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# Isomorphisms of graded skew Clifford algebras

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A regular algebra of global dimension $n+1$ is often called a quantum $\mathbb{P}^n$. In 2011, Nafari, Vancliff and Zhang showed that graded skew Clifford algebras (GSCAs) could be used to classify most quadratic quantum $\mathbb{P}^2$s. Some time later, Chandler, Tomlin and Vancliff used their work with certain families of GSCAs to develop a conjecture on the quantum space of a generic quadratic quantum $\mathbb{P}^3$. These results suggest that (the isomorphism classes of) GSCAs are likely to play a fundamental role in the classification of quadratic quantum $\mathbb{P}^3$s. In this article, we will discuss some of these results on GSCAs and discuss new results on isomorphisms between GSCAs.

## Introduction

The notion of a graded skew Clifford algebra was defined in [Cassidy and Vancliff 2010] as a generalization of graded Clifford algebras (see [Le Bruyn 1995]). Therein, it was shown that GSCAs had many properties analogous to those of graded Clifford algebras. In addition, several families of quadratic regular graded skew Clifford algebras of global dimension 4 that were not twists of graded Clifford algebras were produced. It was shown that many of the algebras in these families had a one-dimensional line scheme [Shelton and Vancliff 2002a; 2002b] and a point scheme [Artin et al. 1990] consisting of twenty distinct points; hence, these algebras became candidates for generic quantum $\mathbb{P}^3$s (in the language of [Chandler and Vancliff 2015]).

The line scheme and point scheme of some of these families were explicitly computed in [Chandler and Vancliff 2015; Tomlin and Vancliff 2018]. The line scheme of the generic member of the family studied in [Chandler and Vancliff 2015] was found to consist of a spatial elliptic curve, four planar elliptic curves and two nonsingular conics; the line scheme of the generic member of the family studied in [Tomlin and Vancliff 2018] was found to consist of four planar elliptic curves and four nonsingular conics. In both cases, the computations suggested that

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the nonsingular conics were coming from a spatial elliptic curve in which one of the defining polynomials factored. This suggests the following conjecture.

**Conjecture 0.1** [Chandler and Vancliff 2015]. There exists a class of generic quadratic quantum $\mathbb{P}^3$s in which any representative member has a line scheme that is isomorphic to the union of two spatial elliptic curves and four planar elliptic curves.

At present, no regular algebra with such a line scheme is known. However, given the results in [Chandler and Vancliff 2015; Tomlin and Vancliff 2018], it is possible that some GSCA may satisfy this property and thus be candidates for generic quantum $\mathbb{P}^3$s.

In [Nafari et al. 2011] it was shown that most quadratic quantum $\mathbb{P}^2$s may be constructed as a twist by an automorphism (in the sense of [Artin et al. 1991]) of either a graded skew Clifford algebra or of an Ore extension (see [Goodearl and Warfield 2004]) of a regular graded skew Clifford algebra of global dimension 2. The only regular algebras of global dimension 3 that were not classified in this way are some that have a point scheme of an elliptic curve (classified as type $E$ and an open subset of type $A$ in [Artin and Schelter 1987]). It is currently unclear as to whether an analogous result holds for quantum $\mathbb{P}^3$s.

These results suggest that GSCAs are likely to play some role in the classification of quantum $\mathbb{P}^3$s (and hence the isomorphism classes of such algebras will be of interest).

The article is outlined as follows. In Section 1, we will introduce preliminary notions that will be relevant to the discussions in the article. In Section 2, we will give our main results. In particular, we will discuss two types of isomorphisms: one in which the degree-1 generators of the algebras are identified and the degree-2 generators are related via matrix multiplication and another in which the degree-2 generators of the algebras are identified and the degree-1 generators are related via a diagonal mapping.

## 1. Preliminaries

Throughout this article, we adopt the following notation:

- $\Bbbk$ denotes an algebraically closed field with char($\Bbbk$) $\neq 2$.
- $\Bbbk^\times$ denotes the nonzero elements of $\Bbbk$.
- $\mathbb{M}_n(\Bbbk)$ denotes the ring of $n \times n$ matrices with entries in $\Bbbk$.
- $M_{ij}$ denotes the $ij$-th entry of the matrix $M$ and $M^T$ denotes the transpose of $M$.
- If $S$ is a positively graded, connected $\Bbbk$-algebra, then $S_n$ denotes the homogeneous degree-$n$ elements of $S$.
- If $V$ is a $\Bbbk$-vector space, then $V^n$ denotes the vector space $\underbrace{V \times V \times \cdots \times V}_{n \text{ times}}$.

**Definition 1.1** [Cassidy and Vancliff 2010]. Let $\mu_{ij} \in \Bbbk^\times$, where $1 \le i, j \le n$, such that $\mu_{ij}\mu_{ji} = 1$ for all $i \ne j$ and $\mu_{ii} = 1$ for all $i$. We write $\mu = (\mu_{ij}) \in \mathbb{M}_n(\Bbbk)$. A matrix $M \in \mathbb{M}_n(\Bbbk)$ is called $\mu$-symmetric if $M_{ij} = \mu_{ij} M_{ji}$ for all $i, j$.

The set of all $n \times n$ $\mu$-symmetric matrices forms a $\Bbbk$-vector space, denoted by $\mathrm{Sym}_n^\mu(\Bbbk)$. The standard basis of $\mathrm{Sym}_n^\mu(\Bbbk)$ is

$$\mathfrak{B} = \{E_{ii} : 1 \le i \le n\} \cup \{E_{ij} + \mu_{ji} E_{ji} : 1 \le i < j \le n\},$$

where $E_{ij} \in \mathbb{M}_n(\Bbbk)$ denotes the matrix with a 1 in the $ij$-th position and zeroes elsewhere.

In [Cassidy and Vancliff 2010], it was shown that the notion of identifying a quadratic form and a symmetric matrix could be generalized to $\mu$-symmetric matrices as follows. Let $\mu \in \mathbb{M}_n(\Bbbk)$ be as in Definition 1.1 and $M \in \mathrm{Sym}_n^\mu(\Bbbk)$. Let $S$ denote the $\Bbbk$-algebra on $z_1, z_2, \dots, z_n$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$ for all $i, j = 1, \dots, n$ so that $S$ is a skew polynomial ring. Then the image of $q = z^T M z$, where $z = [z_1\, z_2\, \cdots\, z_n]^T$, in $S$ is a quadratic form. Furthermore, if $q = \sum_{i \le j} a_{ij} z_i z_j \in S_2$, one may associate to $q$ an $M \in \mathrm{Sym}_n^\mu(\Bbbk)$ defined by $M_{ii} = a_{ii}$ for all $i$ and, for $i < j$, $M_{ij} = a_{ij}/2$.

**Definition 1.2** [Cassidy and Vancliff 2010]. Let $\mu \in \mathbb{M}_n(\Bbbk)$ be as in Definition 1.1 and $M_1, \dots, M_n \in \mathrm{Sym}_n^\mu(\Bbbk)$. A graded skew Clifford algebra $A = A(\mu, M_1, \dots, M_n)$ associated to $\mu$ and $M_1, \dots, M_n$ is a graded $\Bbbk$-algebra on degree-1 generators $x_1, \dots, x_n$ and on degree-2 generators $y_1, \dots, y_n$ with defining relations given by

- $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^n (M_k)_{ij} y_k$ for all $i, j = 1, \dots, n$; and,
- any additional (degree-3 or degree-4) relations necessary in order to guarantee the existence of a normalizing sequence that spans $\bigoplus_{k=1}^n \Bbbk y_k$.

Note that if a graded skew Clifford algebra is regular, then it will only have relations of the form $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^n (M_k)_{ij} y_k$ for all $i, j = 1, \dots, n$.

## 2. Main results

We first examine the case where the degree-1 generators of each algebra are identified and the degree-2 generators are related via a mapping by matrix multiplication. Our algebras will be $\Bbbk$-algebras whose defining relations are of the form $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^n (M_k)_{ij} y_k$. Each such algebra will map onto a graded skew Clifford algebra since the defining relations of a graded skew Clifford algebra will include these relations and (possibly) additional relations necessary to guarantee the existence of a normalizing sequence, as required by Definition 1.2.

**Theorem 2.1.** *Let $\mu \in \mathbb{M}_n(\Bbbk)$ be as in Definition 1.1 and let*

$$M_1, M_2, \dots, M_n, N_1, N_2, \dots, N_n \in \mathrm{Sym}_n^\mu(\Bbbk).$$

*Let $A$ be a $\Bbbk$-algebra on degree-1 generators $x_1, x_2, \dots, x_n$ and degree-2 generators $y_1, y_2, \dots, y_n$ with defining relations*

$$x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k \quad \textit{for all } i, j.$$

*Similarly, let $B$ be a $\Bbbk$-algebra on degree-1 generators $X_1, X_2, \dots, X_n$ and degree-2 generators $Y_1, Y_2, \dots, Y_n$ with defining relations*

$$X_i X_j + \mu_{ij} X_j X_i = \sum_{k=1}^{n} (N_k)_{ij} Y_k \quad \textit{for all } i, j.$$

*Given a nonsingular matrix $P \in \mathbb{M}_n(\Bbbk)$, $A$ is isomorphic to $B$ under a map $\varphi : A \to B$ defined by*

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \overset{\varphi}{\longmapsto} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \overset{\varphi}{\longmapsto} P \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$

*if and only if $N = P^T M$, where $N = [N_1\ N_2\ \cdots\ N_n]^T$, $M = [M_1\ M_2\ \cdots\ M_n]^T \in (\mathrm{Sym}_n^{\mu}(\Bbbk))^n$.*

*Proof.* Let $\mathfrak{B}$ denote the standard $\Bbbk$-basis of $\mathrm{Sym}_n^{\mu}(\Bbbk)$. We may write the defining relations of $A$ as the matrix equation

$$\begin{bmatrix} 2x_1^2 \\ x_1x_2+\mu_{12}x_2x_1 \\ \vdots \\ 2x_n^2 \end{bmatrix} = \begin{bmatrix} (M_1)_{11} & (M_2)_{11} & \cdots & (M_n)_{11} \\ (M_1)_{12} & (M_2)_{12} & \cdots & (M_n)_{12} \\ \vdots & \vdots & \ddots & \vdots \\ (M_1)_{nn} & (M_2)_{nn} & \cdots & (M_n)_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Applying the map $\varphi$ to the above equation we obtain the defining relations of $B$:

$$\begin{bmatrix} 2X_1^2 \\ X_1X_2+\mu_{12}X_2X_1 \\ \vdots \\ 2X_n^2 \end{bmatrix} = \begin{bmatrix} (M_1)_{11} & (M_2)_{11} & \cdots & (M_n)_{11} \\ (M_1)_{12} & (M_2)_{12} & \cdots & (M_n)_{12} \\ \vdots & \vdots & \ddots & \vdots \\ (M_1)_{nn} & (M_2)_{nn} & \cdots & (M_n)_{nn} \end{bmatrix} P \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}.$$

Hence

$$\begin{bmatrix} (M_1)_{11} & (M_2)_{11} & \cdots & (M_n)_{11} \\ (M_1)_{12} & (M_2)_{12} & \cdots & (M_n)_{12} \\ \vdots & \vdots & \ddots & \vdots \\ (M_1)_{nn} & (M_2)_{nn} & \cdots & (M_n)_{nn} \end{bmatrix} P = \begin{bmatrix} (N_1)_{11} & (N_2)_{11} & \cdots & (N_n)_{11} \\ (N_1)_{12} & (N_2)_{12} & \cdots & (N_n)_{12} \\ \vdots & \vdots & \ddots & \vdots \\ (N_1)_{nn} & (N_2)_{nn} & \cdots & (N_n)_{nn} \end{bmatrix}.$$

The $k$-th columns of the matrices in the previous equation are the $\mathfrak{B}$-coordinate vector of $M_k$ and $N_k$, respectively. Thus,

$$\begin{bmatrix} N_1 & \cdots & N_n \end{bmatrix} = \begin{bmatrix} M_1 & \cdots & M_n \end{bmatrix} P \quad \Longrightarrow \quad \begin{bmatrix} N_1 \\ \vdots \\ N_n \end{bmatrix} = P^T \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix}$$
$$\Longrightarrow \quad N = P^T M. \qquad \square$$

**Example 2.2.** Let

$$\mu = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 2 & -i \\ -1 & 2 \end{bmatrix}, \quad N_2 = \begin{bmatrix} i & i \\ 1 & i \end{bmatrix}.$$

Then $A = A(\mu, M_1, M_2)$ is given by

$$A = \frac{\Bbbk\langle x_1, x_2, y_1, y_2 \rangle}{\langle 2x_1^2 - y_1, 2x_2^2 - y_1, x_1x_2 + ix_2x_1 - iy_2 \rangle}$$

and $B = B(\mu, N_1, N_2)$ is given by

$$B = \frac{\Bbbk\langle X_1, X_2, Y_1, Y_2 \rangle}{\langle 2X_1^2 - 2Y_1 - iY_2, 2X_2^2 - 2Y_1 - iY_2, X_1X_2 + iX_2X_1 + iY_1 - iY_2 \rangle}.$$

One can compute that

$$\begin{cases} N_1 = 2M_1 - M_2, \\ N_2 = iM_1 + M_2 \end{cases} \qquad \Longrightarrow \quad \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ i & 1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}.$$

Hence,

$$P = \begin{bmatrix} 2 & i \\ -1 & 1 \end{bmatrix}.$$

The above theorem implies that $A \cong B$ under the map $x_i \mapsto X_i$ for all $i$, $y_1 \mapsto 2Y_1 + iY_2$ and $y_2 \mapsto -Y_1 + Y_2$, which is easily verified.

**Corollary 2.3.** *Let $\mu$, $A$, $B$, $M$, $N$, $\varphi$ and $P$ be as in Theorem 2.1. If $q_k, p_k \in S$ denote the quadratic forms associated to $M_k$ and $N_k$, respectively, then $A$ is isomorphic to $B$ under $\varphi$ if and only if*

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = P^T \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}.$$

*Proof.* The result follows immediately from multiplying each entry of the vectors $N$ and $M$ in the equation $N = P^T M$ by $z^T$ and $z$ on the left and right, respectively. $\square$

**Example 2.4.** Continuing Example 2.2, the quadratic forms associated to $M_1$, $M_2$, $N_1$ and $N_2$ are $q_1 = z_1^2 + z_2^2$, $q_2 = 2iz_1z_2$, $p_1 = 2z_1^2 - 2iz_1z_2 + 2z_2^2$ and $p_2 = iz_1^2 + 2iz_1z_2 + iz_2^2$, respectively. It is easily verified that

$$\begin{cases} p_1 = 2q_1 - q_2, \\ p_2 = iq_1 + q_2 \end{cases} \quad \Longrightarrow \quad \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 2 & -i \\ i & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

We now examine the case where the degree-2 generators of each algebra are identified and the degree-1 generators of the algebra are related via a diagonal mapping.

**Theorem 2.5.** *Let $\mu \in \mathbb{M}_n(\Bbbk)$ be as in Definition 1.1 and let*

$$M_1, M_2, \dots, M_n, N_1, N_2, \dots, N_n \in \mathrm{Sym}_n^{\mu}(\Bbbk).$$

*Let $A$ be a $\Bbbk$-algebra on degree-1 generators $x_1, x_2, \dots, x_n$ and degree-2 generators $y_1, y_2, \dots, y_n$ with defining relations*

$$x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k \quad \textit{for all } i, j.$$

*Similarly, let $B$ be a $\Bbbk$-algebra on degree-1 generators $X_1, X_2, \dots, X_n$ and degree-2 generators $Y_1, Y_2, \dots, Y_n$ with defining relations*

$$X_i X_j + \mu_{ij} X_j X_i = \sum_{k=1}^{n} (N_k)_{ij} Y_k \quad \textit{for all } i, j.$$

*Then $\varphi : A \to B$ is an isomorphism defined $x_i \overset{\varphi}{\longmapsto} \alpha_i X_i$ and $y_i \overset{\varphi}{\longmapsto} Y_i$ for $\alpha_i \in \Bbbk^{\times}$ and for all $i$ if and only if $(N_k)_{ij} = (M_k)_{ij}/(\alpha_i \alpha_j)$ for all $i, j, k$.*

*Proof.* Applying the map $\varphi$ to the defining relations of $A$ yields the defining relations of $B$:

$$\alpha_i \alpha_j X_i X_j + \mu_{ij} \alpha_i \alpha_j X_j X_i = \sum_{k=1}^{n} (M_k)_{ij} Y_k \quad \Longrightarrow \quad X_i X_j + \mu_{ij} X_j X_i = \sum_{k=1}^{n} \frac{(M_k)_{ij}}{\alpha_i \alpha_j} Y_k.$$

Hence, $(N_k)_{ij} = (M_k)_{ij}/(\alpha_i \alpha_j)$. □

**Example 2.6.** Let

$$\mu = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & -1 \\ i & 0 \end{bmatrix}.$$

Then $A = A(\mu, M_1, M_2)$ is given by

$$A = \frac{\Bbbk\langle x_1, x_2, y_1, y_2 \rangle}{\langle 2x_1^2 - y_1, 2x_2^2 - y_1, x_1x_2 + ix_2x_1 - iy_2 \rangle}$$

and $B = B(\mu, N_1, N_2)$ is given by

$$B = \frac{\Bbbk\langle X_1, X_2, Y_1, Y_2\rangle}{\langle 2X_1^2 - Y_1, 2X_2^2 + Y_1, X_1X_2 + iX_2X_1 + Y_2\rangle}.$$

In checking to see if $(N_k)_{ij} = (M_k)_{ij}/(\alpha_i\alpha_j)$, we arrive at the equations

$$\begin{cases} 1 = 1/\alpha_1^2, \\ -1 = 1/\alpha_2^2, \\ -1 = i/(\alpha_1\alpha_2), \\ i = 1/(\alpha_1\alpha_2) \end{cases} \quad\Longrightarrow\quad \begin{cases} \alpha_1 = \pm 1, \\ \alpha_2 = \mp i. \end{cases}$$

The above theorem implies that $A \cong B$ under both the maps $\varphi_+$ and $\varphi_-$ defined by $\varphi_\pm : x_1 \mapsto \pm X_1,\ x_2 \mapsto \mp i X_2$ and $y_k \mapsto Y_k$, for all $k$.

**Corollary 2.7.** *Let $\mu$, $A$, $B$, $M_1, M_2, \ldots, M_n$, $N_1, N_2, \ldots, N_n$, and $\varphi$ be as in Theorem 2.5. If $q_k = \sum_{i \le j} a_{k_{ij}} z_i z_j \in S$ and $p_k = \sum_{i \le j} b_{k_{ij}} z_i z_j \in S$ denote the quadratic forms associated to $M_k$ and $N_k$, respectively, then $A$ is isomorphic to $B$ under $\varphi$ if and only if $b_{k_{ij}} = a_{k_{ij}}/(\alpha_i\alpha_j)$.*

*Proof.* The result follows immediately from $(N_k)_{ij} = (M_k)_{ij}/(\alpha_i\alpha_j)$ and the definition of the quadratic form in [Cassidy and Vancliff 2010]. □

**Example 2.8.** Continuing 2.6, the quadratic forms associated to $M_1$, $M_2$, $N_1$ and $N_2$ are $q_1 = z_1^2 + z_2^2$, $q_2 = 2iz_1z_2$, $p_1 = z_1^2 - z_2^2$ and $p_2 = -2z_1z_2$, respectively. It is easily verified that $b_{k_{ij}} = a_{k_{ij}}/(\alpha_i\alpha_j)$ holds for these forms.

The reader should note that the more general case in which the degree-1 generators of an algebra are related via a mapping by matrix multiplication is much more complicated. Such maps can transform graded skew Clifford algebras into isomorphic algebras whose defining relations cannot be directly written in the form $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^n (M_k)_{ij} y_k$ without changing the presentation of the algebra, as illustrated in the following example.

**Example 2.9.** Let

$$\mu = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}.$$

Then $A = A(\mu, M_1, M_2)$ is given by

$$A = \frac{\Bbbk\langle x_1, x_2, y_1, y_2\rangle}{\langle 2x_1^2 - y_1, 2x_2^2 - y_1, x_1x_2 + ix_2x_1 - iy_2\rangle}.$$

Consider the isomorphism $\varphi : A \to \varphi(A)$ defined by $x_1 \mapsto X_1 - X_2,\ x_2 \mapsto X_2$, and $y_k \mapsto Y_k$ for all $k$. Applying $\varphi$ to the defining relations of $A$ yields the relations

$$\begin{aligned} 2X_1^2 + 2X_2^2 - 2X_1X_2 - 2X_2X_1 - Y_1 = 0,\\ 2X_2^2 - Y_1 = 0,\\ X_1X_2 + iX_2X_1 - (1+i)X_2^2 - iY_2 = 0. \end{aligned}$$

These relations cannot all be written in the form $X_iX_j + \mu_{ij}X_jX_i = \sum_{k=1}^{n}(N_k)_{ij}Y_k$ without a change of presentation; hence, the image of the algebra does not present as a graded skew Clifford algebra.

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