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## APPROXIMATE ANALYSIS OF INTERACTION OF CLOSELY SPACED CRACKS

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An approximate method of stress analysis in elastic solids with multiple cracks is proposed to improve the accuracy of the Kachanov method in analyzing closely spaced cracks. Classical Kachanov method assumed that traction in each crack can be represented as the sum of a uniform component and a nonuniform one, and that the interaction among the cracks is only due to the uniform components. The assumptions simplify considerably the mathematics. However, they may not be valid when the cracks are very close and overlap along the direction of load, because each crack may be embedded in the stress-amplifying region as well as the stress-shielding region of the other cracks at this time. To improve the accuracy of the Kachanov method, a new asymptotic method, in which the influence on a crack of the quadratic parabola pseudotractions (QPPTs) rather than the average ones on the other crack are taken into account, is proposed. Applications to the problem of three collinear cracks and two offset parallel closely spaced cracks are considered to validate the accuracy of the new method.

### 1. Introduction

As one of the key factors in determining the component life of a structure, the interaction effect among cracks has received significant attention in recent years. Various methods [Sneddon and Lowengrub 1969; Erdogan 1983; Chen 1984a; 1984b; Horii and Nemat-Nasser 1985b; Chudnovsky et al. 1987a; 1987b; Kachanov 1987; Lam and Phua 1991; Kachanov 1992; Chen 1995; Basista and Gross 2000; Gorbatikh and Kachanov 2000; Wang et al. 2000; Li et al. 2003] have been developed for the analyses. As the boundary conditions can be more readily defined for traction free cracks, crack interaction problems were formulated in terms of the interaction tractions rather than in terms of the displacement discontinuities. Based on the stress superposition technique, Kachanov [1987] proposed a simple method for calculation of stress intensity factors (SIFs) and effective elastic properties of solids with multiple cracks. The solution can also be used to construct the stress and displacement fields in the solid [Kachanov 1987; Gorbatikh and Kachanov

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*Keywords:* crack interaction, stress intensity factor, multiple cracks, quadratic parabola pseudotraction.

2000]. Combining the Kachanov method with the sliding crack model of brittle deformation, Basista and Gross [2000] extended the method to two-dimensional crack interaction problems under overall compression.

In Kachanov's stress superposition method, the key assumption is that the traction in a crack is composed of a uniform and a nonuniform component. The magnitude of each component is computed under the assumed condition that the resultant force of the nonuniform component is zero. The influence of the nonuniform component on crack interaction is ignored. Therefore, the accuracy of the Kachanov method depends on the configuration of crack distribution, and especially on crack spacing [Kachanov 1992; Wang et al. 2000; Kachanov 2003; Li et al. 2003].

To avoid the limitation of the Kachanov method in analyzing closely spaced cracks, Li et al. [2003] proposed a new approximate method for the analysis of cracked solids. The linearly varying component of traction rather than the uniform one is taken into account. It is shown that the accuracy of the modified method is better than Kachanov's. Recently, Kachanov [2003] gave a short overview of various approaches to crack interaction problems in which solutions for closely spaced cracks were discussed. It's pointed out that in 2-D problems of closely spaced cracks the accuracy of Kachanov's method — and even the method of [Li et al. 2003] — may not be sufficient. For example, the problem of two closely spaced cracks overlapping along the direction of load can not be accurately analyzed using such methods. This problem is of great interest because of its relevance to propagation and coalescence of two close cracks [Horii and Nemat-Nasser 1985a; Wang et al. 1996; Li et al. 2003].

One of the original objectives of the Kachanov method is to analyze the meso-damage of solids with multiple cracks first, and then to compute the macro response of the solids. The propagation and coalescence of cracks are not considered. But the accuracy of the Kachanov method and even the linear approximate method are not satisfactory for closely spaced and overlapping cracks in solids, such as the cracks in rock-like materials under compression, in which the propagation and coalescence of cracks have to be taken into account. To improve the accuracy, a new approximate method, in which the influence of the quadratic parabola pseudo-tractions (DPPTs) of cracks on other cracks, is proposed in this paper. It's shown that DPPTs are able to describe adequately the interaction among closely spaced cracks. Numerical examples of the interactions of three collinear cracks and a pair of offset parallel cracks are analyzed in detail to assess the accuracy of the proposed method.

## 2. The generalised Kachanov method

Consider an infinite  $xy$ -plane containing  $N$  cracks (Figure 1) under remote stresses  $\sigma^\infty$ . It is equivalent to the problem of  $N$  cracks with tractions

$$-n_i \cdot \sigma^\infty \quad (i = 1, 2, \dots, N).$$

The equivalent one can be further decomposed to  $N$  sub-problems with only one crack each, on which the tractions  $t_i = \{p_{ni} \ p_{ti}\}^T$  are acting, where  $p_{ni}$  and  $p_{ti}$  are the normal and tangential components of the tractions  $t_i$ . The tractions  $t_i$  are the summation of the initial tractions  $-n_i \cdot \sigma^\infty$  and the additional interaction tractions induced by other cracks. We also have

$$\mathbf{t}_i = -\mathbf{n}_i \cdot \sigma^\infty + \sum_{j \neq i} \Delta \mathbf{t}_{ji}, \quad (1)$$

where  $\Delta \mathbf{t}_{ji}$  is the additional traction on the  $i$ -th crack due to the presence of the  $j$ -th crack. The key point is how to solve the unknown additional tractions adequately. The Kachanov method [1987] decomposes the unknown tractions into two parts, an average traction and a higher-order one, and only the influence of the average one is considered, while that of the higher-order one is ignored. In the modified method [Li et al. 2003], the linearly varying traction, instead of a uniform one, is used in an effort to improve the accuracy. However, both methods may be erroneous when applied to a problem where cracks are closely spaced and overlapping. Therefore, a modified method considering the influence of QPPTs is carried out in the next section.

## 3. The quadratic parabola pseudotraction method

First, one only considers the contribution to the other cracks of the tractions

$$t_i^{(0)} = \{ p_{ni}^{(0)} \ p_{ti}^{(0)} \} = -n_i \cdot \sigma^\infty$$

(the subscript ‘ $n$ ’ denotes ‘normal to crack line’ and ‘ $t$ ’ means ‘tangential to crack line’) without regard to the additional tractions:

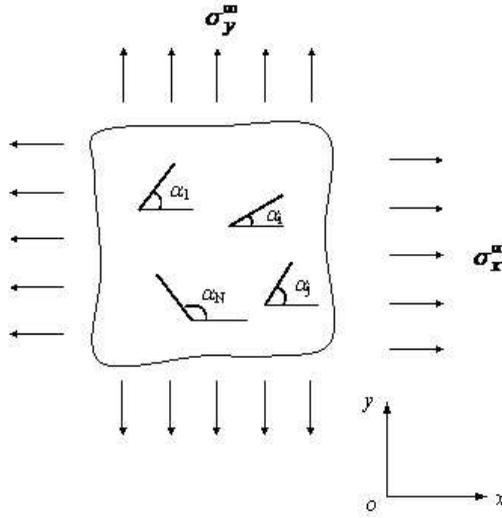
$$\mathbf{t}_i^{(1)} = \{ p_{ni}^{(1)} \ p_{ti}^{(1)} \}^T = -n_i \cdot \sigma^\infty + \sum_{j=1, j \neq i}^N \Delta \mathbf{t}_{ji}^{(1)} \quad (i = 1, 2, \dots, N), \quad (2)$$

where

$$\Delta \mathbf{t}_{ji}^{(1)}(i, j = 1, 2, \dots, N)$$

is the additional traction acting on the  $i$ -th crack due to the traction

$$\mathbf{t}_j^{(0)} = -\mathbf{n}_j \cdot \sigma^\infty$$



**Figure 1.** An infinite plane containing  $N$  cracks

on the  $j$ -th crack. This additional traction can be obtained from the elemental solutions of elastic mechanics [Sneddon and Lowengrub 1969]:

$$t_i^{(1)} = \begin{Bmatrix} p_{ni}^{(1)} \\ p_{ti}^{(1)} \end{Bmatrix} = \begin{Bmatrix} p_{ni}^{(0)} \\ p_{ti}^{(0)} \end{Bmatrix} + \sum_{j=1, j \neq i}^N \begin{bmatrix} f_{ji}^{nm} & f_{ji}^{tn} \\ f_{ji}^{nt} & f_{ji}^{tt} \end{bmatrix} \begin{Bmatrix} p_{nj}^{(0)} \\ p_{tj}^{(0)} \end{Bmatrix} \quad (3)$$

( $i = 1, 2, \dots, N$ ), where  $f_{ji}^{rs}$  ( $i, j = 1, 2, \dots, N$ ;  $j \neq i$ ;  $r, s = n, t$ ) are those solutions. The explicit forms for  $f_{ji}^{rs}$  are given in the Appendix. In fact, the solutions above are only first-order approximations because the contributions of the additional items are ignored.

When considering further the influences of quadratic parabola pseudotractions (QPPTs), it is assumed that the unknown tractions  $t_i^{(2)} = \{p_{ni}^{(2)} \ p_{ti}^{(2)}\}^T$  can be decomposed into

$$t_i^{(2)} = \tilde{t}_i^{(2)} + \Delta t_i^{(2)} = -\mathbf{n}_i \cdot \sigma^\infty + \sum_{j=1, j \neq i}^N \Delta t_{ji}^{(2)} \quad (i = 1, 2, \dots, N), \quad (4)$$

where  $\tilde{t}_i^{(2)}$  are QPPTs. Its normal and tangential components are respectively

$$\tilde{t}_i^{(2)} = \begin{Bmatrix} \tilde{p}_{ni}^{(2)} \\ \tilde{p}_{ti}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \langle p_{ni}^{(2)} \rangle (1 + T_{ni1} \xi_i + T_{ni2} (\xi_i^2 - \frac{1}{3} l_i^2)) \\ \langle p_{ti}^{(2)} \rangle (1 + T_{ti1} \xi_i + T_{ti2} (\xi_i^2 - \frac{1}{3} l_i^2)) \end{Bmatrix} \quad (5)$$

( $i = 1, 2, \dots, N$ ). In these equations,  $\xi_i \in (-l_i \ l_i)$  denotes the coordinate of a point in the  $i$ -th crack line with half-length  $l_i$ , and  $T_{ni1}$ ,  $T_{ni2}$ ,  $T_{ti1}$  and  $T_{ti2}$  are the

dimensionless coefficients of tractions distributions. In (4), the item  $\Delta t_{ji}^{(2)}$  is the additional tractions on the  $i$ -th crack due to the tractions  $t_j^{(2)}$  of the  $j$ -th crack, and  $\Delta t_i^{(2)}$  denotes the higher-order items of tractions. Averaging (4) and (5) along the crack line, one can know the averages of the higher-order tractions are zero, that is

$$\langle \mathbf{t}_i^{(2)} \rangle = \langle \tilde{\mathbf{t}}_i^{(2)} \rangle = \begin{Bmatrix} \langle p_{ni}^{(2)} \rangle \\ \langle p_{ti}^{(2)} \rangle \end{Bmatrix} = \begin{Bmatrix} \langle \tilde{p}_{ni}^{(2)} \rangle \\ \langle \tilde{p}_{ti}^{(2)} \rangle \end{Bmatrix} \langle \Delta \mathbf{t}_i^{(2)} \rangle = 0, \quad (6)$$

( $i = 1, 2, \dots, N$ ).

Assume that the additional tractions

$$\Delta t_{ji}^{(2)}$$

of the  $i$ -th crack are contributed by the tractions  $\tilde{t}_j^{(2)}$  of the  $j$ -th crack, while the higher-order tractions

$$\Delta t_i^{(2)}$$

are ignored. Clearly, if the tractions  $\tilde{t}_i^{(2)}$  were known, the

$$\Delta t_{ji}^{(2)}$$

could be determined. However, the average tractions as well as the dimensionless coefficients  $T_{ni1}$ ,  $T_{ni2}$ ,  $T_{ti1}$  and  $T_{ti2}$  are unknown so far. So, it is reasonable to assume that these coefficients can be estimated by the known tractions

$$t_i^{(1)} = \{ p_{ni}^{(1)} \ p_{ti}^{(1)} \}^T$$

to be

$$\begin{aligned} T_{ni1} &= \frac{p_{ni}^{(1)}(l_i) - p_{ni}^{(1)}(-l_i)}{2l_i \langle p_{ni}^{(1)} \rangle}, \\ T_{ni2} &= \frac{p_{ni}^{(1)}(l_i) + p_{ni}^{(1)}(-l_i) - 2p_{ni}^{(1)}(0)}{2l_i^2 \langle p_{ni}^{(1)} \rangle} \quad (i = 1, 2, \dots, N), \end{aligned} \quad (7)$$

where the  $\langle p_{ni}^{(1)} \rangle$  are the average normal tractions, which can be obtained by integration:

$$\langle p_{ni}^{(1)} \rangle = \frac{1}{2l_i} \int_{-l_i}^{l_i} p_{ni}^{(1)}(\xi_i) d\xi_i, \quad (8)$$

( $i = 1, 2, \dots, N$ ). The tangential coefficients  $T_{ti1}$  and  $T_{ti2}$  and the average tangential tractions

$$\langle p_{ti}^{(1)} \rangle$$

can be obtained by replacing the subscript  $n$  by  $t$  in (7) and (8).

Now the expressions of the additional tractions  $\Delta t_i^{(2)}$  of the  $i$ -th crack due to QPPTs on the other cracks can be given as

$$t_i^{(2)} = \begin{Bmatrix} p_{ni}^{(2)} \\ p_{ti}^{(2)} \end{Bmatrix} = \begin{Bmatrix} p_{ni}^{(0)} \\ p_{ti}^{(0)} \end{Bmatrix} + \sum_{j=1, j \neq i}^N \begin{bmatrix} \hat{f}_{ji}^{nn} & \hat{f}_{ji}^{tn} \\ \hat{f}_{ji}^{nt} & \hat{f}_{ji}^{tt} \end{bmatrix} \begin{Bmatrix} p_{nj}^{(2)} \\ p_{tj}^{(2)} \end{Bmatrix} \quad (i = 1, 2, \dots, N). \quad (9)$$

In (9), the coefficients

$$\hat{f}_{ji}^{rs} \quad (i, j = 1, 2, \dots, N; j \neq i; r, s = n, t)$$

denote the interaction coefficients. For example,  $\hat{f}_{ji}^{tn}$  is the normal ( $n$ ) traction of the  $i$ -th crack contributed by the tangential ( $t$ ) QPPT

$$\left( \left( 1 + T_{tj1} \xi_i + T_{tj2} (\xi_j^2 - \frac{1}{3} l_j^2) \right) \right)$$

of the  $j$ -th crack. The QPPT has a unit mean value along the crack line.

Averaging (9) along the  $i$ -th crack line, one gets

$$\begin{Bmatrix} \langle p_{ni}^{(2)} \rangle \\ \langle p_{ti}^{(2)} \rangle \end{Bmatrix} = \begin{Bmatrix} \langle p_{ni}^{(0)} \rangle \\ \langle p_{ti}^{(0)} \rangle \end{Bmatrix} + \sum_{j=1, j \neq i}^N \begin{bmatrix} \Lambda_{ji}^{nn} & \Lambda_{ji}^{tn} \\ \Lambda_{ji}^{nt} & \Lambda_{ji}^{tt} \end{bmatrix} \begin{Bmatrix} \langle p_{nj}^{(2)} \rangle \\ \langle p_{tj}^{(2)} \rangle \end{Bmatrix} \quad (10)$$

( $i = 1, 2, \dots, N$ ).

In these equations, the coefficients

$$\Lambda_{ji}^{rs} \quad (i, j = 1, 2, \dots, N; j \neq i; r, s = n, t)$$

denote the interaction coefficients. For example,  $\Lambda_{ji}^{tn}$  is the average normal traction of the  $i$ -th crack contributed by the tangential QPPT of the  $j$ -th crack. The explicit forms for  $\hat{f}_{ji}^{rs}$  in (9) and  $\Lambda_{ji}^{rs}$  in (10) are listed in the Appendix.

For each crack, one can establish a set of (10). Therefore, there will be  $2N$  equations with  $2N$  unknown average tractions

$$\langle p_{ni}^{(2)} \rangle \quad \text{and} \quad \langle p_{ti}^{(2)} \rangle \quad (i = 1, 2, \dots, N).$$

By solving these equations, one can determine the average tractions.

Once the average tractions are determined, one can then obtain the QPPTs

$$\tilde{t}_i^{(2)}$$

acting on these cracks and calculate the tractions  $t_i^{(2)}$  in (4) by the elemental solutions of elastic mechanics. The SIFs of each crack can be computed as follows:

$$\left. \begin{Bmatrix} K_I(\pm l_i) \\ K_{II}(\pm l_i) \end{Bmatrix} \right\} = -\frac{1}{\sqrt{\pi l_i}} \int_{-l_i}^{l_i} \sqrt{\frac{l_i \pm \xi_i}{l_i \mp \xi_i}} \begin{Bmatrix} p_{ni}^{(2)}(\xi_i) \\ p_{ti}^{(2)}(\xi_i) \end{Bmatrix} d\xi_i. \quad (11)$$

In fact, it can be seen from (5) that, if the coefficients  $T_{ni2}$  and  $T_{ti2}$  are zero, the present formulation is similar to that of the linear approximate method [Li et al. 2003] except that the way to estimate the coefficients  $T_{ni1}$  and  $T_{ti1}$  is slightly different. Furthermore, if all of the coefficients are assumed to be zero, the present formulation is then identical to the original Kachanov method, in which only the average tractions are considered.

#### 4. Example: three collinear cracks

To evaluate the accuracy of the present method, the problem of three collinear cracks in an infinite medium under a uniform tensile stress normal to the crack line as shown in Figure 2 is taken as a test case here. The present results are compared with those available from the literature [Sih 1965] and those obtained by using the original Kachanov method [1987] and the linear approximate method [Li et al. 2003]. The errors are also listed in the table. It can be seen from Table 1 that very good agreement has been achieved between the present method and the exact solutions. On the other hand, the errors of the Kachanov results at the inner tip are significant when three cracks are closely spaced. In addition, the discrepancies of the SIFs of the outer tips are not as obvious as those of the inner tips.

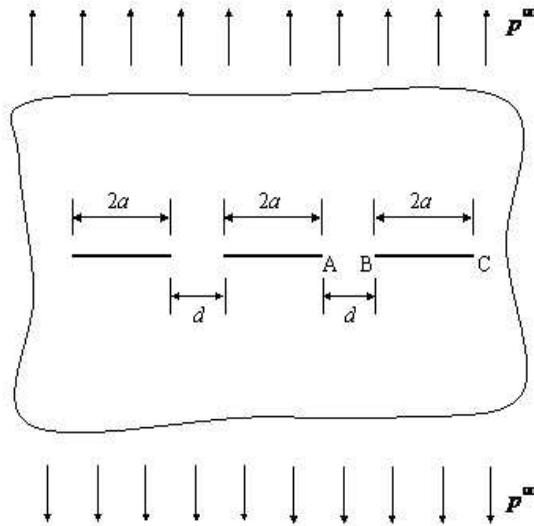
Though the computational time required to obtain the solutions by the present method is slightly more than with Kachanov's method, the much improved accuracy makes it attractive for the analysis of interaction cracks, especially when those cracks are closely spaced.

It is interesting also to compare the improvement in accuracy of the two modified methods. This comparison will be done in Section 6 after another numerical example for two overlapping cracks is given.

#### 5. A pair of offset parallel cracks

In this section, the same approach is used to analyse a pair of offset parallel cracks. In this case, it is slightly more complicated than in the case of collinear cracks, but one can still obtain the interaction factors by integrating the equations given in the Appendix. Having  $\Lambda_{ij}^{r,s}$  determined, one can obtain the Kachanov tractions and the improved tractions accordingly. To investigate the influence of the distance between the two overlapping cracks, two parameters are employed:  $h/a$  and  $a/b$ , as shown in Figures 3 and 4.

The exact solutions, the present solutions and the Kachanov solutions are given together in Figures 3 and 4. Note that  $K_0$  is the stress intensity factor for an isolated crack under normal remote stress and is given by  $K_0 = \sigma \sqrt{\pi a}$ . It is obvious that the results of the proposed method are in good agreement with those of Rooke and Cartwright [1976] even when the two cracks overlap ( $0.5 < h/a < 1.0$ ). When



**Figure 2.** Three collinear cracks.

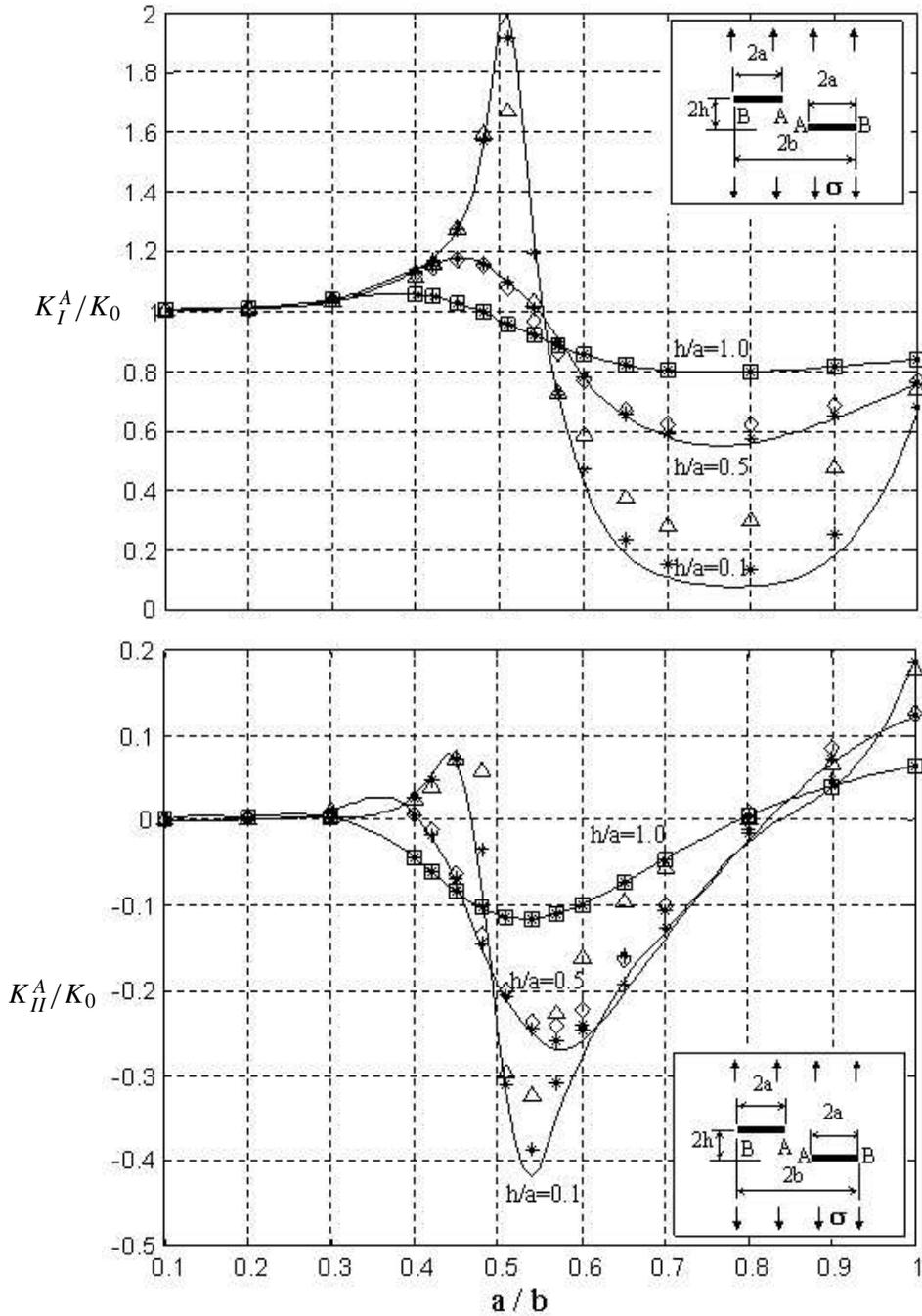
the two cracks are remotely spaced (e.g.,  $h/a = 1.0$ ) or not overlapping ( $a/b < 0.5$ ), the Kachanov solutions are good approximations. However, the Kachanov solutions become increasingly more erroneous as the two cracks get closer (e. g.,  $h/a = 0.5, 0.1$ ) and overlapping ( $0.5 < h/a < 1.0$ ) along the load direction. In addition, the discrepancies of the SIFs of the inner tips are more remarkable than those of the outer tips.

So far, there are three Kachanov-type methods for interaction of multiple cracks. An interesting question is when and why it is worth using the present method instead of the linear one [Li et al. 2003] and Kachanov's. Therefore, comparison of this QPPTs method with the other two Kachanov-type methods and also other polynomial approximation techniques is given in next section.

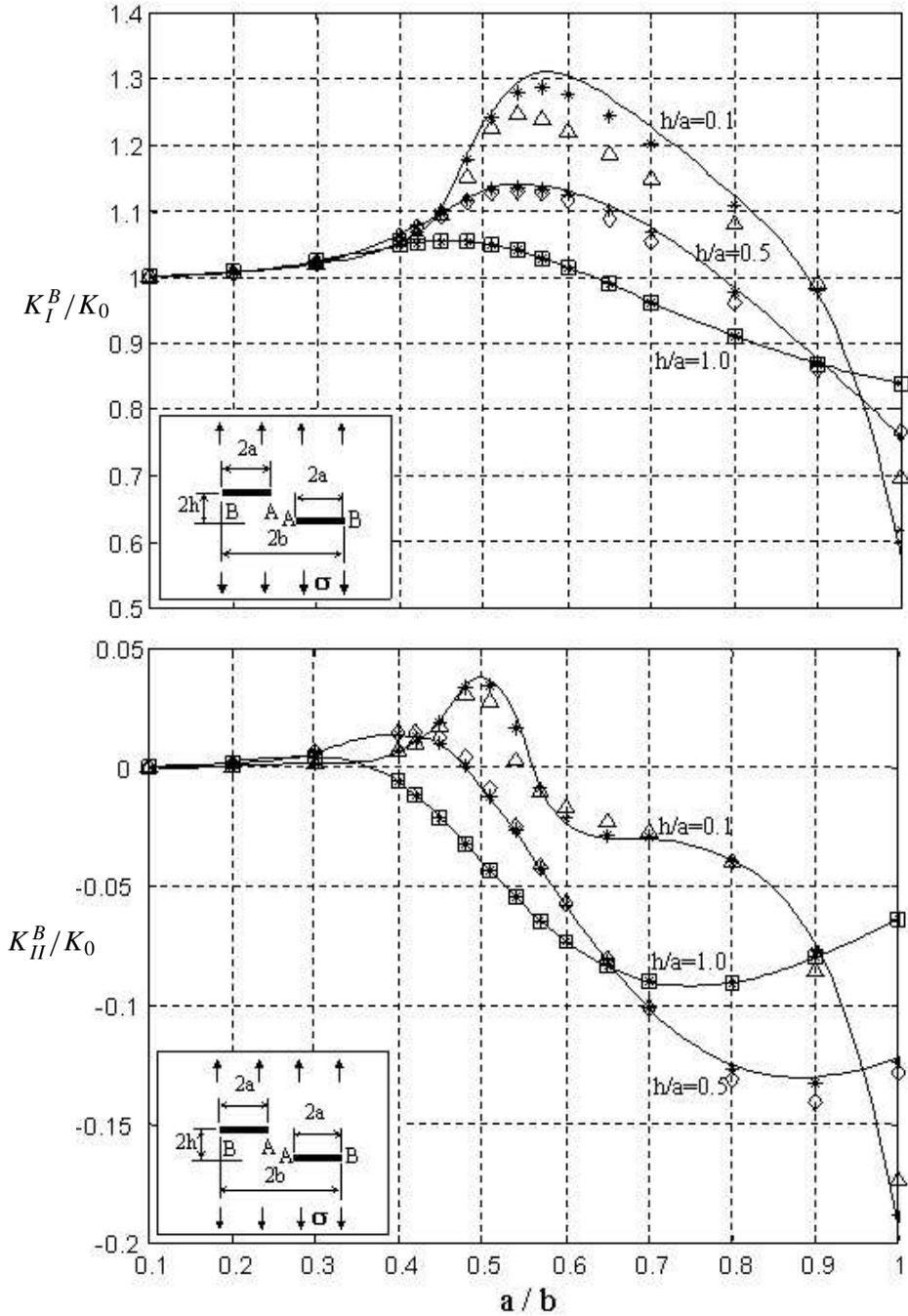
## 6. Comparison with other techniques

First, we compare the original Kachanov method with its two modifications. The essential difference among the three lies in that they use approximate polynomial functions of different orders —  $p = 0, 1$ , and  $2$ .

The results of the three methods in Table 1 indicate that the linear method ( $p = 1$ ) and the present quadratic method ( $p = 2$ ) yield smaller errors than does the Kachanov method ( $p = 0$ ). One can conclude that a higher-degree method gives better results than a lower one, though the present method does not generate as



**Figure 3.** A pair of offset parallel cracks, Tip A. Code: solid line, Rooke and Cartwright [1976]; \*, present method with  $h/a = 0.1, 0.5, 1.0$ ;  $\Delta$ , Kachanov method with  $h/a = 0.1$ ;  $\diamond$ , Kachanov method with  $h/a = 0.5$ ;  $\square$ , Kachanov method with  $h/a = 1.0$ .



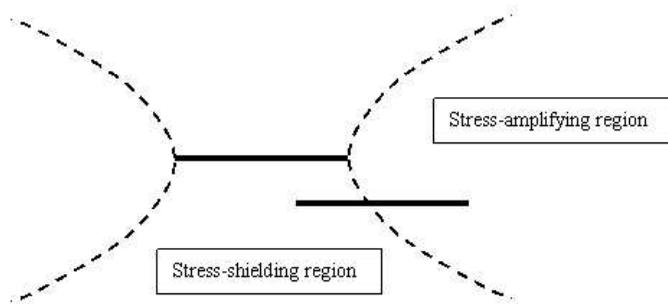
**Figure 4.** A pair of offset parallel cracks, Tip B. Code: solid line, Rooke and Cartwright [1976]; \*, present method with  $h/a = 0.1, 0.5, 1.0$ ;  $\Delta$ , Kachanov method with  $h/a = 0.1$ ;  $\diamond$ , Kachanov method with  $h/a = 0.5$ ;  $\square$ , Kachanov method with  $h/a = 1.0$ .

		$d/a \rightarrow$	1.0	0.5	0.2	0.1	0.05
		$K_I/K_0$	Tip A	Exact*	1.1674	1.3214	1.6542
Kachanov method	1.1669 0.04%			1.3179 0.26%	1.6309 1.4%	1.9601 3.6%	2.3657 7.4%
Linear method	1.1672 0.02%			1.3210 0.03%	1.6649 0.65%	2.0591 1.3%	2.6252 2.8%
Present method	1.1673 0.01%			1.3211 0.02%	1.6604 0.37%	2.0453 0.6%	2.5825 1.1%
Tip B	Exact		1.1387	1.2836	1.6119	1.9923	2.5185
	Kachanov method		1.1382 0.04%	1.2797 0.30%	1.5877 1.5%	1.9086 4.2%	2.3095 8.3%
	Linear method		1.1384 0.03%	1.2833 0.02%	1.6202 0.51%	2.0262 1.7%	2.5941 3.0%
	Present method		1.1385 0.02%	1.2834 0.02%	1.6201 0.50%	2.0093 0.8%	2.5563 1.5%
Tip C	Exact		1.0687	1.1103	1.1714	1.2167	1.2587
	Kachanov method		1.0686 0.00%	1.1081 0.20%	1.1667 0.40%	1.2094 0.60%	1.2486 0.80%
	Linear method		1.0686 0.00%	1.1101 0.02%	1.1741 0.23%	1.2224 0.47%	1.2637 0.40%
	Present method		1.0686 0.00%	1.1102 0.01%	1.1728 0.12%	1.2195 0.23%	1.2617 0.24%

**Table 1.** Results for three collinear cracks (after Sih [1965]).

much of an improvement over the linear one as the latter does over the original. For example, comparing the results for the closest cracks ( $d/a = 0.05$ ) at crack Tip B, the errors are 8.3% for the Kachanov method, 3.0% for the linear method and 1.5% for the present method.

One can compare in succession the results of the three methods for the offset cracks case. In fact, to avoid confusion, the results from [Li et al. 2003] for the same example are not shown in Figures 3 and 4. Therefore, here, the SIFs of the inner tips for the closer case ( $h/a = 0.2$ ) not shown in the figures are listed in Table 2 for comparison of the results of the three methods. From Table 2 and Figures 3 and 4, one can see that the discrepancies of the Kachanov solutions become increasingly more serious as the two cracks get closer and overlap along the loading direction.



**Figure 5.** Stress-amplifying region and stress-shielding region of a crack.

The maximal error reaches 30%. By comparison with the Kachanov method, the linear method generates better results with errors below 14.5%. For the same case, the present method yields practically imperceptible errors of only 4% or less. The accuracy of the results is adequate for the analysis of propagation and coalescence of several cracks (see Section 1).

One can examine this phenomenon as illustrated in Figure 5. For the problem of closely spaced cracks overlapping along the loading direction, each crack is embedded in the stress-amplifying region as well as in the stress-shielding region of the other one. The pseudotractions acting on cracks may vary intensely, in which case the use of constant or linearly varying approximation for the tractions would result in significant errors in the SIFs. Then it is reasonable that a higher-order approximation would yield a remarkable improvement in accuracy.

Therefore, we suggest that: for the problems of remotely-spaced cracks, the three Kachanov-type methods all can generate satisfactory results for SIFs; the most simple Kachanov method is preferred; for the problems of cracks closely spaced but not overlapping each other, the linear method is adequate for analysis; for the problem of cracks closely spaced and overlapping each other, the present method instead of the other two methods should be chosen.

These Kachanov-type methods can be considered as further developments of the ideas from [Chen 1984a; Horii and Nemat-Nasser 1985b; Chudnovsky et al. 1987a; 1987b], where tractions on individual cracks were interrelated by self-consistence conditions. Kachanov [1987] compared his method with previous works in detail, so only some statements are repeated here. In [Horii and Nemat-Nasser 1985b; Chudnovsky et al. 1987a; 1987b], Taylor or Chebyshev polynomials centered at the crack center were applied to the configuration “crack-microcrack array” and to the general 2-D crack array. Horii and Nemat-Nasser [1985b] considered the test problem of two collinear cracks. They found that the degree of the approximating polynomials increases rapidly as the spacing between cracks becomes smaller. The

$a/b \rightarrow$	0.1	0.2	0.4	0.48	0.54	0.60	0.70	0.80	0.90
Exact*	1.0017	1.0093	1.1611	1.6988	1.0805	0.5791	0.3023	0.2987	0.4312
Kachanov method	1.0011 0.00%	1.0085 0.10%	1.1260 3.0%	1.4640 13.8%	1.0663 1.3%	0.6438 11.2%	0.3641 20.4%	0.3890 30.0%	0.5502 27.6%
Linear method	1.0016 0.01%	1.0090 0.03%	1.1499 0.97%	1.6049 5.5%	1.0743 0.58%	0.6024 4.0%	0.3295 9.0%	0.3420 14.5%	0.4883 13.3%
Present method	1.0016 0.01%	1.0091 0.02%	1.1558 0.45%	1.6612 2.2%	1.0770 0.33%	0.5895 1.8%	0.3116 3.1%	0.3104 3.9%	0.4479 3.9%

**Table 2.**  $K_I^A/K_0$  for two offset parallel cracks  $h/a = 0.2$  (after Rooke and Cartwright [1976]).

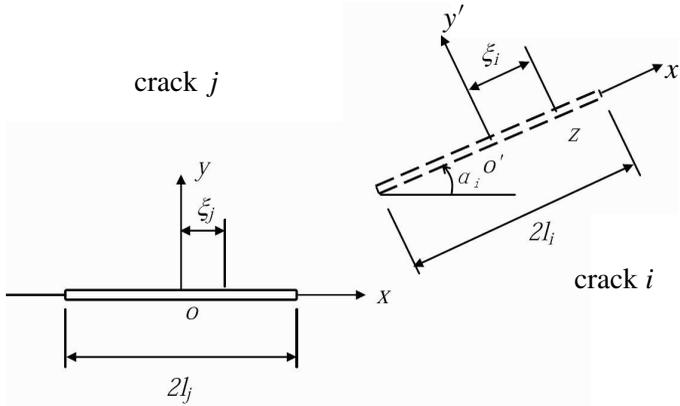
closest distance considered is 1/4 of the crack length (1/10 otherwise in the present work); at this distance the polynomials' degree is 28. At distances one order of magnitude smaller than the crack length, the polynomial's degree in their method can be expected to be several times higher. Kachanov [1987] also pointed out that in the Chebyshev polynomial technique, as with the Taylor polynomial technique, convergence gets increasingly difficult as the spacings between cracks become smaller. At a close distance (1/10 of the crack length), Chebyshev polynomials of the sixth degree yield a 10.7% error in SIFs, while the Kachanov method yields only a 1.5% error.

Note also that the technique of polynomial approximation [Chen 1984a] is based on complex variable representations and is not, therefore, easily extensible to 3-D configurations. The main drawback of the polynomial technique — a large number of polynomial coefficients (rapidly increasing as spacings between cracks become smaller) — will, however, become much more severe in the 3-D problems.

The key idea of Kachanov-type methods — to neglect the impact on a given crack of higher-order traction nonuniformities — is reminiscent of Saint-Venant's principle of elasticity and may be interpreted as a Saint-Venant-type principle for solids with cracks. Moreover, since the SIFs are given by integrals of tractions, the effect of higher-order nonuniformities is neglected not in the pointwise sense but in a milder, integral sense. This may explain why further refinements of the original Kachanov method work well for closely spaced cracks.

## 7. Conclusions

To improve the accuracy of the Kachanov method in analyzing closely spaced and overlapping cracks, a QPPT method for the analysis of solids with multiple



**Figure 6.** Cracks  $i$  and  $j$ .

cracks is proposed. The pseudotraction in a crack is decomposed into a quadratic-varying component and a higher-order one. The higher-order component has a zero average. It is further assumed that the interaction effects among the cracks are only due to the quadratic-varying component. Mathematical relations for the tractions in the cracks are derived and the solutions are obtained for various cases. It is shown through two numerical examples that the results of the method are in close agreement with the exact solutions, even when closely spaced cracks overlap along the direction of loading.

### Appendix: The interaction coefficients

Consider two cracks  $i$  and  $j$  as shown in Figure 6; the interaction coefficients in Equations (3), (9) and (10) can be obtained thus [Chen 1984a; Cheung et al. 1992]:

$$\begin{aligned}
 f_{ij}^{nn} - \mathbf{i}f_{ij}^{tn} &= \int_{-l_j}^{l_j} (\bar{f}_{ij}^{nn} - \mathbf{i}\bar{f}_{ij}^{tn}) d\xi_j, \\
 f_{ij}^{tn} - \mathbf{i}f_{ij}^{tt} &= \int_{-l_j}^{l_j} (\bar{f}_{ij}^{tn} - \mathbf{i}\bar{f}_{ij}^{tt}) d\xi_j, \\
 \hat{f}_{ij}^{nn} - \mathbf{i}\hat{f}_{ij}^{tn} &= \int_{-l_j}^{l_j} (\bar{f}_{ij}^{nn} - \mathbf{i}\bar{f}_{ij}^{tn}) \left(1 + T_{nj1}\xi_j + T_{nj2}(\xi_j^2 - \frac{1}{3}l_j^2)\right) d\xi_j, \\
 \hat{f}_{ij}^{tn} - \mathbf{i}\hat{f}_{ij}^{tt} &= \int_{-l_j}^{l_j} (\bar{f}_{ij}^{tn} - \mathbf{i}\bar{f}_{ij}^{tt}) \left(1 + T_{ij1}\xi_j + T_{ij2}(\xi_j^2 - \frac{1}{3}l_j^2)\right) d\xi_j, \\
 \Lambda_{ij}^{rs} &= \frac{1}{2a_i} \int_{-a_i}^{a_i} f_{ij}^{rs} d\xi_i \quad (r, s = n, t),
 \end{aligned}$$

where

$$\begin{aligned} \bar{f}_{ij}^{mn} - \mathbf{i}\bar{f}_{ij}^{tn} &= \frac{\sqrt{l_j^2 - \xi_j^2}}{2\pi} (G(z) + \overline{G(z)} + e^{-2i\alpha_i} (z - \bar{z}) \overline{G'(z)}), \\ \bar{f}_{ij}^{tn} - \mathbf{i}\bar{f}_{ij}^{tt} &= -\frac{\sqrt{l_j^2 - \xi_j^2}}{2\pi i} (\overline{G(z)}(1 - 2e^{-2i\alpha_i}) - G(z) + e^{-2i\alpha_i} (z - \bar{z}) \overline{G'(z)}); \end{aligned}$$

here  $z = z(\xi_i)$  is the complex coordinate in the  $xy$  coordinate system of the point  $(\xi_i, 0)$  in crack  $i$  and

$$G(z) = \frac{1}{(z - \xi_j)\sqrt{z^2 - l_j^2}}, \quad G'(z) = \frac{l_j^2 + \xi_j z - 2z^2}{(z - \xi_j)^2 (\sqrt{z^2 - l_j^2})^3}.$$

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