Appendix: Proof of Theorem 4.7 and Theorem 4.20

Supplement to "Strongly nonzero points and elliptic pseudoprimes"

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Theorem. *Suppose $s_1 \le r_1$ and $s_2 \le r_2$ and $r_1 \ge 1$. Then $h(s_1, r_1) \cdot h(s_2, r_2)$ has the largest value when $r_1 = r_2 = s_1 = s_2 = 1$, where $h(s, r)$ is as in Definition 4.6.*

First we will show that if any of $r_1, r_2, s_1, s_2 > 1$, then decreasing some of the values (r_1, r_2, s_1, s_2) will result in increasing the value of $h(r_1, s_1) \cdot h(r_2, s_2)$. Thus the maximum value of $h(s_1, r_1) \cdot h(s_2, r_2)$ is achieved when $s_1 = r_1 = s_2 = r_2 = 1$. In the rest of this section we will use the notation $u_i v_i$ to denote the product of the i^{th} coordinates of two vectors u and v. The next several lemmas will be used in the proof of Theorem 4.7.

Lemma A.1. *Suppose $r_1 > r_2$ and $r_1 > s_1$. Then $h(s_1, r_1) \cdot h(s_2, r_2) < h(s_1, r_1 - 1) \cdot h(s_2, r_2)$.*

Proof. Let $r_1 > r_2$ and $r_1 > s_1$. We prove the lemma by doing some simple computations

$$\begin{aligned} h(s_1, r_1) \cdot h(s_2, r_2) &= \frac{1}{2} h(s_1, r_1 - 1) \cdot h(s_2, r_2) + h(s_1, r_1)_{r_1} h(s_2, r_2)_{r_1} \\ &= \frac{1}{2} h(s_1, r_1 - 1) \cdot h(s_2, r_2). \end{aligned}$$

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Lemma A.2. *Suppose $r_1 > r_2$ and $r_1 = s_1$. Then $h(s_1, r_1) \cdot h(s_2, r_2) < h(s_1 - 1, r_1 - 1) \cdot h(s_2, r_2)$.*

Proof. Let $r_1 > r_2$ and $r_1 = s_1$. We prove the lemma by doing some simple computations

$$\begin{aligned} h(s_1, r_1) \cdot h(s_2, r_2) &= \frac{1}{4} h(s_1 - 1, r_1 - 1) \cdot h(s_2, r_2) + h(s_1, r_1)_{r_1} h(s_2, r_2)_{r_1} \\ &= \frac{1}{4} h(s_1 - 1, r_1 - 1) \cdot h(s_2, r_2). \end{aligned}$$

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Remark A.3. *The following equation will be used to prove that the proportion of viable points is increasing as the values of r are decreasing. Suppose $r \ge s_2 \ge s_1$. Then*

$$\begin{aligned} h(s_1, r) \cdot h(s_2, r) &= \frac{1}{2^{2r+s_1+s_2}} \left(1 + \sum_{i=1}^{s_1} 9\left(2^{4i-4}\right) + \sum_{i=s_1+1}^{s_2} 3\left(2^{3i-3+s_1}\right) + \sum_{i=s_2+1}^{r} 2^{2i-2+s_1+s_2} \right) \\ &= \frac{1}{2^{2r+s_1+s_2}} \left(\frac{2}{5} + \frac{6}{35}\left(2^{4s_1}\right) + \frac{2}{21}\left(2^{s_1+3s_2}\right) + \frac{1}{3}\left(2^{2r+s_1+s_2}\right) \right) \end{aligned}$$

Lemma A.4. *Suppose $r_1 = r_2 = s_1 = s_2$. Then $h(s_1, r_1) \cdot h(s_2, r_2) < h(s_1 - 1, r_1 - 1) \cdot h(s_2 - 1, r_2 - 1)$.*

Proof. Let $r_1 = r_2 = s_1 = s_2$. Then simple computations show that

$$15h(s_1, r_1) \cdot h(s_2, r_2) = \frac{6}{2^{4r_1}} + 9 < \frac{6}{2^{4(r_1-1)}} + 9 = 15h(s_1 - 1, r_1 - 1) \cdot h(s_2 - 1, r_2 - 1).$$

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Lemma A.5. *Suppose $r_1 = r_2 = s_2 \ge s_1$. Then $h(s_1, r_1) \cdot h(s_2, r_2) < h(s_1, r_1 - 1) \cdot h(s_2 - 1, r_2 - 1)$.*

Proof. Let $r_1 = r_2 = s_2 \geq s_1$. Then simple computations show that

$$7h\left(s_1, r_1\right) \cdot h\left(s_2, r_2\right) = \frac{1}{2^{2r_1+s_1+s_2}} \left(\frac{14}{5} + \frac{6}{5}\left(2^{4s_1}\right) + 3\left(2^{3r_1+s_1}\right)\right)$$
$$> \frac{1}{2^{2r_1+s_1+s_2}} \left(3\left(2^{3r_1+s_1}\right)\right) = 8h\left(s_1, r_1\right)_{r_1} h\left(s_2, r_2\right)_{r_1}.$$

This implies that

$$h\left(s_1, r_1\right) < 8h\left(s_1, r_1\right) \cdot h\left(s_2, r_2\right) - 8h\left(s_1, r_1\right)_{r_1} h\left(s_2, r_2\right)_{r_1} = h\left(s_1, r_1 - 1\right) \cdot h\left(s_2 - 1, r_2 - 1\right).$$

□

Lemma A.6. *Suppose $r_1 = r_2 > s_2 \geq s_1$. Then $h\left(s_1, r_1\right) \cdot h\left(s_2, r_2\right) < h\left(s_1, r_1 - 1\right) \cdot h\left(s_2, r_2 - 1\right)$.*

Proof. Let $r_1 = r_2 > s_2 \geq s_1$. Then simple computations show that

$$3h\left(s_1, r_1\right) \cdot h\left(s_2, r_2\right) = \frac{1}{2^{2r_1+s_1+s_2}} \left(\frac{6}{5} + \frac{18}{35}\left(2^{4s_1}\right) + \frac{2}{7}\left(2^{3s_2+s_1}\right) + \left(2^{2r_1+s_2+s_1}\right)\right)$$
$$> \frac{1}{2^{2r_1+s_1+s_2}} \left(2^{2r_1+s_2+s_1}\right) = 4h\left(s_1, r_1\right)_{r_1} h\left(s_2, r_2\right)_{r_1}.$$

This implies that

$$h\left(s_1, r_1\right) \cdot h\left(s_2, r_2\right) < 4h\left(s_1, r_1\right) \cdot h\left(s_2, r_2\right) - 4h\left(s_1, r_1\right)_{r_1} h\left(s_2, r_2\right)_{r_1} = h\left(s_1, r_1 - 1\right) \cdot h\left(s_2, r_2 - 1\right).$$

□

Proof of Theorem 4.7. Let S denote the set $S = \{h(1,1) \cdot h(1,1), h(0,1) \cdot h(1,1), h(0,1) \cdot h(0,1)\}$. A simple computation using Remark A.3 yields that $h(1,1) \cdot h(1,1)$ is the $\sup(S)$ and $h(1,1) \cdot h(1,1) = 5/8$. Without loss of generality we assume that $r_1 \geq r_2$.
Case 1: $r_2 = 0$ and $s_1 < r_1$. In this case Lemma A.1 implies that

$$h(s_1, r_1) \cdot h(s_2, 0) = \frac{1}{2} h(s_1, r_1 - 1) \cdot h(s_2, 0) \leq 1/2 < 5/8.$$

Case 2: $r_2 = 0$ and $s_1 = r_1$. In this case Lemma A.2 implies that

$$h(s_1, r_1) \cdot h(s_2, 0) = \frac{1}{4} h(s_1 - 1, r_1 - 1) \cdot h(s_2, 0) \leq 1/4 < 5/8.$$

Case 3: $r_2 > 0$ and $s_1 \leq r_2$. Using Lemma A.1 we have

$$h(s_1, r_1) \cdot h(s_2, r_2) \leq h(s_1, r_2) \cdot h(s_2, r_2).$$

Then applying Lemmas A.4, A.5 and A.6 we get

$$h(s_1, r_2) \cdot h(s_2, r_2) \leq \sup S.$$

Case 4: $r_2 > 0$ and $r_2 \leq s_1$.
From Lemma A.1 we have that

$$h(s_1, r_1) \cdot h(s_2, r_2) \leq h(s_1, s_1) \cdot h(s_2, r_2)$$

and from Lemma A.2 we have that

$$h(s_1, s_1) \cdot h(s_2, r_2) \leq h(r_2, r_2) \cdot h(s_2, r_2).$$

Then Lemmas A.4, A.5 and A.6 show that

$$h(r_2, r_2) \cdot h(s_2, r_2) \leq \sup S.$$

□

Proof of Theorem 4.20 In the rest of this section we give the proof of the following theorem.

Theorem. *Suppose* $s_1 \le r_1$ *and* $s_2 \le r_2$ *and* $r_1 \ge 1$ *and* $2^{s_2} \cdot 2^{r_2} \cdot t_2 \cdot w_2 > 1$. *Then* $h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2) \le h'(1, 1, t_1, w_1) \cdot h'(1, 1, t_2, w_2)$, *where* $h'(s, r, t, w)$ *is as in Definition 4.19.*

The following lemmas will be used to prove Theorem 4.20.

Lemma A.7. *Suppose* $r_1 > r_2$ *and* $r_1 > s_1$. *Then* $h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2) < h'(s_1, r_1 - 1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2)$.

Proof. Let $r_1 > r_2$ and $r_1 > s_1$. Then simple computations show that

$$\begin{aligned}
&h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2)\\
&= \frac{(2^{r_1+s_1-1})t_1w_1 - 1}{(2^{r_1+s_1})t_1w_1 - 1} h'(s_1, r_1 - 1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2) + h'(s_1, r_1, t_1, w_1)_{r_1} h'(s_2, r_2, t_2, w_2)_{r_1}\\
&= \frac{(2^{r_1+s_1-1})t_1w_1 - 1}{(2^{r_1+s_1})t_1w_1 - 1} h'(s_1, r_1 - 1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2)\\
&< h'(s_1, r_1 - 1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2).
\end{aligned}$$

□

Lemma A.8. *Suppose* $r_1 > r_2$ *and* $r_1 = s_1$. *Then* $h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2) < h'(s_1 - 1, r_1 - 1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2)$.

Proof. Let $r_1 > r_2$ and $r_1 = s_1$. Then simple computations show that

$$\begin{aligned}
&h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2)\\
&= \frac{(2^{r_1+s_1-2})t_1w_1 - 1}{(2^{r_1+s_1})t_1w_1 - 1} h'(s_1 - 1, r_1 - 1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2) + h'(s_1, r_1, t_1, w_1)_{r_1} h'(s_2, r_2, t_2, w_2)_{r_1}\\
&= \frac{(2^{r_1+s_1-2})t_1w_1 - 1}{(2^{r_1+s_1})t_1w_1 - 1} h'(s_1 - 1, r_1 - 1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2)\\
&< h'(s_1 - 1, r_1 - 1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2).
\end{aligned}$$

□

Remark A.9. *The following equation will be used in the proof of Theorem 4.20. Suppose* $r \ge s_2 \ge s_1$. *Then*

$$\begin{aligned}
&h'(s_1, r, t_1, w_1) \cdot h'(s_2, r, t_2, w_2) =\\
&= \frac{1}{(2^{r+s_1}t_1w_1 - 1)(2^{r+s_2}t_2w_2 - 1)} \Bigg((t_1w_1 - 1)(t_2w_2 - 1) + \sum_{i=1}^{s_1} 9(2^{4i-4})t_1w_1t_2w_2\\
&+ \sum_{i=s_1+1}^{s_2} 3(2^{3i-3+s_1})t_1w_1t_2w_2 + \sum_{i=s_2+1}^{r} (2^{2i-2+s_1+s_2})t_1w_1t_2w_2 \Bigg)\\
&= \frac{1}{(2^{r+s_1}t_1w_1 - 1)(2^{r+s_2}t_2w_2 - 1)} \Bigg((t_1w_1 - 1)(t_2w_2 - 1) - \frac{3}{5}(t_1w_1t_2w_2) + \frac{6}{35}(2^{4s_1}t_1w_1t_2w_2)\\
&+ \frac{2}{21}(2^{s_1+3s_2}t_1w_1t_2w_2) + \frac{1}{3}(2^{2r+s_1+s_2}t_1w_1t_2w_2) \Bigg).
\end{aligned}$$

Note that if $r = s_i = 0$ *and* $t_i = w_i = 1$ *for* $i = 1$ *or* $i = 2$, *one of the groups is the trivial group, which has no strongly non-zero points, so the proportion of viable points is vacuously equal to 1.*

Lemma A.10. *Suppose* $r_1 = r_2 = s_1 = s_2$ *and* $t_1, w_1, t_2, w_2 > 0$ *are all odd integers. Then*

$$h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2) < h'(s_1 - 1, r_1 - 1, t_1, w_1) \cdot h'(s_2 - 1, r_2 - 1, t_2, w_2).$$

Proof. We will prove the lemma by considering several cases.
Case 1: ($t_1w_1 = 1$ and $r_1 > 1$) or ($t_2w_2 = 1$ and $r_2 > 1$). We may assume that $t_1w_1 = 1$ and let $r = r_2 = r_1 > 1$.

$$\frac{2^{2r}t_2w_2 - 1}{2^{2r-2}t_2w_2 - 1} > 4 > \frac{2^{2r@} + 1}{2^{2r-2} + 1}$$
$$\frac{2^{2r-2} + 1}{2^{2r-2}t_2w_2 - 1} > \frac{2^{2r} + 1}{2^{2r}t_2w_2 - 1}$$
$$\frac{\frac{3}{5}(2^{2r-2} - 1)(2^{2r-2} + 1)}{(2^{2r-2} - 1)(2^{2r-2}t_2w_2 - 1)} > \frac{\frac{3}{5}(2^{2r} - 1)(2^{2r} + 1)}{(2^{2r} - 1)(2^{2r}t_2w_2 - 1).}$$

Therefore,

$$\begin{aligned} h'(s_1 - 1, r_1 - 1, t_1, w_1) \cdot h(s_2 - 1, r_2 - 1, t_2, w_2) &= \frac{\frac{3}{5}t_2w_2(2^{4r-4} - 1)}{(2^{2r-2} - 1)(2^{2r-2}t_2w_2 - 1)} \\ &> \frac{\frac{3}{5}t_2w_2(2^{4r} - 1)}{(2^{2r} - 1)(2^{2r}t_2w_2 - 1)} \\ &= h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2). \end{aligned}$$

Case 2: $6t_1w_1t_2w_2 - 15t_1w_1 - 15t_2w_2 + 15 > 0$. In this case we have the following

$$15t_1w_1t_2w_2 - 15t_1w_1 - 15t_2w_2 + 15 - 9t_1w_1t_2w_2 > 0$$
$$15((t_1w_1 - 1)(t_2w_2 - 1) - \frac{3}{5}t_1w_1t_2w_2) > 0$$
$$16((t_1w_1 - 1)(t_2w_2 - 1) - \frac{3}{5}t_1w_1t_2w_2) > (t_1w_1 - 1)(t_2w_2 - 1) - \frac{3}{5}t_1w_1t_2w_2$$
$$16((t_1w_1 - 1)(t_2w_2 - 1) - \frac{3}{5}t_1w_1t_2w_2 + \frac{3}{5}t_1w_12^{4r-4}) > (t_1w_1 - 1)(t_2w_2 - 1) - \frac{3}{5}t_1w_1t_2w_2 + \frac{3}{5}t_1w_12^{4r}$$

From Remark A.9, we see that

$$16(2^{2r-2}t_1w_1 - 1)(2^{2r-2}t_2w_2 - 1)h'(s_1 - 1, r_1 - 1, t_1, w_1) \cdot h'(s_2 - 1, r_2 - 1, t_2, w_2) \tag{1}$$
$$> (2^{2r}t_1w_1 - 1)(2^{2r}t_2w_2 - 1)h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2). \tag{2}$$

Note that

$$\frac{(2^{2r}t_1w_1 - 1)(2^{2r}t_2w_2 - 1)}{(2^{2r-2}t_1w_1 - 1)(2^{2r-2}t_2w_2 - 1)} > 16$$

Using this inequality in the Equation 1, we have the following

$$\begin{aligned} &(2^{2r}t_1w_1 - 1)(2^{2r}t_2w_2 - 1)h'(s_1 - 1, r_1 - 1, t_1, w_1) \cdot h'(s_2 - 1, r_2 - 1, t_2, w_2) \\ &> (2^{2r}t_1w_1 - 1)(2^{2r}t_2w_2 - 1)h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2) \\ &h'(s_1 - 1, r_1 - 1, t_1, w_1) \cdot h'(s_2 - 1, r_2 - 1, t_2, w_2) \\ &> h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2). \end{aligned}$$

The inequality $6t_1w_1t_2w_2 - 15t_1w_1 - 15t_2w_2 + 15 > 0$ is true if one of the following conditions hold:

- $t_1w_1 \geq 5$ and $t_2w_2 \geq 5$
- $t_1w_1 = 3$ and $t_2w_2 > 10$
- $t_1w_1 > 10$ and $t_2w_2 = 3$.

Case 3: $t_1w_1 = 3$ and $3 \geq t_2w_2 \geq 9$. By inspection.

Case 4: $3 \geq t_1w_1 \geq 9$ and $t_2w_2 = 3$. By inspection. $\square$

Lemma A.11. *Suppose $r_1 = r_2 = s_2 > s_1$. Then $h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2) < h'(s_1, r_1 - 1, t_1, w_1) \cdot h'(s_2 - 1, r_2 - 1, t_2, w_2)$.*

Proof. We prove the lemma by considering several cases.
Case 1: $s_1 = 0$, $t_1w_1 = 1$, and $r_1 > 1$.

$$2(2^{2r-2} + 2^{r-1} + 1)(2^r + 1) > (2^{2r} + 2^r + 1)(2^{r-1} + 1)$$
$$\frac{2^r t_2w_2 - 1}{2^{r-1}t_2w_2 - 1} > 2$$
$$\frac{2^r t_2w_2 - 1}{2^{r-1}t_2w_2 - 1}(2^{2r-2} + 2^{r-1} + 1)(2^r + 1) > (2^{2r} + 2^r + 1)(2^{r-1} + 1)$$
$$\frac{(2^{2r-2} + 2^{r-1} + 1)}{(2^{r-1} + 1)(2^{r-1}t_2w_2 - 1)} > \frac{(2^{2r} + 2^r + 1)}{(2^r + 1)(2^r t_2w_2 - 1)}$$
$$\frac{(2^{r-1} - 1)(2^{2r-2} + 2^{r-1} + 1)}{(2^{r-1} - 1)(2^{r-1} + 1)(2^{r-1}t_2w_2 - 1)} > \frac{(2^r - 1)(2^{2r} + 2^r + 1)}{(2^r - 1)(2^r + 1)(2^r t_2w_2 - 1)}$$
$$\frac{(2^{3r-3} - 1)}{(2^{2r-2} - 1)(2^{r-1}t_2w_2 - 1)} > \frac{(2^{3r} - 1)}{(2^{2r} - 1)(2^r t_2w_2 - 1)}.$$

$$\frac{(\frac{3}{7}t_1w_1t_2w_2(2^{3r-3} - 1))}{(2^{2r-2} - 1)(2^{r-1}t_2w_2 - 1)} > \frac{(\frac{3}{7}t_1w_1t_2w_2(2^{3r} - 1))}{(2^{2r} - 1)(2^r t_2w_2 - 1)} \tag{3}$$

Note that

$$\frac{1}{(2^{2r-2} - 1)(2^{r-1}t_2w_2 - 1)} > \frac{1}{(2^{2r} - 1)(2^r t_2w_2 - 1)} \tag{4}$$
$$\frac{(\frac{2}{21}t_1w_1t_2w_2)}{(2^{2r-2} - 1)(2^{r-1}t_2w_2 - 1)} > \frac{(\frac{2}{21}t_1w_1t_2w_2)}{(2^{2r} - 1)(2^r t_2w_2 - 1)} \tag{5}$$

Using the inequalities (5) and (3) we get that

$$\frac{(\frac{2}{21}t_1w_1t_2w_2 + \frac{3}{7}t_1w_1t_2w_2(2^{3r-3} - 1))}{(2^{2r-2} - 1)(2^{r-1}t_2w_2 - 1)} > \frac{(\frac{2}{21}t_1w_1t_2w_2 + \frac{3}{7}t_1w_1t_2w_2(2^{3r} - 1))}{(2^{2r} - 1)(2^r t_2w_2 - 1)}$$
$$\frac{(-\frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2 + \frac{3}{7}t_1w_1t_2w_2(2^{3r-3}))}{(2^{2r-2} - 1)(2^{r-1}t_2w_2 - 1)} > \frac{(-\frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2 + \frac{3}{7}t_1w_1t_2w_2(2^{3r}))}{(2^{2r} - 1)(2^r t_2w_2 - 1)}$$
$$\frac{(-\frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2 2^{s_1} + \frac{3}{7}t_1w_1t_2w_2(2^{3r-3}))}{(2^{2r-2} - 1)(2^{r+s_1-1}t_2w_2 - 1)} > \frac{(-\frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2 2^{s_1} + \frac{3}{7}t_1w_1t_2w_2(2^{3r}))}{(2^{2r} - 1)(2^{r+s_1}t_2w_2 - 1)}$$
$$h'(s_1, r_1 - 1, 1, 1) \cdot h'(s_2 - 1, r_2 - 1, t_2, w_2) > h'(s_1, r_1, 1, 1) \cdot h'(s_2, r_2, t_2, w_2).$$

Case 2: $s_1 = 0$, $t_2w_2 = 1$, and $r_2 > 1$.

$$4(2^{2r-2} + 2^{r-1} + 1) > 2^{2r} + 2^r + 1$$

$$\frac{2^{2r}t_1w_1 - 1}{2^{2r-2}t_1w_1 - 1} > 4$$

$$\frac{2^{2r}t_1w_1 - 1}{2^{2r-2}t_1w_1 - 1}(2^{2r-2} + 2^{r-1} + 1) > 4(2^{2r-2} + 2^{r-1} + 1) > 2^{2r} + 2^r + 1$$

$$\frac{(2^{2r-2} + 2^{r-1} + 1)}{(2^{2r-2}t_1w_1 - 1)} > \frac{(2^{2r} + 2^r + 1)}{(2^{2r}t_1w_1 - 1)}$$

$$\frac{(2^{2r-2} + 2^{r-1} + 1)(2^{r-1} - 1)}{(2^{2r-2}t_1w_1 - 1)(2^{r-1} - 1)} > \frac{(2^{2r} + 2^r + 1)(2^r - 1)}{(2^{2r}t_1w_1 - 1)(2^r - 1)}$$

(6) $$\frac{\frac{3}{7}t_1w_1t_2w_2(2^{3r-3} - 1)}{(2^{2r-2}t_1w_1 - 1)(2^{r-1} - 1)} > \frac{\frac{3}{7}t_1w_1t_2w_2(2^{3r} - 1)}{(2^{2r}t_1w_1 - 1)(2^r - 1).}$$

Note that

(7) $$\frac{1}{(2^{2r-2}t_1w_1 - 1)(2^{r-1})} > \frac{1}{(2^{2r}t_1w_1 - 1)(2^r - 1)}$$

(8) $$\frac{(\frac{2}{21}t_1w_1t_2w_2)}{(2^{2r-2}t_1w_1 - 1)(2^{r-1} - 1)} > \frac{(\frac{2}{21}t_1w_1t_2w_2)}{(2^{2r}t_1w_1 - 1)(2^r - 1)}$$

Using the inequalities (8) and (6) we get that

$$\frac{(\frac{2}{21}t_1w_1t_2w_2 + \frac{3}{7}t_1w_1t_2w_2(2^{3r-3} - 1))}{(2^{2r-2}t_1w_1 - 1)(2^{r-1} - 1)} > \frac{(\frac{2}{21}t_1w_1t_2w_2 + \frac{3}{7}t_1w_1t_2w_2(2^{3r} - 1))}{(2^{2r}t_1w_1 - 1)(2^r - 1)}$$

$$\frac{(-\frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2 + \frac{3}{7}t_1w_1t_2w_2(2^{3r-3}))}{(2^{2r-2}t_1w_1 - 1)(2^{r-1} - 1)} > \frac{(-\frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2 + \frac{3}{7}t_1w_1t_2w_2(2^{3r}))}{(2^{2r}t_1w_1 - 1)(2^r - 1)}$$

$$\frac{(-\frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_22^{4s_1} + \frac{3}{7}t_1w_1t_2w_2(2^{3r-3}))}{(2^{2r-2}t_1w_1 - 1)(2^{r+s_1-1} - 1)} > \frac{(-\frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_22^{4s_1} + \frac{3}{7}t_1w_1t_2w_2(2^{3r}))}{(2^{2r}t_1w_1 - 1)(2^{r+s_1} - 1)}$$

$$h'(s_1, r_1 - 1, t_1, w_1) \cdot h'(s_2 - 1, r_2 - 1, 1, 1) > h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, 1, 1).$$

Case 3: ($t_1w_1 = 1$ or $t_2w_2 = 1$) and $s_1 \geq 1$.

(9) $$\frac{(2^{2r}t_1w_1 - 1)(2^{r+s_1}t_2w_2 - 1)}{(2^{2r-2}t_1w_1 - 1)(2^{r+s_1-1}t_2w_2 - 1)} > 8$$

(10) $$\frac{\frac{3}{7}t_1w_1t_2w_22^{3r+s_1-3}}{(2^{2r-2}t_1w_1 - 1)(2^{r+s_1-1}t_2w_2 - 1)} > \frac{\frac{3}{7}t_1w_1t_2w_22^{3r+s_1}}{(2^{2r}t_1w_1 - 1)(2^{r+s_1}t_2w_2 - 1)}$$

Note that

(11) $$-\frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_22^{4s_1} > 0$$

(12) $$\frac{1}{(2^{2r-2}t_1w_1 - 1)(2^{r+s_1-1}t_2w_2 - 1)} > \frac{1}{(2^{2r}t_1w_1 - 1)(2^{r+s_1}t_2w_2 - 1)}$$

(13) $$\frac{(\frac{-3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_22^{4s_1})}{(2^{2r-2}t_1w_1 - 1)(2^{r+s_1-1}t_2w_2 - 1)} > \frac{(\frac{-3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_22^{4s_1})}{(2^{2r}t_1w_1 - 1)(2^{r+s_1}t_2w_2 - 1)}$$

Taking the sum of (13) and (10) we get the following inequality

$$\frac{(\frac{-3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2 2^{4s_1} + \frac{3}{7}t_1w_1t_2w_2 2^{3r+s_1-3})}{(2^{2r-2}t_1w_1 - 1)(2^{r+s_1-1}t_2w_2 - 1)} >$$
$$\frac{(\frac{-3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2 2^{4s_1} + \frac{3}{7}t_1w_1t_2w_2 2^{3r+s_1})}{(2^{2r}t_1w_1 - 1)(2^{r+s_1}t_2w_2 - 1)}$$
$$h'(s_1, r_1 - 1, t_1, w_1) \cdot h'(s_2 - 1, r_2 - 1, t_2, w_2) > h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2).$$

Case 4: $t_1w_1 \geq 3$ and $t_2w_2 \geq 3$.

$$\begin{aligned} 0 &< 7t_1w_1t_2w_2 - 7t_1w_1 - 7t_2w_2 + 7 - 3t_1w_1t_2w_2 \\ &= 7\left((t_1w_1 - 1)(t_2w_2 - 1) - \frac{3}{7}t_1w_1t_2w_2\right) \\ &= 7\left((t_1w_1 - 1)(t_2w_2 - 1) - \frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2\right) \end{aligned}$$

$$\begin{aligned} &8((t_1w_1 - 1)(t_2w_2 - 1) - \frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2 2^{4s_1}) \\ &\quad > (t_1w_1 - 1)(t_2w_2 - 1) - \frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2 2^{4s_1} \\ &8((t_1w_1 - 1)(t_2w_2 - 1) - \frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2 2^{4s_1} + \frac{3}{7}t_1w_1t_2w_2 2^{3r+s_1-3}) \\ &\quad > (t_1w_1 - 1)(t_2w_2 - 1) - \frac{3}{5}t_1w_1t_2w_2 + \frac{6}{35}t_1w_1t_2w_2 2^{4s_1} + \frac{3}{7}t_1w_1t_2w_2 2^{3r+s_1}. \end{aligned}$$

From Remark A.9, we see that

$$8(2^{2r-2}t_1w_1 - 1)(2^{r+s_1-1}t_2w_2 - 1)h'(s_1, r_1 - 1, t_1, w_1) \cdot h'(s_2 - 1, r_2 - 1, t_2, w_2) \tag{14}$$
$$> (2^{2r}t_1w_1 - 1)(2^{r+s_1}t_2w_2 - 1)h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2). \tag{15}$$

Note that

$$\frac{(2^{2r}t_1w_1 - 1)(2^{r+s_1}t_2w_2 - 1)}{(2^{2r-2}t_1w_1 - 1)(2^{r+s_1-1}t_2w_2 - 1)} > 8.$$

Using this inequality in (14), we get the following

$$\begin{aligned} &(2^{2r}t_1w_1 - 1)(2^{r+s_1}t_2w_2 - 1)h'(s_1, r_1 - 1, t_1, w_1) \cdot h'(s_2 - 1, r_2 - 1, t_2, w_2) \\ &\ > (2^{2r}t_1w_1 - 1)(2^{r+s_1}t_2w_2 - 1)h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2) \\ &h'(s_1, r_1 - 1, t_1, w_1) \cdot h'(s_2 - 1, r_2 - 1, t_2, w_2) \\ &\ > h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2). \end{aligned}$$

□

Lemma A.12. *Suppose* $r_1 = r_2 > s_2 \geq s_1$. *Then* $h'(s_1, r_1, t_1, w_1) \cdot h'(s_2, r_2, t_2, w_2) < h'(s_1, r_1 - 1, t_1, w_1) \cdot h'(s_2, r_2 - 1, t_2, w_2)$.

Proof. We prove the lemma by considering several cases.
Case 1: $s_1 = s_2 = 0$ and ($t_1w_1 = 1$ and $r_1 > 1$) or ($t_2w_2 = 1$ and $r_2 > 1$).
We may assume that $t_1w_1 = 1$. Note that

$$\frac{2^r t_2w_2 - 1}{2^{r-1}t_2w_2 - 1} > 2 > \frac{2^r + 1}{2^{r-1} + 1}.$$

In particular, if we cross-multiply the left-hand side and the right-hand side, we get

$$\frac{2^{r-1}+1}{2^{r-1}t_2w_2-1} > \frac{2^r+1}{2^rt_2w_2-1}.$$

By multiplying the left-hand side by $\frac{1}{3}t_1w_1t_2w_2\frac{2^{r-1}-1}{2^{r-1}-1}$, and the right-hand side by $\frac{1}{3}t_1w_1t_2w_2\frac{2^r-1}{2^r-1}$ we get that

$$\frac{\frac{1}{3}t_1w_1t_2w_2(2^{2r-2}-1)}{(2^{r-1}-1)(2^{r-1}t_2w_2-1)} > \frac{\frac{1}{3}t_1w_1t_2w_2(2^{2r}-1)}{(2^r-1)(2^rt_2w_2-1).}$$

By expanding the numerator, we have

$$\frac{t_1w_1t_2w_2}{(2^{r-1}-1)(2^{r-1}t_2w_2-1)}\left(-\frac{3}{5}+\frac{6}{35}+\frac{2}{21}+\frac{1}{3}2^{2r-2}\right) > \frac{t_1w_1t_2w_2}{(2^r-1)(2^rt_2w_2-1)}\left(-\frac{3}{5}+\frac{6}{35}+\frac{2}{21}+\frac{1}{3}2^{2r}\right).$$

Using Remark A.9 the above inequality simplifies to

$$h'(0,r-1,1,1)\cdot h'(0,r-1,t_2,w_2) > h'(0,r,1,1)\cdot h'(0,r,t_2,w_2),$$

as desired.

Case 2: $s_2 \geq 1$ and $t_1w_1 = 1$ or $t_2w_2 = 1$. We may assume that $t_1w_1 = 1$. First, note that

$$\frac{(2^{r+s_1}-1)(2^{r+s_1}t_2w_2-1)}{(2^{r+s_1-1}-1)(2^{r+s_1-1}t_2w_2-1)} > 4.$$

which is the same as

$$\frac{\frac{1}{3}t_1w_1t_2w_2 2^{2r+s_1+s_2-2}}{(2^{r+s_1-1}-1)(2^{r+s_1-1}t_2w_2-1)} > \frac{\frac{1}{3}t_1w_1t_2w_2 2^{2r+s_1+s_2}}{(2^{r+s_1}-1)(2^{r+s_1}t_2w_2-1).}$$

Note that

$$\begin{gathered}
t_1w_1t_2w_2\left(-\frac{3}{5}+\frac{6}{35}2^{4s_1}+\frac{2}{21}2^{3s_2+s_1}\right) \geq t_1w_1t_2w_2\left(-\frac{3}{5}+\frac{6}{35}+\frac{16}{21}\right) > 0 \\
\frac{t_1w_1t_2w_2(-\frac{3}{5}+\frac{6}{35}2^{4s_1}+\frac{2}{21}2^{3s_2+s_1})}{(2^{r+s_1-1}-1)(2^{r+s_1-1}t_2w_2-1)} > \frac{t_1w_1t_2w_2(-\frac{3}{5}+\frac{6}{35}2^{4s_1}+\frac{2}{21}2^{3s_2+s_1})}{(2^{r+s_1}-1)(2^{r+s_1}t_2w_2-1)} \\
\frac{t_1w_1t_2w_2(-\frac{3}{5}+\frac{6}{35}2^{4s_1}+\frac{2}{21}2^{3s_2+s_1}+\frac{1}{3}2^{2r+s_1+s_2-2})}{(2^{r+s_1-1}-1)(2^{r+s_1-1}t_2w_2-1)} > \frac{t_1w_1t_2w_2(-\frac{3}{5}+\frac{6}{35}2^{4s_1}+\frac{2}{21}2^{3s_2+s_1}+\frac{1}{3}2^{2r+s_1+s_2})}{(2^{r+s_1}-1)(2^{r+s_1}t_2w_2-1)} \\
h'(s_1,r-1,1,1)\cdot h'(s_2,r-1,t_2,w_2) > h'(s_1,r,1,1)\cdot h'(s_2,r,t_2,w_2).
\end{gathered}$$

Case 3: $t_1w_1 \geq 3$ and $t_2w_2 \geq 3$.

$$\begin{aligned}
0 &< 3t_1w_1t_2w_2 - 3t_1w_1 - 3t_2w_2 + 3 - t_1w_1t_2w_2 \\
&= 3((t_1w_1-1)(t_2w_2-1) - \frac{3}{5}t_1w_2t_2w_2 + \frac{6}{35}t_1w_2t_2w_2 + \frac{2}{21}t_1w_2t_2w_2) \\
&\leq 3((t_1w_1-1)(t_2w_2-1) - \frac{3}{5}t_1w_2t_2w_2 + \frac{6}{35}t_1w_2t_2w_2 2^{4s_1} + \frac{2}{21}t_1w_2t_2w_2 2^{3s_2+s_1}).
\end{aligned}$$

$$
\begin{aligned}
&4((t_1w_1-1)(t_2w_2-1)-\frac{3}{5}t_1w_2t_2w_2+\frac{6}{35}t_1w_2t_2w_22^{4s_1}+\frac{2}{21}t_1w_2t_2w_22^{3s_2+s_1})\\
&>(t_1w_1-1)(t_2w_2-1)-\frac{3}{5}t_1w_2t_2w_2+\frac{6}{35}t_1w_2t_2w_22^{4s_1}+\frac{2}{21}t_1w_2t_2w_22^{3s_2+s_1}
\end{aligned}
$$

$$
\begin{aligned}
&4((t_1w_1-1)(t_2w_2-1)-\frac{3}{5}t_1w_2t_2w_2+\frac{6}{35}t_1w_2t_2w_22^{4s_1}+\frac{2}{21}t_1w_2t_2w_22^{3s_2+s_1}+\frac{1}{3}t_1w_1t_2w_22^{2r+s_1+s_2-2})\\
&>(t_1w_1-1)(t_2w_2-1)-\frac{3}{5}t_1w_2t_2w_2+\frac{6}{35}t_1w_2t_2w_22^{4s_1}+\frac{2}{21}t_1w_2t_2w_22^{3s_2+s_1}+\frac{1}{3}t_1w_1t_2w_22^{2r+s_1+s_2}.
\end{aligned}
$$

From Remark A.9, we see that

$$4(2^{r+s_1-1}-1)(2^{r+s_1-1}t_2w_2-1)h'(s_1,r-1,t_1,w_1)\cdot h'(s_2,r-1,t_2,w_2) \tag{16}$$

$$>(2^{r+s_1}-1)(2^{r+s_1}t_2w_2-1)h'(s_1,r,t_1,w_1)\cdot h'(s_2,r,t_2,w_2). \tag{17}$$

Note that

$$\frac{(2^{r+s_1}-1)(2^{r+s_1}t_2w_2-1)}{(2^{r+s_1-1}-1)(2^{r+s_1-1}t_2w_2-1)}>4$$

Using this inequality in (16), we get the following

$$
\begin{aligned}
&(2^{r+s_1}-1)(2^{r+s_1}t_2w_2-1)h'(s_1,r-1,t_1,w_1)\cdot h'(s_2,r-1,t_2,w_2)\\
&>(2^{r+s_1}-1)(2^{r+s_1}t_2w_2-1)h'(s_1,r,t_1,w_1)\cdot h'(s_2,r,t_2,w_2)\\
&h'(s_1,r-1,t_1,w_1)\cdot h'(s_2,r-1,t_2,w_2)\\
&>h'(s_1,r,t_1,w_1)\cdot h'(s_2,r,t_2,w_2)
\end{aligned}
$$

□

Proof of Theorem 4.20. First we define the following sets

$$
\begin{aligned}
S_1&=\{h'(1,1,t_1,w_1)\cdot h'(1,1,t_2,w_2)\mid t_1,t_2,w_1,w_2\in\mathbb{N}\}\\
S_2&=\{h'(0,1,t_1,w_1)\cdot h'(1,1,t_2,w_2)\mid t_1,t_2,w_1,w_2\in\mathbb{N}\}\\
S_3&=\{h'(0,1,t_1,w_1)\cdot h'(0,1,t_2,w_2)\mid t_1,t_2,w_1,w_2\in\mathbb{N}\}\\
S&=S_1\cup S_2\cup S_3
\end{aligned}
$$

A simple computation yields that $\sup S = 9/11$ noting that the max value occurs in S_3 when $t_1 = w_1 = t_2 = w_2 = 1$. As in the proof of Theorem 4.7, it is easy to see that $h'(s_1, r_1, t_1, w_1)\cdot h'(s_2, r_2, t_2, w_2) \le \sup S$ when $r_1, r_2 \ge 1$. This leaves the case when $r_2 = 0$. Since $s_2 \le r_2$ and $2^{s_2}2^{r_2}t_2w_2 > 1$, we must have $t_1w_1 \ge 3$. Then

$$h'(s_1,r_1,t_1,w_1)\cdot h'(0,0,t_2,w_2)=\frac{(t_2w_2-1)(t_1w_1-1)}{(t_2w_2-1)(2^{s_1}2^{r_1}t_1w_1-1)}\le\frac{t_1w_1-1}{2(t_1w_1-1)+1}\le 1/2.$$

□