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Counting profile strings from rectangular tilings

Anthony Petrosino, Alissa Schembor and Kathryn Haymaker

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We study the enumeration of profiles, which are outlines that occur when tiling a rectangular board with squares, dominoes, and trominoes. Profiles of length m correspond to a special subset of the set $\{0, 1, 2, 3\}^m$, called profile strings. Profiles and their corresponding strings first appeared in the enumeration of the tilings of rectangular $2\times n$ and $3\times n$ boards with squares, dominoes, and trominoes. Profiles also play a role in enumerating the tilings of an $m\times n$ board for any fixed $m \geq 2$. We describe how profiles arise when enumerating tilings, and we prove that the number of profile strings of length m equals $m \cdot 3^{m-1}$.

1. Background

A *tiling* of an $m\times n$ board using square tiles (1×1), domino tiles (1×2 or 2×1), and I-tromino tiles (1×3 or 3×1) is a configuration of tiles that covers every position in the board. In this paper we will use the word "tromino" to mean exclusively an I-tromino. Figure 1 shows a tiling of a 3×4 board using two squares, two dominoes, and two trominoes. Throughout this paper, the variable x denotes the placement of a square tile, y denotes the placement of a domino tile, and z denotes the placement of a tromino tile.

Read [1982] and Haymaker and Robertson [2017] determined recursions for the number of tilings of a $2\times n$ board with squares and dominoes, and squares, dominoes, and trominoes, respectively, using the technique of building a tiling from the leftmost, topmost untiled position. The outline of the right-hand side of a partially tiled board is called a *profile*, and counting the number of profiles of length m is the main goal of this paper.

We first define a building process that begins on the leftmost, topmost open square.

Definition 1. Consider a blank or partially tiled $m\times n$ board. A *building move* is when a tile is placed in the leftmost column that has an uncovered square, in the topmost square that is uncovered.

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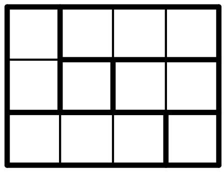

Figure 1. An example tiling of a 3×4 board with squares, dominoes, and trominoes.

There is a one-to-one correspondence between tilings of an $m \times n$ rectangular board and sequences of building moves that result in a complete $m \times n$ tiling [Read 1982]. That is, every tiling can be obtained from a unique sequence of building moves.

Definition 2. The *target square* of a partial tiling being built using building moves is the topmost square in the leftmost column with an uncovered square.

For an example of a tiling obtained using building moves, see Figure 2. In each partial tiling in Figure 2, the "×" denotes the position of the target square.

Definition 3. The m-row *profile* of a partial tiling of an $m \times n$ board is the outline of the rightmost edge of the partial tiling created by building moves. A *profile string* of an m-row profile is a string of length m where positions are assigned values of 0, 1, 2, or 3 depending on how many squares to the right of the rightmost fully tiled column are covered with tiles. We define the *length* of the profile to be m, the number of rows in the profile.

For example, the profile of the second partial tiling shown in Figure 2 has corresponding profile string 110.

Definition 4. The length-m *A profile* is the profile whose outline is a vertical line. The *A profile string* is $\underbrace{0 \cdots 0}_{m}$, the all-zero string of length m.

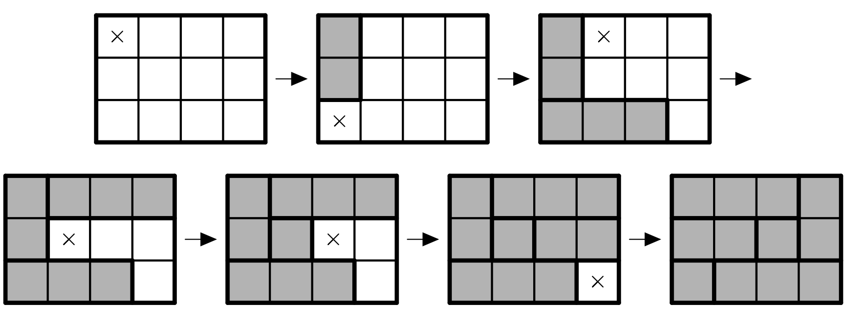

Figure 2. Sequence of profile strings from the building process for this 3×4 tiling: $000 \to 110 \to 002 \to 302 \to 201 \to 110 \to 000$.

Next, we give an example of how building moves result in a sequence of profiles and profile strings.

Example 5 (illustrating profile strings with the 3×4 tiling). Figure 2 begins with an untiled 3×4 board, placing the target square in the leftmost-topmost position, indicated by the "$\times$". This profile is represented by the profile string 000 (the A profile). This is followed by the placement of a vertical domino, causing the target square to move down to the bottom row, and resulting in a profile string of 110. Next, a horizontal tromino is placed on top of the target square position, causing the target square to now move to the top row. Although this profile string appears to be 113, it should be noted that strings are denoted by the number of squares beyond the rightmost fully tiled column. Because there is one fully tiled column, the profile string reduces to 002. The process of constructing a tiling arrangement proceeds with the placement of squares, dominoes, or trominoes on top of the target square. Once the construction is complete, it will result in a 000 profile string, which is exactly how the example began.

Remark 6. The profile of a partially tiled board is the *shape* of the right-hand side of the board; the same shape/profile can result from different tile configurations. In fact, for any profile, it is possible to construct the profile shape using only "horizontal" tiles. To see this, consider a partial tiling that contains a vertical domino, for example. A specific example is the partial tiling on the far left of the middle row in Figure 2. This partial tiling can be changed into a different partial tiling with only horizontal tiles by replacing the vertical domino with two squares stacked vertically. The outline of the right side of the board remains the same; i.e., the profile remains the same. Similarly, if a partial tiling contains a vertical tromino, that piece can be replaced by three vertically stacked squares, yielding the same profile. This property of profiles becomes important in Section 3 when we count profiles using a construction method with only horizontal tiles.

2. Profile-type strings

To describe the shape of profiles in a numerical sense, we define profile-type strings. We use profile-type strings in order to further analyze the behavior of profiles resulting from the tiling of an $m\times n$ board. In Section 3, we will show that profile-type strings give a combinatorial description of all possible profiles that could result from the tiling of an $m\times n$ board using square tiles, domino tiles, and I-tromino tiles.

Definition 7. A string $x_1, x_2, \dots, x_m$ in $\{0, 1, 2, 3\}^m$ is called a *profile-type* string if it satisfies the following conditions:

(1) There exists an $i \in \{1, 2, \dots, m\}$ such that $x_i = 0$.

(2) If $x_i = 0$, then $x_j \neq 3$ for all $j > i$.

In words, a profile-type string is a string with at least one 0, where no 3 occurs to the right of a 0 in the string.

For example, the following are profile-type strings with $m = 5$: 31202, 11302, 00122. On the other hand, the following strings of length 5 are not profile-type strings: 12321, 00130, 00123.

The $m = 2$ case contains six profile-type strings

$$\begin{array}{c} 00,\ 20,\ 10, \\ 30,\ 01,\ 02, \end{array} \tag{2-1}$$

while the $m = 3$ case contains 27 profile-type strings

$$\begin{array}{c} 000,\ 300,\ 200,\ 110,\ 100,\ 330,\ 320,\ 310,\ 230, \\ 220,\ 210,\ 002,\ 001,\ 120,\ 130,\ 101,\ 202,\ 201, \\ 011,\ 010,\ 102,\ 302,\ 301,\ 012,\ 022,\ 021,\ 020. \end{array} \tag{2-2}$$

Proposition 8. *There are* 108 *profile-type strings for the* $m = 4$ *case.*

Proof. We enumerate the profile-type strings in $\{0, 1, 2, 3\}^4$ as follows. Let a, b, c, d represent a profile-type string. Then we count the following cases:

(1) If $a = 0$, then $b, c, d \in \{0, 1, 2\}$ by Definition 7, so there are $3^3 = 27$ such strings.

(2) If $a \in \{1, 2, 3\}$ and $b = 0$, then $c, d \in \{0, 1, 2\}$, so there are 27 such strings.

(3) If $a, b \in \{1, 2, 3\}$ and $c = 0$, then $d \in \{0, 1, 2\}$, so there are 27 such strings.

(4) If $a, b, c \in \{1, 2, 3\}$ and $d = 0$, there are 27 strings of this type, given the options for a, b, and c.

The total number of such strings is $4(27) = 108$. □

Theorem 9. *The number of profile-type strings of length* m *is equal to* $m \cdot 3^{m-1}$.

Proof. We will go through each position of a length-m profile-type string, anchor a 0 onto each position, and see how many different values are possible for all of the positions other than the fixed position. If there is a 0 anchored on a fixed position, we will say that there cannot be any 0 on any position before the fixed position, in order to avoid repeats in the enumeration.

Let x_i represent the i-th entry in the profile-type string.

The cases begin as follows:

(1) Use x_1 as the anchor, with $x_1 = 0$; then $x_2, \ldots, x_m \in \{0, 1, 2\}$ as there cannot be a 3 following a 0 in a profile-type string, by definition.

(2) Next use x_2 as the anchor position, so $x_2 = 0$; then $x_1 \in \{1, 2, 3\}$ as there are no restrictions on the first position, besides not having 0 to avoid double

counting. Further, $x_3, \ldots, x_m \in \{0, 1, 2\}$ as there cannot be a 3 following a 0 in a profile-type string, by definition.

In general, for $k = 1, \ldots, m$, the k-th case is to anchor $x_k = 0$, and assume $x_i \in \{1, 2, 3\}$ for each i satisfying $1 \le i < k$. Therefore by the definition of profile-type strings, $x_j \in \{0, 1, 2\}$ for each j satisfying $k < j \le m$.

Each of these m cases has 3^{m-1} strings: there are three options for every position, other than in the fixed position in which we anchored a 0; as there are $m-1$ positions with three options (every position but the fixed position in a length-m profile-type string), there are 3^{m-1} options. There are m cases, so the total number of profile-type strings is $m \cdot 3^{m-1}$. □

3. Counting profiles

In this section, we prove that the profile-type strings of length m are in one-to-one correspondence with the profile strings of length m. That is, we can drop the word "type" in the name profile-type strings!

As a motivating example, we first look at the case of $m = 2$.

Example 10 (profiles and profile-type strings for $m = 2$). Notice that the length-2 strings that satisfy the conditions in Definition 7 are given in (2-1).

On the other hand, the profiles with two rows are in Figure 3, and their corresponding profile strings are

$$A : 00, \quad B : 20, \quad C : 10, \quad D : 30, \quad E : 01, \quad F : 02.$$

By inspection, we see that the profile strings for profiles A–F in Figure 3 are precisely the profile-type strings listed in (2-1).

Definition 11. A *profile construction* is the process of sequentially adding squares, dominoes, and trominoes to the A-profile until the desired profile results. The order in which squares, dominoes, and trominoes are added must correspond with proper

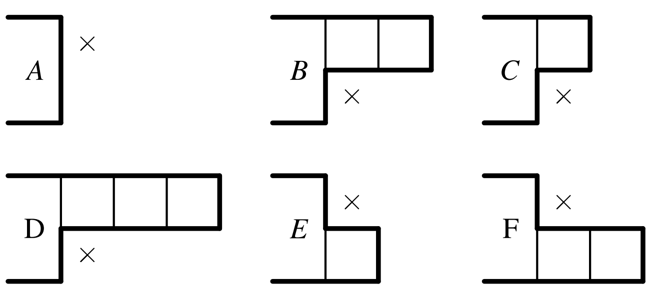

Figure 3. This figure shows all six profiles for $m = 2$. We see that there is a one-to-one correspondence between these profiles and the profile-type strings for $m = 2$.

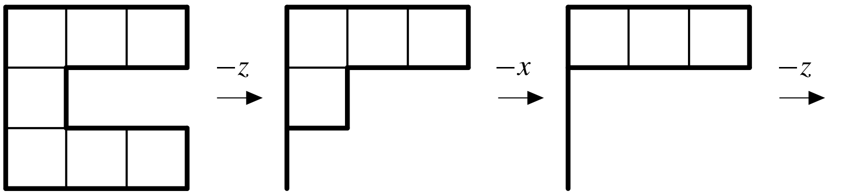

Figure 4. Deconstruction of the length-3 profile 202 to the A-profile in the -1 column position.

building moves; that is, at each step a square, domino, or tromino must be added onto the target square.

Definition 12. A *profile deconstruction* is the process of sequentially removing squares, dominoes, and trominoes from a profile until all that is left is the A-profile. The order in which squares, dominoes, and trominoes are removed must correspond with the construction of a profile; that is, squares, dominoes, and trominoes must be removed based on where the previous target square was located.

Remark 13. Profile deconstructions are not unique; there are different ways to deconstruct the same profile.

Example 14 (deconstruction and reconstruction with an additional row). We will shortly see an induction proof that involves adding a row to a profile. In this example we show a deconstruction of the profile with string 202 to the A-profile in column -1. Then we add a row and construct the profile with string 2020, as a motivating example for the inductive step.

In Figure 4, the initial profile will be deconstructed to the -1 column position. In order to do so, the deconstruction first removes a tromino from the bottom row, followed by the removal of a square from the middle row, and finally the removal of a tromino from the top row. This allows for the figure to be fully deconstructed to the A-profile in column -1.

In Figure 5, an additional row is then added to the profile. This begins the process of reconstructing the profile 2020 by adding the following (in this particular order, from top to bottom): tromino, square, tromino, square.

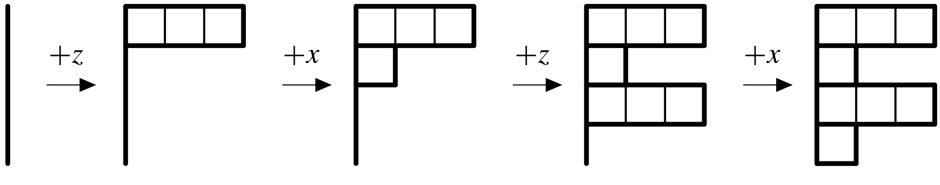

Figure 5. Construction of the $m = 4$ profile 2020 by undoing the deconstruction in Figure 4, with an additional row.

Lemma 15. *Every profile of length m can be deconstructed to the A-profile of length m in column* -1.

Proof. The following algorithm deconstructs a profile of length m to the A-profile in the -1 column. Let $\mathcal{P}_m \subseteq \{0, 1, 2, 3\}^m$ be the set of profile strings of length m.

Input: $x_1, x_2, \ldots, x_m \in \mathcal{P}_m$. The algorithm runs through $k = m, m-1, \ldots, 1$. If $x_k \in \{0, 1, 2\}$, remove a length-$(x_k + 1)$-tile from the k-th row and proceed to row $k-1$. If $x_k = 3$, remove a combined square and tromino from the k-th row and proceed to row $k-1$. The algorithm terminates when $k = 1$. The resulting profile is an A-profile at the -1 column. □

To reconstruct the profile from A at the -1 column, place the removed pieces from top to bottom; in the rows consisting of a square followed by a tromino, place the squares first. Then in the 0 column, place the trominoes from top to bottom. These steps comprise two phases of reconstruction.

Two-phase reconstruction, starting with the A-profile in the -1 column:

- Phase 1: If $x_k \in \{0, 1, 2\}$, then place a length-(x_k+1) tile in row k. If $x_k = 3$, phase 1 is to place a square tile in row k.
- Phase 2: Place trominoes in all rows where $x_k = 3$.

Notice that this two-phase process is designed to follow the rules of building moves and target squares, while resulting in the desired profile.

Example 16 (two-phase reconstruction of 3120).

- Phase 1: In the first portion of Figure 6, we see the result of phase 1 of construction to obtain the profile with string 3120. Note that for the desired profile, $x_1 = 3,\ x_2 = 1,\ x_3 = 2$, and $x_4 = 0$. Begin phase 1 by placing a square (length-1 tile) in row 1, as $x_1 = 3$. Next, place a domino (length-2 tile) in row 2, as $x_2 = 1$. Next, place a tromino (length-3 tile) in row 3, as $x_3 = 2$. Next, place a square (length-1 tile) in row 4, as $x_4 = 0$.
- Phase 2: Place a tromino (length-3 tile) in row 1, as $x_1 = 3$. Phase 2 is also shown in Figure 6.

Next we give a lemma that contributes to establishing the one-to-one correspondence between profiles and profile-type strings.

Lemma 17. *If* $x_1, \ldots, x_m, x_{m+1}$ *is a profile-type string of length* $m+1$ *where at least one of* $x_1, \ldots, x_m$ *equals* 0, *then the string* $\sigma = x_1, x_2, \ldots, x_m$ *is a profile-type string of length* m.

Proof. Assume that $x_1, \ldots, x_m, x_{m+1}$ is a profile-type string, and at least one of $x_1, \ldots, x_m$ equals 0. Let σ denote the string $x_1, x_2, \ldots, x_m$. Condition (1) in Definition 7 holds for the string σ, since it contains at least one zero.

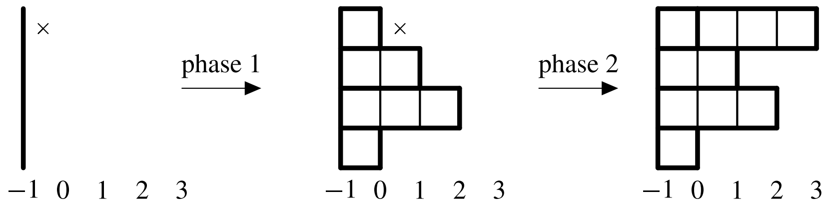

Figure 6. Two phase construction of the profile with string 3120, starting at the -1 column position.

To show that condition (2) in Definition 7 holds, we proceed by contradiction. Suppose that the string σ is not a profile-type string of length m because it violates condition (2). If σ violates condition (2), then there exist i, j such that $1 \leq i < j \leq m$, and $x_i = 0$ and $x_j = 3$ in σ. The entries x_i and x_j are identical in the original string $x_1, x_2, \ldots, x_m, x_{m+1}$, which implies that it also violates condition (2). This is a contradiction as we assumed that the original string was a profile-type string. □

The main result of this paper is next.

Theorem 18. *The m-row profiles are in one-to-one correspondence with the profile-type strings of length m.*

Proof. The theorem contains two claims:

(1) An m-row profile string is a profile-type string of length m (i.e., a string that satisfies Definition 7).

(2) A profile-type string of length m corresponds to an m-row profile.

To prove the first claim, let $x_1, \ldots, x_m$ be a profile string of length m corresponding to a profile P. By the definition of profile strings, at least one entry x_j must be 0, so the string satisfies condition (1) in Definition 7. Suppose that $x_j = 0$ and $x_k = 3$ for some $k > j$. Then in the construction of the profile P, a square, domino, or tromino must have been placed in the k-th row, while the j-th row contained an empty square in a position farther up and to the left. This contradicts the building move process, so no such j and k exist.

To prove the second claim, that every profile-type string corresponds to a profile string, we consider two cases and use induction.

We use induction on m. The base case is $m = 2$. See Example 10 for details of the one-to-one correspondence in the base case.

The inductive hypothesis is that there is a one-to-one correspondence between m-row profiles and profile-type strings of length m.

We will show the correspondence holds for $m + 1$.

Case 1: Suppose that the profile-type string $S = x_1, x_2, \ldots, x_{m+1}$ has $x_{m+1} = 0$, and $x_j \neq 0$ for all $j = 1, \ldots, m$. Then we construct a profile with the profile string S as follows:

Algorithm for constructing a length-$(m+1)$ profile, Case 1:

(1) Start with the A-profile of length $m + 1$.

(2) From top to bottom, place tiles of length x_i horizontally in row i for $i = 1, 2, \ldots, m$.

(3) Leave row $m + 1$ unchanged.

The resulting profile has corresponding string $x_1, x_2, \ldots, x_m, x_{m+1}$, where x_{m+1} is the only entry that equals 0. (Notice that this case does not involve the inductive hypothesis.)

Case 2: For this case, consider the profile-type string $S = x_1, x_2, \ldots, x_m, x_{m+1}$, where at least one of $x_1, \ldots, x_m$ equals 0.

Thus $x_1, \ldots, x_m$ is a profile-type string of length m by Lemma 17, so by the inductive hypothesis there is a profile P of length m that corresponds to it.

Algorithm for constructing a length-$(m+1)$ profile, Case 2:

(1) Deconstruct the profile P to the A-profile.

(2) Add a row to the A-profile. We are now working with an $(m+1)\times n$ array.

(3) Declare the column that the A-profile is on as column -1. The outline of the resulting profile will be at most four columns to the right of this column.

(4) Implement phase 1 of the profile reconstruction process for $x_1, \ldots, x_m$ in the first m rows. As $x_{m+1} \in \{0, 1, 2\}$, place a length-$(x_{m+1}+1)$ tile in row $m + 1$.

(5) Implement phase 2 of the profile reconstruction process for $x_1, \ldots, x_m$ in the first m rows. (Since $x_{m+1} \neq 3$, there is no phase 2 for row $m + 1$).

This algorithm shows that there is a profile P with profile string S, proving our claim that every profile-type string S corresponds to a profile. □

Corollary 19. *The number of m-row profiles is $m \cdot 3^{m-1}$.*

Proof. By Theorem 18, the profiles are in one-to-one correspondence with profile-like strings, and by Theorem 9 there are $m \cdot 3^{m-1}$ profile-type strings of length m. □

Remark 20. Notice that the case where the only zero in the string S is $x_{m+1} = 0$ does not rely on the deconstruction and reconstruction of $x_1, \ldots, x_m$. This is because constructing a profile where only the final entry is 0 requires only one phase, working from top to bottom and placing exactly the length-x_i tile in row i.

4. Conclusions and open questions

In this paper we proved the one-to-one correspondence of profile-type strings of length m with profile strings of length m, partly using induction. This understanding allowed for the creation of a basic formula for the number of profiles in terms of m for any given $m \times n$ board.

An open question that stems from this work is: how can we generalize the profile-type string definition for tilings with more types of tiles, for example, L-trominoes or quadrominoes?

Acknowledgments

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