

*Journal of*  
***Mechanics of  
Materials and Structures***

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FOR BCC METALS AND POSSIBLY OTHER  
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***Volume 1, N° 1***

***January 2006***

 mathematical sciences publishers



## **YIELD FUNCTIONS AND PLASTIC POTENTIALS FOR BCC METALS AND POSSIBLY OTHER MATERIALS**

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Yield functions and plastic potentials are expressed in terms of the invariants of the stress tensor for polycrystalline metals and other isotropic materials. The plastic volume change data of Richmond is used to evaluate the embedded materials properties for some bcc metals and one polymer. A general form for the plastic potential is found that is intended to represent and cover a wide range of materials types.

### **1. Introduction**

The present work is concerned with the yield functions describing the departure from ideal, linear elastic conditions, and with the plastic potentials which are used to describe the ensuing plastic flow which occurs after the yield functions have been traversed. The definitive theoretical work in this area was formalized by Hill in his early and insightful book [1950] and his many later contributions such as [Hill 1959; 1968a; 1968b; Hill and Rice 1972]. The definitive experimental work was given by Richmond and colleagues (to be cited later), based mainly upon body centered cubic (bcc) metals. The present work follows the lead of these two valuable sources. In the intervening time, most efforts to use yield functions and plastic potentials have proceeded by taking whatever forms were expedient for the particular application of immediate interest. A main objective here is to deduce general representations for yield functions and plastic potentials that have a minimum number of embedded parameters (properties) in order to have the most reasonably useful forms for application to a wide range of full density materials. The resulting forms will be evaluated for various materials types. We begin with the consideration of very ductile metals.

Face centered cubic (fcc) metals provide the backbone of ideal elastic-plastic behavior. Such metals as copper, nickel, aluminum, silver, gold and lead constitute the basis for ideal plastic flow, whether that flow be described at the dislocation level or the continuum level using so-called  $J_2$  plasticity theory. The first significant evidence for the nonideal behavior not adequately described by  $J_2$  theory is the class of bcc metals: chromium, molybdenum, tantalum, tungsten, vanadium, iron

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*Keywords:* yield functions, plastic potentials, BCC Metals.

and most steels. These bcc metals provide the perfect test bed for studying the inception of nonideal plastic effects, with the ultimate aim of generalizing beyond this class to much broader classes of materials such as polymers and ceramics.

The present work is at the macroscopic level, but it is helpful to rationalize controlling effects at a more basic level. There are at least two possible sources for the departure of most bcc metals from the ideal behavior exhibited by most fcc metals. One is the far-from-ideal form of grain boundaries on the atomic scale. This state of disorder quite naturally implies a state of nonuniformity and heterogeneity in the strength properties of grain boundaries. The other possible source of nonideal behavior for bcc metals is the fact that the core structure of dislocations spreads over many atomic layers of glide planes [Hirsch 1960; Christian 1983; Vitek 1975]. This fact greatly decreases the mobility of the dislocations, and results in a greater sensitivity to temperature (and pressure) dependent behavior. Other explanations are certainly possible for the nonideal behavior of bcc metals; dislocation dynamics studies related to these matters are rapidly evolving and likely will ultimately provide new insights. Until that time, however, only the two sources just mentioned will be further considered here.

The nonuniformity of strength of grain boundaries in fcc materials is of little importance because the great mobility of the dislocation structures implies that the loads on the grain boundaries are insufficient to cause any disruption of the grain boundary. However, in bcc metals the grain boundaries are much more highly stressed than in fcc metals. Our interest here is with initially isotropic materials, so only polycrystalline aggregates of bcc crystals will be considered. The actual behavior on the grain scale involves variability from grain to grain, and progressive and accumulating degrees of irreversible damage. Because of this variability, the slip on the grain boundaries and the slip systems within the grains may coordinate and interact in some grains. A macroscopic description is necessarily an average over all grains. Probably the grain boundary behavior is much more variable than that of the grain-to-grain form.

The grain failure itself and the grain boundary failure are not necessarily independent and competing physical events. They can be interactive with the grain boundaries, operating to some extent as slip systems in conjunction with those within the crystal. In the macroscopic view, sufficiently general descriptors must be used to cover these possibilities. Even if only shear stresses are needed for the individual crystals, both shear and normal stresses are needed for the grain boundary failure. Macroscopically this then requires both shear and normal stresses.

The behavior of the polycrystalline aggregate thus depends not only upon the shear stress on the slip planes in the individual grains, but also upon normal stresses acting within the grains and upon the grain boundaries. The corresponding macroscopic characteristics involved are the shear stresses and the mean normal stress.

For the failure of isotropic materials, we will then use the invariants that involve the shear stresses and the mean normal stress.

The formal statement of the yield function is given by

$$f(\sigma_{ij}) \leq 1.$$

The plastic potential  $G(\sigma_{ij})$  describes the plastic flow through the standard flow form

$$\dot{\varepsilon}_{ij}^p = \lambda \frac{\partial G}{\partial \sigma_{ij}}, \quad (1)$$

where the strain is decomposed into elastic and plastic components. The associative form of (1) is that which occurs when the yield function and plastic potential are taken to be identical:

$$G(\sigma_{ij}) = f(\sigma_{ij}).$$

As is well known, this ideal associative form occurs only in the extremely ductile limit for application to most fcc metals.

Some of the complications to be considered in the following work will include the following effects. All materials except the ideal case, show an asymmetry in the uniaxial tensile and compressive yield values,  $T \neq C$ . Also, all except ideal materials show a pressure dependency. Is one of these effects a fundamental effect and the other a following consequence? An answer to this question will be sought. Since the materials are initially isotropic, most approaches utilize the invariants of the stress tensor,  $I_1$ ,  $J_2$  and  $J_3$ . The first two,  $I_1$  and  $J_2$ , are commonly used, but the third invariant,  $J_3$ , is sometimes also argued to be important. The present work will seek to clarify the significance or lack thereof of  $J_3$  for use in yield functions and plastic potentials. We also will look for a unifying method by which to treat plastic potentials for materials other than bcc metals. In the present context, the term yield is interpreted to mean the stress value at the point of major deviation from the preceding linear elastic region, not at some hypothetical, initial deviation point, which can be extremely difficult to identify. We begin by considering the three standard invariants.

## 2. Invariants

Consider the eigenvalue problem used to find the principal values of the stress tensor. The notation will follow that of [Wilson 2002]. The invariants  $I_1$ ,  $I_2$  and  $I_3$  follow from the characteristic equation

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0,$$

where

$$\begin{aligned} I_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33}, \\ I_2 &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}, \\ I_3 &= \sigma_{11}\sigma_{22}\sigma_{33}. \end{aligned}$$

Alternatively, the deviatoric stress is taken as

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3}\sigma_{kk}, \quad (2)$$

and the invariants for  $s_{ij}$  are given through

$$\lambda^3 + J_1\lambda^2 - J_2\lambda - J_3 = 0,$$

where

$$\begin{aligned} J_1 &= 0, \\ J_2 &= \frac{1}{6}((\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2), \\ J_3 &= (\sigma_{11} - \sigma_m)(\sigma_{22} - \sigma_m)(\sigma_{33} - \sigma_m), \\ \sigma_m &= \frac{1}{3}I_1. \end{aligned} \quad (3)$$

The invariants for  $\sigma_{ij}$  and  $s_{ij}$  are related through

$$\begin{aligned} J_2 &= \frac{1}{3}I_1^2 - I_2, \\ J_3 &= \frac{2}{27}I_1^3 - \frac{1}{3}I_1I_2 + I_3 = -\frac{1}{27}I_1^3 + \frac{1}{3}I_1J_2 + I_3. \end{aligned} \quad (4)$$

Using the identities (4) it is completely equivalent to take the three independent invariants as either the grouping  $I_1$ ,  $I_2$  and  $I_3$  or  $I_1$ ,  $J_2$  and  $J_3$ . It is advantageous to use the latter combination since the two invariants  $J_2$  and  $J_3$  are independent of mean normal stress, which then comes in only through  $I_1$ . These invariants will be taken to be those that will be used to specify yield functions for isotropic materials.

Begin by considering yield functions for isotropic materials. The two widely recognized features of nonideal yield behavior are, first, the asymmetry in the uniaxial tensile and compressive yield values  $T$  and  $C$ , thus  $T \leq C$ . The other feature is the dependence of the yield function upon mean normal stress, in all cases except the ductile limit described by the Mises form. Consider cases where these two effects are taken to be independent of each other. In particular take cases having  $T \neq C$  but no dependence upon mean normal stress.

Consider a possible yield function of the form

$$aJ_2 + bJ_3 \leq 1, \quad (5)$$

which, because of independence of  $I_1$ , has no mean normal stress dependence. The yield stress asymmetry  $T \neq C$  can however be accommodated by (5). Evaluate  $a$

and  $b$  in (5) to give the uniaxial yield values  $T$  and  $C$ . The form that (5) then takes is

$$3 \left( \frac{1 + (C/T)^3}{1 + C/T} \right) \frac{J_2}{C^2} + \frac{27}{2} \left( \frac{-1 + (C/T)^2}{1 + T/C} \right) \frac{J_3}{C^3} \leq 1. \tag{6}$$

Now specialize (6) to a biaxial stress state with only principal stresses  $\sigma_1$  and  $\sigma_2$  but  $\sigma_3 = 0$ . Then

$$J_2 = \frac{1}{3}(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2)$$

and

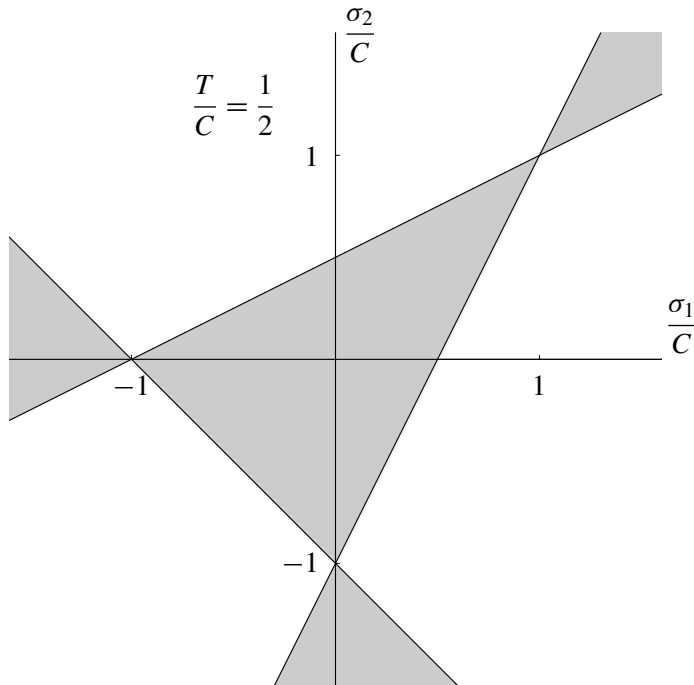
$$J_3 = \frac{1}{27}(2\sigma_1 - \sigma_2)(\sigma_1 - 2\sigma_2)(\sigma_1 + \sigma_2). \tag{7}$$

Take the particular case of  $T/C = 1/2$ , which is well within the range of possibility. Then the yield function (6) becomes

$$\frac{J_2}{C^2} + 3 \frac{J_3}{C^3} \leq \frac{1}{9}. \tag{8}$$

It can be shown analytically that the yield envelopes in the equation of (8) with (7) are the linear (line) segments as shown in Figure 1, going through the tensile and compressive values along the axes.

As seen from Figure 1 the yield function at  $T/C = 1/2$  has a strongly nonconvex character. In the limit of  $T/C = 1$ , the yield function is the Mises form with a



**Figure 1.**  $J_2$ - $J_3$  yield criterion, Equation (6).

completely convex character. But as the value of  $T/C$  diminishes, a nonconvex form develops, becoming that of Figure 1 at  $T/C = 1/2$ . The yield form shown in Figure 1 is also physically unrealistic in other respects. It predicts unlimited yield strength in a state of equibiaxial tension, as well as other nonrealistic features.

Continuing the examination of yield forms which have the strength asymmetry characteristic but no dependence upon mean normal stress leads to forms such as

$$aJ_2^{1/2} + bJ_3^{1/3} \leq 1, \quad aJ_2 + bJ_3^{1/3} \leq 1, \quad aJ_2^{3/2} + bJ_3 \leq 1.$$

All of these cases were been examined (together with Dr. A. Arsenlis) and were found to reveal the emergence of a nonconvex character somewhere in the range  $1/2 \leq T/C < 1$ . The nonconvex character results as the direct consequence of the third invariant  $J_3$ .

Now consider the opposite situation, namely, yield functions which have no strength asymmetry but do have a dependence upon mean normal stress. For example, the yield form

$$aI_1^2 + bJ_2 \tag{9}$$

does have a dependence on mean normal stress through  $I_1^2$  but it does not allow  $T \neq C$ . However, the form (9) must be excluded from consideration because it is independent of whether the mean normal stress is tensile or compressive, which is a known and strong physical effect.

The conclusion from examining these yield function cases, which are somewhat arbitrarily tailored to reflect a particular physical effect, is that this is an unproductive approach. In the next section, in connection with bcc metals data, a more organized approach to yield functions (and plastic potentials) will be taken, one which assures convexity of the related forms, and interrelates the tension-compression asymmetry and the pressure dependence.

### 3. Polynomial expansion for BCC Metals, $0 \leq \alpha \leq 1$

Both yield functions  $f(\sigma_{ij})$  and plastic potentials  $G(\sigma_{ij})$  for isotropic materials will now be considered. For either of these, perform a polynomial expansion in the invariants. Take an expansion in the invariants of the stress tensor, giving

$$F(\sigma_{ij}) = a_1 I_1 + a_2 I_1^2 + a_3 J_2 + a_4 I_1^3 + a_5 I_1 J_2 + a_6 J_3 + \dots, \tag{10}$$

where  $F()$  represents either  $f()$  or  $G()$ .

Rewrite this form explicitly designating the different possible levels of truncation through third degree terms

$$F = a_1 I_1 \left| \begin{array}{l} + a_2 I_1^2 \\ + a_3 J_2 \end{array} \right| + a_4 I_1^3 + a_5 I_1 J_2 + a_6 J_3 \left| \begin{array}{l} \\ \\ \end{array} \right| \tag{11}$$

1st degree                      2nd degree                      3rd degree



At the first-degree level there is one parameter to be evaluated, at the second-degree level there are three parameters and at the third degree level there are six parameters. Obviously the first-degree level cannot give the operative physical effects; this leaves either the second degree level or the third-degree level to be examined. It is convenient to rewrite (11) normalized by the modulus  $E$  so that the coefficients are in preferred nondimensional form. This gives

$$F = b_1 \frac{I_1}{E} + b_2 \frac{I_1^2}{E^2} + b_3 \frac{J_2}{E^2} + b_4 \frac{I_1^3}{E^3} + b_5 \frac{I_1 J_2}{E^3} + b_6 \frac{J_3}{E^3}. \quad (12)$$

Whatever the level of truncation, the form (12) gives a rationale for neglecting the terms of higher order beyond that level. Obviously the lowest degree level that can adequately reflect the physical effects of interest must be used. The second-degree level will be considered in this work. If it does not successfully capture the requisite physical effects, then the third degree level with six parameters must be considered.

Using the form (11) rather than (12) for convenience, the second degree form is

$$F = a_1 I_1 + a_2 I_1^2 + a_3 J_2. \quad (13)$$

In application to yield functions and plastic potentials, if it is assumed that there can be no plastic response under a state of purely hydrostatic compressive stresses then it can be shown that the coefficient  $a_2$  in (13) must vanish. This condition will be used in this work, leaving (13) as

$$F = a_1 I_1 + a_3 J_2. \quad (14)$$

The form (14) has a considerable history, described in [Christensen 2004].

The form (14) thus excludes the third invariant from participation in the process. It is quite logical that the third invariant not be involved with yielding and plastic flow even though it is involved in the eigenvalue problem of principal stresses. Stress is a  $3 \times 3$  matrix and the characteristic equation necessarily involves the third invariant. In the present approach, there are only two relevant stress states for isotropic materials. These are dilatation and shear, and it is these that are directly involved with the first and second invariants. The third invariant cannot be visualized as a specific and independent stress state having  $I_1 = J_2 = 0$ . For these reasons the polynomial expansion is truncated at terms of 2<sup>nd</sup> degree, not bringing in  $J_3$ . It may also be noted that the form (14) always produces a convex surface. Finally, since the expansion directly involves the mean normal stress,  $I_1$ , it is concluded that the dependence upon mean normal stress is a primary effect, and that the tension compression asymmetry is merely a consequence of that.

As shown in [Christensen 2004] the form (14) as a yield function can be written as

$$\alpha \hat{\sigma}_{kk} + \frac{3}{2}(1 + \alpha) \hat{s}_{ij} \hat{s}_{ij} \leq 1, \quad (15)$$

where the stress is nondimensionalized by the uniaxial compressive yield strength as

$$\hat{\sigma}_{ij} = \frac{\sigma_{ij}}{\kappa},$$

where

$$\kappa = C$$

and

$$\alpha = \frac{C}{T} - 1.$$

It is advantageous to use the nondimensional parameter  $\alpha$ , and in this section it will be restricted to cover the range  $0 \leq \alpha \leq 1$  which does cover most bcc metals. Either of the two properties groupings  $T$  and  $C$  or  $\alpha$  and  $\kappa$  will be used, as appropriate.

Now, using the representation (14) for the plastic potential,  $G$ , in flow form (1) gives

$$G = \beta \kappa \sigma_{kk} + \frac{3}{2} s_{ij} s_{ij}. \quad (16)$$

The 3/2 factor in (16) could be absorbed into  $\lambda$  (1) but it is retained here for scaling convenience, and parameter  $\beta$  remains to be determined from data. Comparing (15) and (16) it is seen that the associative form of the plastic potential is given by

$$\beta = \frac{\alpha}{1 + \alpha}, \quad (\text{associative}).$$

The deviatoric term in the yield function can be written in terms of stress components as

$$\hat{s}_{ij} \hat{s}_{ij} = \frac{1}{3} ((\hat{\sigma}_{11} - \hat{\sigma}_{22})^2 + (\hat{\sigma}_{22} - \hat{\sigma}_{33})^2 + (\hat{\sigma}_{33} - \hat{\sigma}_{11})^2) + 2(\hat{\sigma}_{12}^2 + \hat{\sigma}_{23}^2 + \hat{\sigma}_{31}^2). \quad (17)$$

The plastic potential is

$$G = \kappa \beta (\sigma_{11} + \sigma_{22} + \sigma_{33}) + \frac{1}{2} ((\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2) + 3(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2). \quad (18)$$

Using the flow rule (1) with (18) gives the plastic strain increments as

$$\begin{aligned} \frac{\dot{\epsilon}_{11}^p}{\lambda} &= \kappa \beta + 2\sigma_{11} - \sigma_{22} - \sigma_{33}, & \frac{\dot{\epsilon}_{22}^p}{\lambda} &= \kappa \beta - \sigma_{11} + 2\sigma_{22} - \sigma_{33} \\ \frac{\dot{\epsilon}_{33}^p}{\lambda} &= \kappa \beta - \sigma_{11} - \sigma_{22} + 2\sigma_{33} & \frac{\dot{\epsilon}_{ij}^p}{\lambda} &= 6\sigma_{ij} \quad \text{for } i \neq j. \end{aligned} \quad (19)$$

The plastic volume change is then given by

$$\dot{\varepsilon}_{kk}^p = 3\lambda\kappa\beta. \quad (20)$$

The volume change in states of uniaxial tension and compression will be used to evaluate the parameter  $\beta$  in the plastic potential (18). From the yield function (15) and (17) it is found that for uniaxial tension

$$\hat{\sigma}_{11}^T = \frac{1}{1 + \alpha}, \quad (21)$$

and for uniaxial compression

$$\hat{\sigma}_{11}^C = -1. \quad (22)$$

Using (21) and (22) in the plastic strain expressions (19) and using the volume change form (20) gives the plastic volume change normalized by the plastic strain increment in the stress direction as

$$\frac{\dot{\varepsilon}_{kk}^p}{\dot{\varepsilon}_{11}^p} = \frac{3}{1 + \frac{2}{\beta(1 + \alpha)}} \quad (\text{tension}) \quad (23)$$

and

$$\frac{\dot{\varepsilon}_{kk}^p}{|\dot{\varepsilon}_{11}^p|} = \frac{3}{1 - 2/\beta} \quad (\text{compression}). \quad (24)$$

At this point, data can be used to evaluate the parameter  $\beta$ . The carefully prepared and evaluated data of Spitzig, Sober and Richmond [Spitzig et al. 1975] on two formulations of steel will be used. The two materials types give the same results, to within experimental accuracy. Spitzig et. al. expressed the  $T/C$  asymmetry through a factor defined as the strength differential (SD). The relation between their strength differential and parameter  $\alpha$  defined above is given by

$$\alpha = \frac{SD}{1 - SD/2}. \quad (25)$$

For the two quenched and tempered (4310 and 4330) steels, the strength differential was determined to be in the range

$$SD = 0.045 - 0.065.$$

The strength differential of  $SD = 0.05$  will be used giving an  $\alpha$  value of

$$\alpha = 0.051.$$

This means that the tensile yield value  $T$  was about 5% less than the compressive value  $C$ . The experimental value for the plastic volume change normalized by the

plastic strain was

$$\frac{\dot{\epsilon}_{kk}^p}{\dot{\epsilon}_{11}^p} = 0.005,$$

with the same values for both the tension and compression cases. This very small plastic volume increase is likely due to the generation of vacancies as dislocation lines cross each other [Hull and Bacon 2001].

Using the  $\alpha$  value and the above plastic volume change value in either (23) or (24) gives  $\beta$  as

$$\beta = 0.0032.$$

The associative value for  $\beta$ ,  $\beta = \alpha/(1 + \alpha)$ , would be given by

$$\beta = 0.049 \quad (\text{associative}).$$

Thus the actual material parameter  $\beta$  in (16) is an order of magnitude less than the associative value for it would be in the case of these bcc metals.

The fact that the tensile and compressive plastic volume changes were indistinguishable experimentally is a form of partial verification for the results (23) and (24) in this range of  $\alpha$  and  $\beta$  values. [Spitzig et al. 1976] tested several more steel formulations and in all cases but one found results compatible with those just used for these first cases. Their main conclusions were that the associative flow rule is in error by about an order of magnitude for these materials. Their results are also compatible with the general forms of yield functions and plastic potentials considered here, namely (15) and (16). It can be said that these forms appear adequately to describe the plastic behavior for these bcc metals.

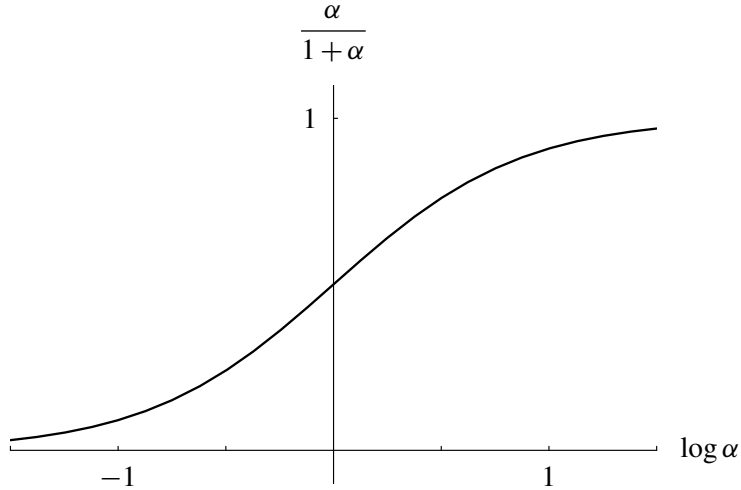
The situation as it stands at this point is that the yield function (15) is completely specified by measurements of the uniaxial tensile and compressive yield strengths for each material of interest. The plastic potential is completely specified by the evaluation of the parameter  $\beta$  in (16) for each material of interest. In the next section we consider a more general approach for specifying the plastic potential, possibly applicable to a much broader class of materials than just bcc metals.

#### 4. General materials, $0 \leq \alpha \leq \infty$

With no certainty of success, we now look for a more general and unifying approach than that of determining parameter  $\beta$  in the plastic potential (16) for each separate material of interest. First some recent results need to be assembled to approach this problem.

Write the yield function (15) in a slightly different form as

$$\frac{\alpha}{1 + \alpha} \hat{\sigma}_{kk} + \frac{3}{2} \hat{s}_{ij} \hat{s}_{ij} \leq \frac{1}{1 + \alpha}, \quad (26)$$



**Figure 2.** Dilatational term coefficient, Equation (26).

with  $\alpha$  now having the full range

$$\alpha = \frac{C}{T} - 1, \quad 0 \leq \alpha \leq \infty.$$

The value  $\alpha = 0$  is that of the perfectly ductile Mises material. The other limit  $\alpha \rightarrow \infty$  is the brittle limit, wherein the tensile yield strength is negligible compared with the compressive yield strength. This limiting case still has material integrity as opposed to that of a granular material. Free flowing granular materials as well as porous materials are separate and distinct classes from the macroscopically homogeneous materials considered here.

The coefficient  $\alpha/(1 + \alpha)$  in (26) has an interesting behavior. It is shown in Figure 2 with a log scale. The log scale is natural to use because  $\alpha$  varies from 0 to  $\infty$  and, as with spectra, this properties variation is best viewed through log scales. Note that at  $\log \alpha = 0$ ,  $\alpha = 1$  the coefficient shown in Figure 2 undergoes a transition (transition of material type) defined by the point of maximum slope (rate of change) with respect to  $\log \alpha$ . Thus, the coefficient  $\alpha/(1 + \alpha)$  in (26) determines the relative weights of the dilatational and distortional terms. Over the range of  $\alpha$ , this goes from no dilatational contribution to the yield function up to a fully interactive dilatational contribution. Then there is the transition between these two extremes at  $\alpha = 1$  for this contribution of the dilatational term.

Christensen [2004] has examined this yield/failure behavior described above and found that at the transition value of  $\alpha = 1$ , a Rankine type fracture criterion must come into effect. Thus the yield function (26) must be augmented by the explicit fracture criterion

$$\sigma_1 \leq T \quad \text{if } \alpha \geq 1, \quad (27)$$

where  $\sigma_1$  is the largest principal stress. The fracture criterion has no effect directly at  $\alpha = 1$ , but as  $\alpha$  is incrementally increased beyond that value, the fracture criterion (27) gains a gradually increasing effect, more limiting than the yield criterion (26) under some conditions. For large values of  $\alpha$  the fracture criterion can be very limiting in effect. See [Christensen 2005] for an elaboration.

With the above background, we can now proceed with the plastic potential problem. For the plastic potential form given in (16) the coefficient  $\beta$  controls the relative weight of the dilatational and distortional terms. Take  $\beta = \beta(\alpha)$  and note that it is required that

$$(i) \quad \beta = 0 \text{ at } \alpha = 0,$$

in order to be consistent with Mises behavior at that limit. Now, assume two more conditions on  $\beta(\alpha)$  in (16) and its applicability over the full range of  $\alpha$ . Take

$$(ii) \quad \beta \rightarrow A \text{ as } \alpha \rightarrow \infty,$$

where  $A$  is some nonzero constant, unknown at this point. Finally require that  $\beta(\alpha)$  have a transition (point of maximum slope versus  $\log \alpha$ ) at the same value as for the yield function, namely at  $\log \alpha = 0$ ,  $\alpha = 1$ . Thus

$$(iii) \quad \beta(\alpha) \text{ has transition at } \alpha = 1.$$

This requirement ensures compatibility between the transition locations in the yield function and the plastic potential.

A standard form for  $\beta(\alpha)$  in the plastic potential (16) would be as an expansion of the type

$$\beta = \sum_{n=1}^{\infty} A_n \left( \frac{\alpha}{1+\alpha} \right)^n.$$

The first term will be explicitly considered here, as a form that directly satisfies conditions (i), (ii) and (iii),

$$\beta = \frac{A\alpha}{1+\alpha}, \quad (28)$$

where  $A$  is a constant with respect to  $\alpha$  and to be determined. Parameter  $\alpha$  is considered to be known from the yield function. Substituting (28) into (16) gives the plastic potential as

$$G = A \left( \frac{\alpha}{1+\alpha} \right) \hat{\sigma}_{kk} + \frac{3}{2} \hat{s}_{ij} \hat{s}_{ij}.$$

Form (28) for the plastic potential in (16) is intimately related to the yield form, being directly proportional to the corresponding coefficient in (26) over the full range of  $\alpha$ . The limits on  $A$  are

$$0 \leq A \leq 1,$$

where

$$A = \begin{cases} 0 & \text{for a purely distortional plastic potential,} \\ 1 & \text{for an associative plastic potential.} \end{cases}$$

The constant  $A$  is unlikely to be universal, but the form (28) could possibly be useful as an approximation. If so, the form (28) would have considerable utility since  $\beta$  would not have to be re-determined for each different material of interest. The possible validity and usefulness of (28) will now be examined.

With the form (28), the results (23) and (24) on uniaxial tension and compression volume change for the plastic response become

$$\frac{\dot{\varepsilon}_{kk}^p}{\dot{\varepsilon}_{11}^p} = \frac{3A\alpha}{2 + A\alpha} \quad (\text{tension}) \quad (29)$$

and

$$\frac{\dot{\varepsilon}_{kk}^p}{|\dot{\varepsilon}_{11}^p|} = \frac{3A\alpha}{2(1 + \alpha) - A\alpha} \quad (\text{compression}). \quad (30)$$

The steel testing data of [Spitzig et al. 1975] given in the previous section just after Equation (25) with the values for  $\beta$  and  $\alpha$  inserted into (28) give constant  $A$  as

$$A = 0.065 = 1/15 \quad (\text{steel}). \quad (31)$$

This result means that the dilatational contribution to the plastic potential is 1/15 the size of the dilatational contribution to the yield function at the same value of  $\alpha$ .

Next, a very different type of material will be considered. Spitzig and Richmond [1979] performed tests on polyethylene, similar to those described for steel. The strength differential was determined to be

$$SD = 0.085,$$

giving  $\alpha$  as

$$\alpha = 0.089.$$

The plastic volume change was measured in uniaxial compression and tension and was found to be

$$\frac{\dot{\varepsilon}_{kk}^p}{\dot{\varepsilon}_{11}^p} = 0.011.$$

Using these data to evaluate  $\beta$  and  $A$  gives

$$\beta = 0.0067,$$

$$A = 0.083 = 1/12 \quad (\text{polyethylene}).$$

Other testing data on polycarbonate by Spitzig and Richmond [1979] were inconclusive.

Compare the  $\beta$  values for steel and polyethylene of 0.0032 and 0.0067, respectively. These differ by more than a factor of two. However, the corresponding  $A$  values of 1/15 and 1/12 are much closer together. The  $\beta$  values difference show the two materials to have strongly different behaviors, but the much smaller  $A$  value differences shows the form (28) for  $\beta(\alpha)$  to have a unifying effect.

The previous two materials examples are of a very ductile type; now an example far removed from this condition will be given, that of cast iron. For grey cast iron the yield values in [Grassi and Cornet 1949; Coffin 1950] give

$$\frac{T}{C} = \frac{1}{3}, \quad \alpha = 2.$$

Using this value of  $\alpha$  in the uniaxial compression result (30) gives

$$\frac{\dot{\epsilon}_{kk}^P}{|\dot{\epsilon}_{11}^P|} = \frac{3A}{3-A} \quad (\text{compression}). \quad (32)$$

The corresponding result for uniaxial tension is not relevant because it is excluded by the fracture criterion (27) for this value of  $\alpha$ . The same situation applies for simple shear stress, at large values of  $\alpha$  plastic flow is subsumed by the fracture criterion (27) and brittle behavior.

Unfortunately, there does not appear to be data for the plastic volume change in compression of cast iron of the quality of that of the previous two examples. An alternative approach using plastic Poisson's ratio will be given instead. Plastic Poisson's ratio is defined in the usual way as

$$\nu^P = -\frac{\dot{\epsilon}_{22}^P}{\dot{\epsilon}_{11}^P}.$$

With  $\dot{\epsilon}_{22}^P = \dot{\epsilon}_{33}^P$ , this then gives

$$\frac{\dot{\epsilon}_{kk}^P}{|\dot{\epsilon}_{11}^P|} = 1 - 2\nu^P. \quad (33)$$

Note that  $\nu^P > 1/2$  must occur to have positive plastic volume change in uniaxial compression. The elastic restrictions on Poisson's ratio do not apply here.

Equating the forms in (32) and (33) gives

$$\nu^P = \frac{1}{2} \left( \frac{3+2A}{3-A} \right), \quad \alpha = 2. \quad (34)$$

The limits of  $A$  give the values for  $\nu^P$  as

$$A = \begin{cases} 0, & \nu^P = 1/2, \\ 1, & \nu^P = 5/4. \end{cases}$$



The extremely large value of  $\nu^p$  at the associative value  $A = 1$  is completely unrealistic, again specifying extremely large plastic volume change. The value of  $A$  must be small,  $A \ll 1$ , to avoid this unrealistic behavior.

Poisson's ratio is difficult to measure accurately unless very precise volumetric measurements are done. As a first approximation, it is widely taken that the deformation is plastically not expandable,  $\nu^p = 1/2$ , which we will loosely refer to as incompressible, since that is common terminology. The error or difference for the plastic deformation to be taken as incompressible, when in fact it is slightly expandable, is given by  $\varepsilon$  as

$$\varepsilon = \frac{\nu^p - 1/2}{\nu^p}, \quad (35)$$

where  $\nu^p$  is the actual value. For example, for  $\nu^p = 0.55$  the error in assuming plastic incompressibility is 9.1%. By this method, taking realistic errors in assuming an incompressible plastic Poisson's ratio to be in the range of 5–10% can then be used to give the value of  $\nu^p$  from (35). With the  $\nu^p$  value, the corresponding constant  $A$  value follows from (34). The results, for this case of  $\alpha = 2$ , are

$$\begin{array}{rcccl} \text{Error} & 6\% & 8\% & 10\% & \\ A = & 1/16.0 & 1/11.8 & 1/9.3. & \end{array}$$

These values of  $A$  are in the same range as those found for steels and the one polymer. The corresponding  $\beta$  values are found from (28). For example for  $A = 1/14$  and at the value  $\alpha = 2$ ,  $\beta = 0.047$ . This is over an order of magnitude larger than the value found for steel of  $\beta = 0.0032$ . Thus the form (28) for  $\beta$  involving  $A$  does appear to be reasonable and realistic in going from the extremely small value of  $\alpha$  in the ductile range to an order of magnitude larger value of  $\alpha = 2$  for cast iron.

Finally, consider the limiting case  $\alpha \rightarrow \infty$ . From (30) it follows that

$$\frac{\dot{\varepsilon}_{kk}^p}{|\dot{\varepsilon}_{11}^p|} = -\frac{3A}{2-A}. \quad (36)$$

Equating (36) and (33) then gives

$$\nu^p = \frac{1+A}{2-A}, \quad \alpha \rightarrow \infty. \quad (37)$$

The associative case with  $A = 1$  gives  $\nu^p = 2$ , a wholly unrealistic condition. Again, constant  $A$  must be small. Following the method just outlined, for a 10% error in assuming  $\nu^p$  to be incompressible compared with its actual value, (35) and (37), give the corresponding  $A$  value as

$$A = \frac{1}{14.0}, \quad \alpha \rightarrow \infty. \quad (38)$$

The plastic potential in (16) has the  $A$  value in (28) to be in the general range of an order of magnitude less than one. This then is consistent with bcc behavior and polymer behavior at the ductile end of the  $\alpha$  scale and also consistent with the above reasoning for behavior in the brittle range at  $\alpha = 2$  and the brittle limit  $\alpha \rightarrow \infty$ .

## 5. Conclusions

The consequence of this assessment is that the dilatational contribution to the plastic potential (16) is found to be much less influential than the dilatational contribution to the yield function (26) over the full range of each. At first it might be surprising that the dilatational contribution to the plastic potential seems so small, by comparison. The present work suggests, however, that the inverse situation is more understandable; that is, the small dilatational contribution to the plastic potential can be reasoned, but the much stronger dilatational contribution to the yield function is the somewhat surprising effect. The plastic flow, as accessed through the plastic potential, seems quite naturally to be highly influenced by the physical effects of shearing (distortional) motion, indeed, that is the essence of ductile behavior. However, the yield function shows a strong departure from distortionally dominated effects for larger values of  $\alpha$ . The source for this effect is the emerging importance of fracture as  $\alpha$  increases. In the present context, the yield function (26) undergoes rather drastic changes of shape as  $\alpha$  increases — changes necessary to ensure compatibility with the fracture criterion (27), as evidenced by their union at  $\alpha = 1$ . This emergence of fracture modes greatly cuts down the size of the domain for plastic flow, as  $\alpha$  increases.

For the examples considered, the values of constant  $A$  in the plastic potential (16) and (28) were found to be in the general range of 1/10 to 1/20. We now inquire as to whether there is any special significance to this magnitude of  $A$  in the plastic potential. The coefficient  $A\alpha/(1 + \alpha)$  controls the size and effect of the dilatational term in the plastic potential. This coefficient at the full extent of its range,  $\alpha \rightarrow \infty$ , just becomes coefficient  $A$  itself, which is about 1/14, to take a specific value from the examples considered earlier. Thus at the limit  $\alpha \rightarrow \infty$  the plastic volume change, (36), at  $A = 1/14$  becomes

$$\dot{\varepsilon}_{kk}^p = \frac{1}{9} |\dot{\varepsilon}_{11}^p| = 0.111 |\dot{\varepsilon}_{11}^p|.$$

The plastic volume change is about 11% of the size of the imposed strain for uniaxial compressive stress, at this limit. This size for the plastic volume change is in the proper range for the effect of dilatancy in highly damaged materials. The dilatancy is most likely due to the nucleation of voids in general and void space at grain boundaries in particular materials of that type. The dilatancy is still a

significant and recognizable effect at this size, but it is not at the unrealistically large size predicted by the associative form. Alternatively, if the constant  $A$  were yet another order of magnitude smaller than the above value, the predicted dilatancy effect would be much too small to be recognizable or significant. Thus the constant  $A$  being about an order of magnitude less than one recovers the proper result for bcc metals approaching the extreme ductile range,  $\alpha \rightarrow 0$ , and it also recovers the proper dilatancy behavior approaching the opposite limit,  $\alpha \rightarrow \infty$ . If a single value for  $A$  were to be taken for general applications, the present work suggests it would be about  $1/15$  as supported by the data of Richmond and colleagues.

The plastic potential apparently would be of purely distortional form were it not for the proclivity of homogeneous materials to generate voids, requiring the small correction found here. In the very ductile range, such as with most bcc metals, the combination of a small value of  $\alpha$  and small  $A$  in (28) means that it is justified, and perhaps obvious, to approximate the plastic potential as being purely distortional, even though the yield function may not be taken to be so. For materials with values of  $\alpha$  that are not small, the smallness of constant  $A$  still provides assurance that the dilatational term in the plastic potential can be neglected in many situations. According to the preceding examples the resulting error would be of the order of constant  $A$ . Thus the present work indicates that for most homogeneous and isotropic materials (not just ductile metals) in stress states allowing plastic flow rather than brittle behavior, the plastic potential is quite well represented by the simple distortional form of (16) having  $\beta \cong 0$ , and as coordinated with the yield function (15) or (26) having interacting distortional and dilatational effects. The competitive fracture mode of failure is controlled by criterion (27). All of these forms are fully specified by two properties: the uniaxial tensile yield (or fracture) strength and the uniaxial compressive yield strength.

### Acknowledgements

I wish to thank Dr. A. Arsenlis for many spirited and helpful discussions and Dr. R. Becker for reacquainting me with the fine work of Richmond. Support from the Office of Naval Research is appreciated, Dr. Y. D. S. Rajapakse, Program Manager.

This work was performed under the auspices of the U.S. Department of Energy by the University of California, Lawrence Livermore National Laboratory under Contract No. W-7405-Eng-48.

### References

[Christensen 2004] R. M. Christensen, "A two-property yield, failure (fracture) criterion for homogeneous, isotropic materials", *J. Eng. Mater. Technol.* **126**:1 (2004), 45–52.

- [Christensen 2005] R. M. Christensen, “Exploration of ductile, brittle failure characteristics through a two-parameter yield/failure criterion”, *Mater. Sci. Eng. A* **394**:1–2 (2005), 417–424.
- [Christian 1983] J. W. Christian, “Some surprising features of the plastic deformation of body-centered cubic metals and alloys”, *Metall. Mater. Trans. A* **14A** (1983), 1237–1256.
- [Coffin 1950] L. F. Coffin, “The flow and fracture of a brittle material”, *J. Appl. Mech.* **17** (1950), 233–248.
- [Grassi and Cornet 1949] R. C. Grassi and I. Cornet, “Fracture of gray cast iron tubes under biaxial stresses”, *J. Appl. Mech.* **16** (1949), 178–182.
- [Hill 1950] R. Hill, *The mathematical theory of plasticity*, Oxford Univ. Press, Oxford, UK, 1950.
- [Hill 1959] R. Hill, “Some basic principles in the mechanics of solids without a natural time”, *J. Mech. Phys. Solids* **7**:3 (1959), 209–225.
- [Hill 1968a] R. Hill, “On constitutive inequalities for simple materials I”, *J. Mech. Phys. Solids* **16**:4 (1968), 229–242.
- [Hill 1968b] R. Hill, “On constitutive inequalities for simple materials II”, *J. Mech. Phys. Solids* **16**:5 (1968), 315–322.
- [Hill and Rice 1972] R. Hill and J. R. Rice, “Constitutive analysis of elastic-plastic crystals at arbitrary strain”, *J. Mech. Phys. Solids* **20**:6 (1972), 401–413.
- [Hirsch 1960] P. B. Hirsch, “Recorded Oral Communication”, in *5th Int. Conf. on Crystallography*, Cambridge, 1960.
- [Hull and Bacon 2001] D. Hull and D. J. Bacon, *Introduction to dislocations*, 4th ed., Butterworth–Heinemann, Oxford, 2001.
- [Spitzig and Richmond 1979] W. A. Spitzig and O. Richmond, “Effect of hydrostatic pressure on the deformation behavior of polyethylene and polycarbonate in tension and in compression”, *Polym. Eng. Sci.* **19**:16 (1979), 1129–1139.
- [Spitzig et al. 1975] W. A. Spitzig, R. J. Sober, and O. Richmond, “Pressure dependence of yielding and associated volume expansion in tempered martensite”, *Acta Metall.* **23**:7 (1975), 885–893.
- [Spitzig et al. 1976] W. A. Spitzig, R. J. Sober, and O. Richmond, “The effect of hydrostatic pressure on the deformation behavior of maraging and HY-80 steels and its implications for plasticity theory”, *Metall. Mater. Trans. A* **7A** (1976), 1703–1710.
- [Vitek 1975] V. Vitek, “Theory of the core structures of dislocations in body centered cubic metals”, *Crystal Lattice Defects* **5** (1975), 1–34.
- [Wilson 2002] C. D. Wilson, “A critical reexamination of classical metal plasticity”, *J. Appl. Mech.* **69**:1 (2002), 63–68.

Received 28 Oct 2005.

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