. We singled out the eigenvector of the transition probability matrix that corresponds to the smallest eigenvalue, normalized so that it has length 1 in L^2_π (weighted by the reversible measure $\boldsymbol{\pi}$). Then, we analyzed the function, which we termed the ideology index, which assigns to each node the corresponding entry in this eigenvector. We should remark that this function gives only a *relative* measure, as opposed to an *absolute* measure, in the sense that, in order to assign a meaning to the value at x, one has to relate it to the values at the other nodes in the graph. On the other hand, it is an objective function, free from any subjective interpretation, being defined entirely from the recorded data.

The [SCDB 2018] contains data compiled by legal scholars indicating whether a case carried ideological implications and the direction (conservative or liberal) the justices voted in every such case. For each justice, we used these data to calculate the frequency with which their votes coincided with conservative ideology, and observed a strong correlation between these frequencies and the entries in the eigenvector corresponding to the smallest eigenvalue. This observation was one of the main motivations for undergoing this study.

For the theoretical investigation, in order to study the eigenvalues of a transition matrix of a reversible Markov chain P_B obtained from a disagreement matrix $B = B(\boldsymbol{p})$ with $\boldsymbol{p} \in [0, 1]^N$, our approach was to study the spectrum of P_T for the matrix $T = \boldsymbol{p}\boldsymbol{q}^\intercal + \boldsymbol{q}\boldsymbol{p}^\intercal$ first. We found explicit formulas for the highest-energy eigenvector of P_T. It would be interesting to investigate whether similar formulas for the highest-energy eigenvector of P_B exist. Even without such formulas, numerical simulations suggest that the two eigenvectors are close to each other, and therefore have similar properties as for example monotonicity in $\boldsymbol{p}$ or boundedness estimates. One should then decide whether such properties grant the entries of the highest-energy eigenvector of P_B the quality of an objective measure of an ideology index.

Confirmation for the validity and robustness of our model can be achieved by selecting appropriately the vector $\boldsymbol{p}$ and comparing the output of the numerical implementation to the results given by formula (3-5) or (3-11). We propose two different ways to obtain an acceptable vector $\boldsymbol{p}$ adapted to the model. The first method is the one we used in order to obtain the y-axis values in the plot in Figure 1.

To obtain the vector $\boldsymbol{p}$, we used the frequencies calculated from the [SCDB 2018], as described above. The ability to calculate such frequencies was dependent upon the case-by-case legal analysis of ideology included in the database. Such comprehensive analysis may not exist for other data sets.

An alternative method that can be used to obtain the vector $\boldsymbol{p}$ is the following procedure. Consider data collected over a continuous time period, spanning the careers of N justices. In the following, by *case* we mean only those cases in the Supreme Court on which there was not an unanimous decision. Let C denote the $N \times N$ symmetric matrix with entries $C(x, y)$ representing the number of cases justices x and y disagreed on. Denote by $n(x, y)$ the number of cases justices x and y served together. For a vector $\boldsymbol{p} \in [0, 1]^N$ define the disagreement matrix $B = B(\boldsymbol{p})$ to be equal to $(\boldsymbol{p}\boldsymbol{q}^\intercal + \boldsymbol{q}\boldsymbol{p}^\intercal)$ off the diagonal, and zero on the diagonal. A natural candidate for $\boldsymbol{p}$ is obtained by minimizing over $\boldsymbol{p} \in [0, 1]^N$ the square distance between the matrix C and the matrix with entries $n(x, y)B(x, y)$. By this we mean minimizing the function

$$f(\boldsymbol{p}) := \sum_{x \neq y} (n(x, y)(p_x + p_y - 2p_x p_y) - C(x, y))^2.$$

The general direction would be to investigate whether one can obtain useful estimates on the minimum value of the function f, and whether these estimates are transferable further to the distance between the highest-energy eigenvector of P_B and that of P_C. Should this program be successful, the highest-energy eigenvector of P_C would provide an objective measure of the proposed *ideology index*.

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