Journal of
Mechanics of
Materials and Structures

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Volume 1, Nº 5 May 2006

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 1, No. 5, 2006

THERMOMECHANICAL FORMULATION OF STRAIN GRADIENT PLASTICITY FOR GEOMATERIALS

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Constitutive formulation of strain gradient plasticity for geomaterials via a thermomechanical approach is investigated in this paper. It is demonstrated that, by defining two thermodynamical potentials (a free-energy function and a rate of dissipation function), the entire constitutive behavior of a decoupled strain-gradient-dependent material may be determined. The elastic relations are dependent on the free-energy function, while the plastic yielding and flow rule are determined by the dissipation function in conjunction with the free-energy function. Yield surfaces in both dissipative stress and true stress spaces may be derived without difficulty. Nonassociative flow rules and possible micromechanical mechanisms for the difference between plastic work and rate of plastic dissipation are interpreted for gradient-dependent materials. Using the obtained formulations and choosing appropriate thermodynamical functions, a wide variety of strain gradient plasticity models in the literature are recovered. Typical features associated with geomaterials, such as pressure and Lode-angle dependency, are addressed in detail. This paper provides a general thermodynamically-consistent framework of developing strain gradient plasticity models for geomaterials.

1. Introduction

A large number of microscale experiments on metallic materials have been done in recent years and have demonstrated strong size effects in solids (see [Fleck et al. 1994; Nix and Gao 1998; Tsagrakis and Aifantis 2002] and references therein). In most cases, the introduction of one or more gradient-dependent internal length scales of the deformation field is necessary to qualitatively and quantitatively interpret the behavior of size effects. Classical continuum theories fail to address the problem due to their local assumptions and lack of length scale(s) in the constitutive descriptions. The same reason accounts for their inability to determine the shear band size and describe the post-localization behavior in localization problems, which are frequently observed in metals and geomaterials. Modeling of strain localization by classical theories may generally lead to such consequences as the loss of ellipticity for the governing equations and spurious mesh-dependency problems in computation. In fact, length scales exist prevalently in a material and the loads applied to it, in the form of either the characteristic size of the homogeneous deformation domain (*internal length scale*), or the wavelength of a harmonic external load (*external characteristic length scale*) [Eringen and Kafadar 1976]. The internal length scale is closely related to the microscopically discontinuous structures or particles underlying a macroscopically continuous material body, such as the atomic lattice spacing in crystals, the grain size of polycrystals, and the grain diameter in granular materials, etc., whereas the external length scale is dependent on the properties of a loading force. The application regime of classical continuum theories is actually bounded

Keywords: strain gradient plasticity, thermomechanical approach, pressure dependency, lode-angle dependency, geomaterials.

by the ratio of the external characteristic length scale over the internal length scale. When this ratio is much larger than 1, the details of the interactions between the microstructures can be neglected, and classical local continuum theories may be successfully applied. However, when this ratio approaches 1, the influence of microstructures is no longer negligible and classical models become inadequate to account for the mechanical behavior. Consequently gradient-enhanced generalized continuum theories are needed to take microstructural effects into account [Chambon et al. 2004].

A variety of gradient-enhanced theories have thus been proposed in the literature to address the problems mentioned above associated with classical theories and to account for the influence of microstructures, such as the Cosserat (micropolar) continuum theory [Cosserat and Cosserat 1909], the theory of micromorphic continua [Toupin 1962; Mindlin 1964; 1965; Germain 1973], the nonlocal gradient plasticity pioneered in [Aifantis 1984; 1987; Zbib and Aifantis 1988b; 1988a] and Fleck and Hutchinson's [1993; 1997] flow theory of gradient plasticity and its further extensions. For a comprehensive review of gradient theories, see [Chambon et al. 2004; Voyiadjis and Abu Al-Rub 2005]. In general, the existing gradient theories typically include higher-order gradient terms with coefficients that represent length-scale measures of microstructural deformation in the constitutive equations, and are mostly employed to investigate metal plasticity behavior. Compared with metals, the gradient-dependent characteristics in geomaterials are also evident. Geomaterials often contain microstructures such as grains, microvoids and microcracks. The basic micromechanical particles in geomaterials are much larger than the corresponding particles in metals, so microstructure is of much greater significance to continuum models of geomaterials than it is to metals. Higher gradient theories with internal length scales that relate the microstructures in geomaterials with the macroscopic mechanical behavior are thus necessary. The gradient dependency in geomaterials is also experimentally evident. For example, the experimental observation of localization phenomena in granular soils has indicated that the deformed material is usually characterized by strong spatial density variation. Formation of shear bands, a finite-thickness material zone with increased porosity, has been observed in sands [Vardoulakis and Graf 1985]. In such a region of strong spatial variation, higher gradients of appropriate physical properties of the material should be of vital importance. In addition, many other features of geomaterials can be attributed to micromechanical behaviors of structures on the microscale. By using gradient-dependent constitutive models with internal length scales, the influence of these microstructures may be appropriately addressed.

However, due to the distinctive features exhibited by geomaterials in contrast with metals, direct application of the gradient theories developed for metals to geomaterials may result in great deviations. Typical geomaterials like rocks, soils and some other granular materials have long been recognized to be associated with special properties and mechanical behaviors, such as the property of multiphase media, nonlinear elasticity, isotropic hardening behavior, pressure-sensitive frictional behavior, plastic volumetric changes during plastic loading course, nonassociated flow laws, and progressive strength degradation. In developing gradient models for geomaterials, these features have to be addressed.

There have been a limited number of gradient-enhanced constitutive models specially developed for and applied to geomaterials. In this connection, micropolar Cosserat continuum theories have long been applied to granular materials [Mühlhaus 1986; Mühlhaus and Vardoulakis 1987; Pijaudier-Cabot and Bazant 1987] by including microrotational terms into constitutive formulations. However, introducing only microrotational terms in a micropolar continuum is not sufficient since it cannot describe the dilative deformation in a shear band or other localized bifurcation modes such as compaction bands

[Olsson 1999]. Vardoulakis and Aifantis [1989; 1991; 1994] extended the gradient plasticity model to account for frictional/dilatant materials by incorporating the second-order strain gradients into the flow rule, yield condition, and/or dilatancy condition. The principle of virtual work is used to derive the appropriate extra boundary conditions. Mühlhaus and Aifantis [1991] derived a variational formulation for the Aifantis' gradient plasticity in connection with shear band analysis. Fremond and Nedjar [1996] proposed a combined gradient and rate-dependent damage model for quasibrittle materials. Chambon et al. [1996; 1998; 2001; 2004] developed a single and multimechanism flow theory of gradient plasticity for geomaterials as a possible approach to the analysis of localized failure, based on the framework of [Germain 1973]. Matsushima et al. [2000; 2002] further presented a finite deformation version of their theory. Aifantis et al. [1999] and Oka et al. [2002] proposed a gradient-dependent viscoplastic constitutive model for water-saturated clay, with gradients of the volumetric viscoplastic strain being included in the constitutive equations, while di Prisco et al. [2002] modified a preexisting elastoviscoplastic model for granular soils according to gradient and nonlocal approaches. Based on the framework of [Mindlin 1964; Fleck and Hutchinson 1993; 1997], Zhou and coworkers [Zhou et al. 2002; Zhao et al. 2005] developed a strain-gradient-enhanced damage model for rock-like geomaterials with application to shear band analysis. [Zhao and Sheng 2006] further constructed a framework of strain gradient plasticity by an internal-variable approach with normality structures to account for the influence of microstructures on the overall macroscopic mechanical behavior of gradient-dependent solids. Effects of strain gradient have also been used to account for the friction-controlled interparticle slip and particle rearrangement for granular materials [Valanis 1996; Walsh and Tordesillas 2004].

In developing the above gradient-dependent constitutive relations for geomaterials, a general routine for constitutive development has been followed by proposing observation-based constitutive relations first and then imposing the laws of thermodynamics on these relations. However, the modern theory of thermomechanics, as expounded, for example, in [Ziegler 1975; 1983; Houlsby 1981; Maugin 1992; Coussy 1995; Collins and Houlsby 1997; Rajagopal and Srinivasa 1998a; 1998b; Houlsby and Puzrin 2000; Puzrin and Houlsby 2001], develops constitutive models by first guaranteeing the fulfillment of these laws. Additional internal variables other than the general state variables such as plastic strain or generalized stress variables will then be used to characterize the material behavior. It is suggested that the total constitutive relations be determined by merely two thermodynamic potentials: a specific free-energy function and a dissipation function. This theory places strong emphasis on the use of internal variables to describe the past history of the material. The first and second laws of thermodynamics are enforced in the formulation to accommodate the requirements for most of the constitutive models. This theory has also been systematically applied to soil mechanics modeling; see [Houlsby 1981; 1982; Collins 2005] and references therein). In particular, the thermomechanical framework has been proven to be useful in accommodating models for geotechnical materials which generally exhibit nonassociated plastic flow behavior, pressure-sensitivity, nonlinear elasticity and dilatancy. A variety of plasticity models like the Drucker–Prager criterion and critical state models have been recovered and extended.

Even though arguments still exist on some issues, this thermomechanical approach has shown great generality in constitutive developments with solid thermodynamic considerations. In recognition of the advantages associated with this approach, we will apply it to develop a framework of gradient theory for geomaterials, covering some of the important features of geomaterials associated with the influence of microstructures. This theory is essentially a continuum one in that the discrete granular structure for granular materials is not specially treated. However, by taking into account gradient terms and introducing a length scale, the microstructures in the materials as well as the microscale inhomogeneities may be appropriately reflected. It is demonstrated that by the thermomechanical formulations developed in the next section, a large range of gradient-dependent constitutive models existing in literature may be recovered and special features of geomaterials may be conveniently addressed.

Due to the introduction of gradient terms, some second-order tensors that correspond to microlevel strains, as well as third-order tensors corresponding to microlevel strain gradients, may be chosen as state variables. However, it should be noted that, because of the dependence of elastic compliances on the stresses and internal variables for some materials ('*coupled*', as will be noted subsequently), the strain rates and strain gradient rates may be dependent on the current values of the microdeformations within the material body, such that their integrations, which make up the total strains and strain gradients, are generally path-dependent. In this case, elastic and plastic strains and strain gradients are no longer appropriate for selection as independent state variables. The only exception is for *decoupled* materials where the elastic compliances are independent of the internal variables. Such materials include most ductile metals and the critical state models with linear relationships in the double logarithmic $\ln e$-$\ln p$ space [Lubliner 1972; Collins and Houlsby 1997]. To avoid the excessive complication of the coupled case for materials, we hereafter restrict our discussions to *decoupled materials only*, for which the free-energy function may be defined by two separate terms that characterize the elastic free energy and stored plastic energy components. In this case, the elastic and plastic terms for both strain and strain gradient may be regarded as state variables.

2. Thermomechanical formulation for gradient-dependent geomaterials

2.1. ***Development of constitutive relations.*** In the following formulation, isothermal, small strain deformation and rate-independent processes are assumed. The mechanical behavior of a material is assumed to be determined once the (Helmholtz) free-energy function Ψ and the rate of dissipation function $\hat{\Phi}$ are specified. The former represents the stored energy, the latter the rate at which energy is being converted irreversibly into heat. Both functions are defined for unit volume. Since the dissipation rate is generally path dependent, it is different from the pure time-rate as expressed by $\dot{\Phi}$. Following the framework of strain gradient theory in [Mindlin 1964; Germain 1973; Fleck and Hutchinson 1993; 1997], the mechanical behavior of the material is assumed to be described by the strain $\boldsymbol{\varepsilon}$ and strain gradients $\boldsymbol{\eta}$, and their respective work-conjugate thermodynamic forces: Cauchy stress $\boldsymbol{\sigma}$ and higher-order stress $\boldsymbol{\tau}$. The first law of thermodynamics states that a variation in the free energy is equal to the variation in the work done on the unit domain and the heat flux into the domain such that

$$\sigma_{ij}\dot{\varepsilon}_{ij} + \tau_{ijk}\dot{\eta}_{ijk} = \dot{\Psi} + \hat{\Phi}, \tag{1}$$

where the free energy is a state function dependent on a set of state variables that can describe the past history of the material: $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_n)$. Here the internal variable $\boldsymbol{\alpha}_i$ $(i = 1, \cdots, n)$ may be in the form of a scalar, vector, or tensor (of second order or higher). The generalized n internal variables in tensor form, as stated in [Collins and Houlsby 1997] and [Puzrin and Houlsby 2001], preserve the ability to model compression and shear effects separately in granular geomaterials, model anisotropy features through use of multiple kinematic hardening laws, and describe the past history of an elastic-plastic material. A more general form of the free-energy function has been suggested by Rice [1971] to be a functional of the time history of the internal variables. The Helmholtz free-energy function $\Psi = \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\eta}, \boldsymbol{\alpha})$ is expressed in the compound space of strain and strain gradients. If expressed in an alternative space of Cauchy stress and higher-order stress, a Gibbs free energy function $\Omega = \Omega(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\alpha})$ may be found, related to the Helmholtz free-energy function by an appropriate Legendre transformation, such as

$$\Omega = \Omega(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\alpha}) = \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\eta}, \boldsymbol{\alpha}) - \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \boldsymbol{\tau} \,\vdots\, \boldsymbol{\eta}. \tag{2}$$

The rate of dissipation function is assumed to be dependent on dissipative internal variables $\boldsymbol{\alpha}$ as well as their rates $\dot{\boldsymbol{\alpha}}$. For rate independent materials, $\hat{\Phi}$ is a homogeneous first-degree functional in the space of $\dot{\boldsymbol{\alpha}}$. For purely frictional materials, the strength parameters are dimensionless frictional angles, which is different from the yield stress for metals. It is thus necessary to include some component(s) of the current stress and/or higher-order stress in the expression for the rate of dissipation function [Collins and Kelly 2002; Collins 2005]. In consequence, the rate of dissipative function for gradient-dependent materials is assumed to have an expression of this form: $\hat{\Phi} = \hat{\Phi}(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})$.

The key assumption of the thermomechanical approach for constructing constitutive relations is that they are fully determined by two functions: a thermodynamic potential, such as the Helmholtz free-energy function, or any of the related potentials derived from Legendre transformations, and the rate of dissipation function. Here it is assumed that all aspects of the isothermal constitutive behavior of such rate-independent gradient-enhanced materials may be uniquely defined by the two functions $\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\eta}, \boldsymbol{\alpha})$ and $\hat{\Phi}(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})$. Thus Equation (1) may be rewritten as

$$\sigma_{ij}\dot{\varepsilon}_{ij} + \tau_{ijk}\dot{\eta}_{ijk} = \frac{\partial \Psi}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial \Psi}{\partial \eta_{ijk}}\dot{\eta}_{ijk} + \left(\frac{\partial \Psi}{\partial \boldsymbol{\alpha}_k} \otimes \dot{\boldsymbol{\alpha}}_k + \frac{\partial \hat{\Phi}}{\partial \dot{\boldsymbol{\alpha}}_k} \otimes \dot{\boldsymbol{\alpha}}_k\right),$$

where Euler's theorem for homogeneous functions has been used to rewrite $\hat{\Phi}$, and where $\otimes$ denotes an appropriate inner tensor operator according to the order of the internal variables. For example, if the second-order tensor is adopted for the internal variables, $\otimes$ implies the dyadic inner production operator '$\boldsymbol{:}$'. It is then simple to obtain

$$\sigma_{ij} = \frac{\partial \Psi}{\partial \varepsilon_{ij}}, \qquad \tau_{ijk} = \frac{\partial \Psi}{\partial \eta_{ijk}}, \qquad \boldsymbol{q}^k = -\frac{\partial \Psi}{\partial \boldsymbol{\alpha}_k} = \frac{\partial \hat{\Phi}}{\partial \dot{\boldsymbol{\alpha}}_k} \tag{3}$$

and

$$\hat{\Phi} = \boldsymbol{q}^k \otimes \dot{\boldsymbol{\alpha}}_k \text{ for } \dot{\boldsymbol{\alpha}}_k \neq 0, \tag{4}$$

where the Cauchy stress $\boldsymbol{\sigma}$ and the higher-order stress $\boldsymbol{\tau}$ are sometimes called quasiconservative stresses [Ziegler and Wehrli 1987], which are conjugate with the state variables of strains and strain gradients, and the dissipative stresses $\boldsymbol{q}^k$ are the work conjugates of the internal variables. The dissipation surfaces represented by $\hat{\Phi}$ in the (compound) space of $\dot{\boldsymbol{\alpha}}_k$ are star-shaped with respect to the origin and convex, and the dissipative stresses $\boldsymbol{q}^k$ lie in their outward normal [Ziegler and Wehrli 1987]. In deriving these relations, a weak form of Ziegler's orthogonality hypothesis has been used.

Recalling the Gibbs free-energy function defined in Equation (2) and the results obtained in (3) and (4), one may easily obtain

$$\sigma_{ij} = \frac{\partial \Psi}{\partial \varepsilon_{ij}}, \quad \eta_{ijk} = \frac{\partial \Psi}{\partial \tau_{ijk}}, \quad \boldsymbol{q}^k = \frac{\partial \Omega}{\partial \boldsymbol{\alpha}_k}. \tag{5}$$

Hence decomposition of the rates of strain and strain gradient leads to

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij}, \quad \dot{\eta}_{ijk} = \dot{\eta}^e_{ijk} + \dot{\eta}^p_{ijk},$$

where

$$\dot{\varepsilon}^e_{ij} = \frac{\partial^2 \Omega}{\partial \sigma_{ij} \partial \sigma_{kl}} \dot{\sigma}_{kl}, \qquad \dot{\eta}^e_{ijk} = \frac{\partial^2 \Omega}{\partial \tau_{ijk} \partial \tau_{lmn}} \dot{\tau}_{lmn}, \tag{6}$$

$$\dot{\varepsilon}^p_{ij} = \frac{\partial^2 \Omega}{\partial \sigma_{ij} \partial \boldsymbol{\alpha}_k} \otimes \dot{\boldsymbol{\alpha}}_k, \quad \dot{\eta}^p_{ijk} = \frac{\partial^2 \Omega}{\partial \tau_{ijk} \partial \boldsymbol{\alpha}_k} \otimes \dot{\boldsymbol{\alpha}}_k, \tag{7}$$

denote the elastic and plastic rates for strains and strain gradients, respectively. The coefficients before $\dot{\sigma}_{kl}$ and $\dot{\tau}_{lmn}$ in Equation (6) are the instantaneous elastic and higher order elastic compliances for the material. They are generally dependent on the Cauchy stresses, higher-order stresses and on the internal variables as well. Total elastic and plastic parts of strains and strain gradients may be obtained by integrations of the expressions (6) and (7) over the entire loading course.

As far as decoupled materials are concerned, we further choose ε^p_{ij} and η^p_{ijk} to be the internal variables. A general form of free-energy function depends on both the elastic and plastic strains and strain gradients like this: $\Psi = \Psi(\varepsilon^e_{ij}, \varepsilon^p_{ij}, \eta^e_{ijk}, \eta^p_{ijk})$. For the decoupled case, it can be written as the sum of an elastic term $\Psi^e(\varepsilon^e_{ij}, \eta^e_{ijk})$, which depends on the elastic strains and elastic strain gradients only, and a plastic term $\Psi^p(\varepsilon^p_{ij}, \eta^p_{ijk})$, which depends only on the plastic strains and plastic strain gradients. That is,

$$\Psi(\varepsilon^e_{ij}, \varepsilon^p_{ij}, \eta^e_{ijk}, \eta^p_{ijk}) = \Psi^e(\varepsilon^e_{ij}, \eta^e_{ijk}) + \Psi^p(\varepsilon^p_{ij}, \eta^p_{ijk}). \tag{8}$$

As ε^p_{ij} and η^p_{ijk} have been chosen as the internal variables, from Equation (3), their thermodynamic conjugate dissipative stresses σ^d_{ij} and τ^d_{ijk} may be written as

$$\sigma^d_{ij} = \frac{\partial \hat{\Phi}}{\partial \dot{\varepsilon}^p_{ij}}, \quad \tau^d_{ijk} = \frac{\partial \hat{\Phi}}{\partial \dot{\eta}^p_{ijk}}. \tag{9}$$

With the preceding assumptions, the instantaneous elastic moduli for both strains and strain gradients are independent of the plastic strains and plastic strain gradients. Consequently, the total work rate may be decomposed into two parts:

$$\hat{W} = \hat{W}^e + \hat{W}^p, \tag{10}$$

where $\hat{W}^e = \dot{\Psi}^e(\varepsilon^e_{ij}, \eta^e_{ijk})$ and $\hat{W}^p = \dot{\Psi}^p(\varepsilon^p_{ij}, \eta^p_{ijk}) + \hat{\Phi}$.

We have

$$\dot{\Psi}^e(\varepsilon^e_{ij}, \eta^e_{ijk}) = \sigma_{ij}\dot{\varepsilon}^e_{ij} + \tau_{ijk}\dot{\eta}^e_{ijk};$$

thus

$$\sigma_{ij} = \frac{\partial \Psi^e}{\partial \varepsilon^e_{ij}}, \quad \tau_{ijk} = \frac{\partial \Psi^e}{\partial \eta^e_{ijk}}. \tag{11}$$

The last several equations since Equation (8) lead to the relations

$$\sigma_{ij} = \rho_{ij} + \sigma^d_{ij}, \quad \tau_{ijk} = \pi_{ijk} + \tau^d_{ijk}, \tag{12}$$

where

$$\rho_{ij} = \frac{\partial \Psi^p}{\partial \varepsilon^p_{ij}} \qquad \text{and} \qquad \pi_{ijk} = \frac{\partial \Psi^p}{\partial \eta^p_{ijk}} \tag{13}$$

are the shift stress and higher-order shift stress, which may present sound physical interpretations in developing anisotropic, kinematic hardening gradient models.

As we can see, several stress measures have been defined in the course of this section, including Cauchy stress, higher-order stress, dissipative stress, shift stress and higher-order shift stress. It is necessary to clarify these stresses here. First, as is well known, in conventional theories, *Cauchy stress* refers to the stress term at infinitesimal deformation, and is used to express Cauchy's first and second laws of motion (for the latter case, together with couple stress). Cauchy stress denotes force per unit area of the deformed solid. Other definitions of stress also appear in constitutive descriptions, including Kirchhoff, nominal (First Piola–Kirchhoff) and Material (Second Piola–Kirchhoff) stress tensors. The latter three stress measures consider forces acting on the undeformed solid, and require knowledge not only of the behavior of the deformed state, but also of the predeformation state. For a problem involving infinitesimal deformation, all the aforementioned stress measures are equal. However, here in this paper, we use *Cauchy stress* to represent the usual stress as in conventional theory, merely in distinguishing from the higher-order stresses that are introduced afterward, not distinguishing it from the aforementioned Kirchhoff and other stresses. In addition, the assumption of small strain made in the beginning of this section also cancels the differences between Cauchy stress and the other three terms.

Second, the term *higher-order stress* follows the usage of [Toupin 1962; Mindlin 1964; 1965; Fleck and Hutchinson 1997], and addresses the additional stresses required by strain gradient theories in addition to Cauchy stress. This higher-order stress is a more general stress that includes the *couple stress* as a subset case, as it may address both cases of rotation gradient and stretch gradient, while couple stress applies to rotation gradient case only. The third point regards *dissipative stress*. In passing, the dissipative stresses are defined as the conjugates of internal variables [Ziegler and Wehrli 1987]. In conventional thermomechanics, it may denote a vector, a second order tensor such as Cauchy stress, or a set of such vectors and tensors. While in the framework of strain gradient theory, third-order tensors may also appear (as is shown in Equation (9)), depending on the choice of internal variables. The occurrence of shift stress and higher-order shift stress is due to the nonidentity of true compound stress space (constituted by Cauchy stress and higher-order stress) and compound dissipative stress space (constituted by usual dissipative stress and higher-order dissipative stress). More information regarding this point can be found in [Collins and Houlsby 1997].

Plastic work and plastic dissipation. The plastic work rate and the rate of dissipation are generally different from each other, contrary to the long-held understanding in soil mechanics [Collins and Kelly 2002; Collins 2005]. The second law of thermodynamics states that the rate of dissipation can never be negative, but the sign of the plastic work increment is not restricted [Mroz 1973; Lubliner 1990]. Thus, the two rates are not equal to each other in the general case. The difference between them was called the *stored plastic work*, and was thought to be related to the frozen elastic energy on the microscale. As for strain gradient characterized materials, the plastic work and dissipation rates are given by

$$\begin{aligned}\hat{W}^p &= \sigma_{ij}\dot{\varepsilon}^p_{ij} + \tau_{ijk}\dot{\eta}^p_{ijk} = \sigma_{ij}\frac{\partial^2\Omega}{\partial\sigma_{ij}\partial\boldsymbol{\alpha}_k}\otimes\dot{\boldsymbol{\alpha}}_k + \tau_{ijk}\frac{\partial^2\Omega}{\partial\tau_{ijk}\partial\boldsymbol{\alpha}_k}\otimes\dot{\boldsymbol{\alpha}}_k,\\ \hat{\Phi} &= \boldsymbol{q}^k\otimes\dot{\boldsymbol{\alpha}}_k = \frac{\partial\Omega}{\partial\boldsymbol{\alpha}_k}\otimes\dot{\boldsymbol{\alpha}}_k.\end{aligned}\tag{14}$$

For a decoupled material, the plastic strain ε^p_{ij} and plastic strain gradient η^p_{ijk} are chosen to be the internal state variables, with

$$\sigma^d_{ij} \quad \text{and} \quad \tau^d_{ijk}$$

being their conjugate dissipative stresses. The difference between the plastic work rate and dissipation rate is

$$\hat{W}^s = \hat{W}^p - \hat{\Phi} = \left(\sigma_{kl}\frac{\partial^2\Omega}{\partial\sigma_{kl}\partial\varepsilon^p_{ij}} - \sigma^d_{ij}\right)\dot{\varepsilon}^p_{ij} + \left(\tau_{lmn}\frac{\partial^2\Omega}{\partial\tau_{lmn}\partial\eta^p_{ijk}} - \tau^d_{ijk}\right)\dot{\eta}^p_{ijk},$$

where $\hat{W}^s$ is the stored plastic work rate for the gradient-dependent material. For the fully decoupled case, $\hat{W}^s = \dot{\Psi}^p$, or alternatively, $\hat{W}^p = \dot{\Psi}^p + \hat{\Phi}$. From a micromechanical point of view, the stored plastic work in gradient-dependent materials may depend on the microscopic material properties as well as the imposed deformation state. As discussed in [Benzerga et al. 2003], for brittle or quasibrittle materials that may be characterized by single crystals, the behavior of stored plastic work during unloading processes is essentially controlled by the density of statistical stored dislocations (SSDs) as well as geometrical necessary dislocations (GNDs), of which the latter have long been attributed to the micromechanical origin of strain gradient effects. As for granular Cosserat materials, part of plastic work may be used to drive the interparticle rotation, and upon unloading, part of the rotation deformation remains unrecovered. The unrecoverable plastic work associated with this rotation part together with the frozen part contributed from strains makes up the total stored plastic work for granular Cosserat materials.

It is easy to see, for arbitrary plastic deformation increments, that the following conditions hold if and only if $\hat{W}^s$ becomes zero:

$$\sigma^d_{ij} = \sigma_{kl}\frac{\partial^2\Omega}{\partial\sigma_{kl}\partial\varepsilon^p_{ij}}, \qquad \tau^d_{ijk} = \tau_{lmn}\frac{\partial^2\Omega}{\partial\tau_{lmn}\partial\eta^p_{ijk}}.\tag{15}$$

Taking (5) into account, Equation (15) may be rewritten as

$$\boldsymbol{q}^k = (\boldsymbol{q}_\sigma, \boldsymbol{q}_\tau), \qquad \boldsymbol{q}_\sigma = \frac{\partial\boldsymbol{q}_\sigma}{\partial\sigma_{ij}}\sigma_{ij}, \qquad \boldsymbol{q}_\tau = \frac{\partial\boldsymbol{q}_\tau}{\partial\tau_{ijk}}\tau_{ijk},\tag{16}$$

where $\boldsymbol{q}_\sigma = \partial\Omega/\partial\boldsymbol{\alpha}_\varepsilon$, $\boldsymbol{q}_\tau = \partial\Omega/\partial\boldsymbol{\alpha}_\eta$. Equation (16) implies that $\boldsymbol{q}^k$ are homogeneous functions of degree one in the compound space of Cauchy stress σ_{ij} and higher-order stress τ_{ijk}. In consequence, the rate of dissipation function $\hat{\Phi}$ is also a homogeneous first degree-function of degree one in the compound space of Cauchy stress and higher-order stress, such that the material may be called a *purely frictional material*. For a purely frictional material, there is no stored plastic work frozen in the elastic energy, and thus the plastic work is totally dissipated. Yield loci of such models always exhibit convex cones in three dimensions, as will be demonstrated in the next section for such criteria as the generalized Coulomb, von Mises, Drucker–Prager and Matsuoka–Nakai criteria in the framework of the strain gradient theory. However, *frictional materials* refer to those whose dissipation function depends not only on the Cauchy stress and higher-order stress, but also on the function(s) of the plastic deformations (such as the consolidation pressure in soil mechanics). In conventional soil mechanics, these materials are generally treated by critical state models with typical yield loci of homothetic curves or surfaces, as in [Collins and Kelly 2002]. It will also be shown in the following sections, in the compound stress space of σ_{ij} and τ_{ijk}, that the yield loci of gradient-dependent frictional materials also exhibit a homothetic feature. A third category of materials is termed as *quasifrictional materials*, which, in addition to the true stresses, depends on material parameters that involve stresses (such as cohesion as for structured soils, or fracture toughness as in particle crushing).

Nonassociative flow rule. The nonassociated flow rule is a distinguished characteristic of geomaterials. In conventional plastic theory, the dissipation function is generally assumed to depend on the plastic strain and the plastic strain rate. Consequently, the assumption that the associated flow rule is normal to the yield surface is frequently made in this theory. However, geomaterials generally exhibit a quite different behavior when the direction of the plastic strain rate vector is not normal to the current yield surface. The most used associated flow rule for metals has been proved to be inadequate for geomaterials. Models with nonassociated flow rules have thus been developed in terms of separate yield functions and plastic potentials [Oda and Iwashita 1999; Lade 1988; Vardoulakis and Sulem 1995]. However, the general theory of thermomechanics can still be applied with necessary nonassociated flow rules whenever the dissipation potential depends on the current stress or on the total strain or elastic strain. It remains true for the case of frictional gradient-dependent geomaterials. As will be shown in the following, whenever the dissipation function has an explicit dependency on the true stresses, the normality property of flow rules in the dissipative stress space will be lost when being transferred to true stress space.

We follow the same procedure used in [Collins 2005] to derive the nonassociative flow rule for gradient-dependent geomaterials. As has been stated in Equation (3), the dissipative stresses $\boldsymbol{q}^k$ are expressed as the derivative of $\hat{\Phi}$ with respect to $\dot{\boldsymbol{\alpha}}_k$. It is assumed a dual function $\tilde{F}(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\alpha}, \boldsymbol{q})$ may be found by a Legendre transformation of $\hat{\Phi}$, such that $\dot{\boldsymbol{\alpha}}_k$ may be expressed in terms of $\boldsymbol{q}^k$. As has been stated, $\hat{\Phi}$ is a homogeneous function of degree one in $\boldsymbol{\alpha}$ for rate-independent materials, since there is no characteristic time. For such a function the above transformation is singular. In such a case, the value of the Legendre dual of $\hat{\Phi}$ is identically zero [Collins and Houlsby 1997]:

$$\tilde{F}(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\alpha}, \boldsymbol{q}) = 0. \tag{17}$$

In addition, the relation between $\dot{\boldsymbol{\alpha}}_k$ and $\boldsymbol{q}^k$ is not unique. Instead of having a unique expression, the dual relation to Equation (3) may be presented in a form for the time derivatives of the internal variables as follows:

$$\dot{\boldsymbol{\alpha}}_k = \dot{\lambda} \frac{\partial \tilde{F}(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\alpha}, \boldsymbol{q})}{\partial \boldsymbol{q}^k}. \tag{18}$$

Equations (17) and (18) present a yield condition and associated flow rule in generalized stress space (or alternatively dissipative stress space). And the yield function in Equation (17) displays an obvious dependence on the true stresses $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, which is inherited from the original form of $\hat{\Phi}$ and is always necessary for frictional materials. In true compound stress space of $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, the yield condition may be obtained as follows:

$$F\big(\boldsymbol{\sigma},\boldsymbol{\tau},\boldsymbol{\alpha}_k\big)=\tilde{F}\left(\boldsymbol{\sigma},\boldsymbol{\tau},\boldsymbol{\alpha}_k,\boldsymbol{q}^k=\frac{\partial\Omega(\boldsymbol{\sigma},\boldsymbol{\tau},\boldsymbol{\alpha}_k)}{\partial\boldsymbol{\alpha}_k}\right)=0. \tag{19}$$

As we have

$$\frac{\partial F}{\partial\boldsymbol{\sigma}}=\frac{\partial\tilde{F}}{\partial\boldsymbol{\sigma}}+\frac{\partial^2\Omega}{\partial\boldsymbol{\sigma}\,\partial\boldsymbol{\alpha}_k}\otimes\frac{\partial\tilde{F}}{\partial\boldsymbol{q}^k},\qquad \frac{\partial F}{\partial\boldsymbol{\tau}}=\frac{\partial\tilde{F}}{\partial\boldsymbol{\tau}}+\frac{\partial^2\Omega}{\partial\boldsymbol{\tau}\,\partial\boldsymbol{\alpha}_k}\otimes\frac{\partial\tilde{F}}{\partial\boldsymbol{q}^k}. \tag{20}$$

In connection with Equations (6), (18), (19) and (20), the flow rule for the plastic rates of strain and strain gradient in the true compound stress space may be obtained as

$$\dot{\varepsilon}^p_{ij}=\dot{\lambda}\frac{\partial F}{\partial\boldsymbol{\sigma}}+\frac{\partial\hat{\Phi}}{\partial\boldsymbol{\sigma}},\qquad \dot{\eta}^p_{ijk}=\dot{\lambda}\frac{\partial F}{\partial\boldsymbol{\tau}}+\frac{\partial\hat{\Phi}}{\partial\boldsymbol{\tau}}, \tag{21}$$

where

$$\frac{\partial\hat{\Phi}}{\partial\boldsymbol{\sigma}}=-\dot{\lambda}\frac{\partial\tilde{F}}{\partial\boldsymbol{\sigma}},\qquad \frac{\partial\hat{\Phi}}{\partial\boldsymbol{\tau}}=-\dot{\lambda}\frac{\partial\tilde{F}}{\partial\boldsymbol{\tau}}.$$

These relations are obtained from the Legendre transformation between $\hat{\Phi}$ and $\tilde{F}$ in connection with Equation (18). As can be seen, when $\partial\tilde{F}/\partial\boldsymbol{\sigma}$ and $\partial\tilde{F}/\partial\boldsymbol{\tau}$ are not zero, Equation (21) clearly exhibits a natural feature of nonassociated flow rule. This is an obvious advantage of thermomechanical approach for addressing geomaterial behaviour.

3. Derivation of a class of $J2$-flow theory of strain gradient plasticity for isotropic incompressible geomaterials

In this section, the thermomechanical formulations obtained in last section are employed to derive a class of strain gradient plasticity models that may be regarded as extensions of J_2-flow theory for typical pressure-insensitive materials. Further implications of the formulations, including features suitable for describing pressure-sensitive geomaterials, will be addressed in the next section. Following [Fleck and Hutchinson 1993], it is assumed that couple stresses exist in the elastic-plastic body, with elastic strain, plastic strain and Cauchy stress being ε^e_{ij}, ε^p_{ij} and σ_{ij}. The strain gradient effects are manifested in a curvature tensor in its elastic and plastic parts:

$$\chi^e_{ij}=e_{ilm}\eta^e_{mjl}\quad\text{ and }\quad\chi^p_{ij}=e_{ilm}\eta^p_{mjl},$$

where $e_{ilm}$is the permutation tensor. The couple stress is denoted by m_{ij}, which may be related to the higher-order stress τ_{ijk} defined in last section as: $m_{ij}=e_{ilm}\tau_{mjl}$. In a conventional J_2 flow theory, it is generally regarded that the plastic strain is incompressible, such that the yield condition may be defined through the von Mises effective stress by the deviatoric part of the Cauchy stress s_{ij}: $\sigma_e=\sqrt{(3s_{ij}s_{ij})/2}$. In the presence of strain gradients and couple stresses, the plastic work conjugate forces for ε^p_{ij} and χ^p_{ij} are s_{ij} and m_{ij} respectively. In addressing incompressibility in the presence of strain gradients, Smyshlyaev and Fleck [1996] (see also [Fleck and Hutchinson 2001]) have made an orthogonal decomposition of the strain gradient tensor and obtained an acute incompressible tensor for strain gradients, in which the incompressibility condition for strain gradients is expressed as: $\eta_{kjj}=0$ for $k=1,2,3$. To avoid the excessive complexity of accounting for the general case of strain gradients, however, here in this paper we assume the curvature tensor in conjunction with its work conjugate couple stress adequately fulfills the incompressibility condition for gradient effects, at least for a particular range of materials (such as Cosserat materials). In fact, due to the symmetrical nature of the strain gradient tensor, $\eta_{ijk} = \eta_{jik}$, the curvature tensor $\chi_{ii} = 0$, which implies the incompressibility of strain gradients.

Free energy function and elastic relations. We first suppose that the material can be characterized by the linear Gibbs free-energy function

$$\Omega = \tfrac{1}{2} C_{ijkl}\sigma_{ij}\sigma_{kl} + \tfrac{1}{2} M_{ijkl} l^{-2} m_{ji} m_{lk}, \tag{22}$$

where

$$C_{ijkl} = \frac{1+\upsilon}{2E}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\upsilon}{E}\delta_{ij}\delta_{kl}, \qquad M_{ijkl} = \frac{1+\upsilon}{E}\left(\frac{l}{l_e}\right)^2 \delta_{ik}\delta_{jl}, \tag{23}$$

and E and υ denote the Young's modulus and Poisson's ratio. The length scale l is interpreted as the free slip distance between statistically stored dislocations, while l_e is introduced to divide the curvature tensor χ_{ij} into elastic and plastic parts. It is assumed $l_e \ll l$ such that the dominant size effect in the material is associated with plastic rather than elastic strain gradients. It is easy to find that this definition of the free-energy function in Equations (22) and (23) results in a dependence of the instantaneous elastic modulus on the elastic state variables only, which corresponds to the special case of decoupled materials, where the shift stresses contributed from free energy are zero and the dissipative stresses coincide with the true stresses. In this case, the elastic behavior of the material may be totally determined by the Gibbs free-energy function, while the plastic behavior may be completely determined by a dissipative rate function. In view of Equation (6), we obtain from (22) the elastic incremental relations

$$\dot{\varepsilon}^e_{ij} = C_{ijkl}\dot{\sigma}_{kl}, \quad \dot{\chi}^e_{ij} = M_{ijkl} l^{-2} \dot{m}_{lk}. \tag{24}$$

Dissipation function and plastic relations. In defining a function for the rate of plastic dissipation, it is further assumed that the plastic parts of the strains and strain gradients ε^p_{ij} and χ^p_{ij}, may be regarded to be the internal variables. The following general form for the rate-dissipation function is assumed to be capable of appropriately describing the dissipative processes for a class of strain-gradient-dependent materials:

$$\hat{\Phi} = B\big(A_1(\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij}) + A_2(l^2\dot{\chi}^p_{ij}\dot{\chi}^p_{ij})\big)^{1/2}, \tag{25}$$

where the coefficients A_1, and A_2 are dimensionless constants. B is a scalar with a dimension of stress, with the possibility of being further extended to a complex function depending on plastic strains and strain gradients. Equation (25) is a generalization of the dissipation function for von Mises materials in [Ziegler and Wehrli 1987, Equation 6.2, p. 208], where the dissipation function is assumed to depend on the second invariant of the plastic strain rate only. In view of Equation (9), the dissipation stresses may be obtained by differentiating the dissipation function (25) with respect to the plastic strain increments and the curvature increments respectively, such that

$$\sigma^d_{ij} = \frac{\partial\hat{\Phi}}{\partial\dot{\varepsilon}^p_{ij}} = \frac{A_1 B^2}{\hat{\Phi}}\dot{\varepsilon}^p_{ij}, \qquad m^d_{ji} = \frac{\partial\hat{\Phi}}{\partial\dot{\chi}^p_{ij}} = \frac{A_2 B^2 l^2}{\hat{\Phi}}\dot{\chi}^p_{ij}. \tag{26}$$

The rate of dissipation function can always be written as

$$\hat{\Phi} = \sigma^d_{ij}\dot{\varepsilon}^p_{ij} + m^d_{ji}\dot{\chi}^p_{ij}. \tag{27}$$

Eliminating the rates of strain and strain gradient from Equations (26) and (27), we obtain for the yield function in dissipative stress space

$$\tilde{F} = \frac{(\sigma_{ij}^d \sigma_{ij}^d)}{A_1} + \frac{(l^{-2} m_{ij}^d m_{ij}^d)}{A_2} - B^2 = 0. \tag{28}$$

We will demonstrate here that a variety of strain gradient plasticity models proposed in the literature may be recovered by selecting different values for the coefficients A_1, A_2 and B for the dissipation function and, in conjunction, the elastic relations presented in (24), We identify several cases:

Case 1. $A_1 = A_2 = 2/3$, B is a scalar and $B \geq 0$.

In this case, the dissipative stresses are identical with the true stresses: $\sigma_{ij}^d = s_{ij}$, $m_{ij}^d = m_{ij}$. Manipulation of Equation (28) easily leads to the following yield surface in the true stress space:

$$F = \Sigma_e - B = 0, \tag{29}$$

where Σ_e is a generalized effective stress defined by

$$\Sigma_e = \sqrt{\tfrac{3}{2} s_{ij} s_{ij} + \tfrac{3}{2} l^{-2} m_{ij} m_{ij}}. \tag{30}$$

In conjunction with (26), the normality of the plastic flow rule of (29) results in

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda} \frac{\partial F}{\partial s_{ij}} = \dot{\lambda} \frac{\partial F}{\partial \Sigma_e} \frac{\partial \Sigma_e}{\partial s_{ij}}, \qquad \dot{\chi}_{ij}^p = \dot{\lambda} \frac{\partial F}{\partial m_{ij}} = \dot{\lambda} \frac{\partial F}{\partial \Sigma_e} \frac{\partial \Sigma_e}{\partial m_{ij}},$$

where $\dot{\lambda} = \dot{\Sigma}_e / h$ and the instantaneous hardening rate $h = h(\Sigma_e)$ are chosen so that the uniaxial tensile response may be reproduced. We thus obtain, for the incremental plastic strains and strain gradients,

$$\dot{\varepsilon}_{ij}^p = \frac{3}{2h} \frac{s_{ij}}{\Sigma_e} \dot{\Sigma}_e, \qquad \dot{\chi}_{ij}^p = \frac{3}{2h} \frac{l^{-1} m_{ji}}{\Sigma_e} \dot{\Sigma}_e. \tag{31}$$

In view of the elastic incremental relations defined in Equation (24), the yield surface defined in (29), and the plastic incremental parts defined in (31), one may readily find that the J_2 flow version of the [Fleck and Hutchinson 1993] strain gradient theory is fully recovered (see Equations (52), (53), (52a), (52b), (52c) and (45) therein). This case can actually be termed an *elastic-perfectly plastic gradient material*, as noted in [Ziegler and Wehrli 1987].

Suppose the principal deviator stresses and principal couple stresses are s_1, s_2, s_3 and m_1, m_2, m_3 respectively. Define

$$\begin{aligned} S_e^{(I)} &= \sqrt{(s_1^2 + l^{-2} m_1^2)}\, \mathrm{sgn}(s_1), \\ S_e^{(II)} &= \sqrt{(s_2^2 + l^{-2} m_2^2)}\, \mathrm{sgn}(s_2), \\ S_e^{(III)} &= \sqrt{(s_3^2 + l^{-2} m_3^2)}\, \mathrm{sgn}(s_3). \end{aligned} \tag{32}$$

Obviously the yield locus defined by (29) is a circle with a radius $\sqrt{6}B/3$ in the π-plane defined by $S_e^{(I)}$, $S_e^{(II)}$ and $S_e^{(III)}$, as is shown in Figure 1. The scalar B serves here as a generalized yield stress in tension, which is $\sqrt{3}$ times the generalized yield stress in pure shear. For the case being considered here, the dissipation stress space coincides with the true stress space so that the yield surfaces in the two stress spaces are identical. Based on similar formulations for the above J_2-flow theories with generalized von Mises yield criteria, other yielding criteria such as Tresca and maximum tension failure can also be formulated with ease, by defining different dissipation rate functions. For example, if we defined the rate of dissipation function as

$$\hat{\Phi} = B\left(\left|\dot{\gamma}_1^p\right| + \left|\dot{\gamma}_2^p\right| + \left|\dot{\gamma}_3^p\right|\right),$$

where $\dot{\gamma}_1^p = \dot{e}_1^p + l\dot{\omega}_1^p$, $\dot{\gamma}_2^p = \dot{e}_2^p + l\dot{\omega}_2^p$, $\dot{\gamma}_3^p = \dot{e}_3^p + l\dot{\omega}_3^p$, and $\dot{e}_i^p$ and $\dot{\omega}_i^p (i = 1, 2, 3)$ are the principal distortion strains and deviatoric curvatures. Denoting the principal deviator stresses and principal couple stresses by s_1, s_2, s_3 and m_1, m_2, m_3, we have

$$s_i = \frac{\partial \hat{\Phi}}{\partial \dot{e}_i^p} = \frac{\partial \hat{\Phi}}{\partial \dot{\gamma}_i^p} \frac{\partial \dot{\gamma}_i^p}{\partial \dot{e}_i^p} = B \operatorname{sgn}\left(\dot{\gamma}_i^p\right),$$

$$m_i = \frac{\partial \hat{\Phi}}{\partial \dot{\omega}_i^p} = \frac{\partial \hat{\Phi}}{\partial \dot{\gamma}_i^p} \frac{\partial \dot{\gamma}_i^p}{\partial \dot{\omega}_i^p} = Bl \operatorname{sgn}\left(\dot{\gamma}_i^p\right).$$

Adopting the same definition as in (32) and noting that $\operatorname{sgn}(s_i) = \operatorname{sgn}(\dot{\gamma}_i^p)$, we have

$$\begin{aligned} S_e^{(I)} - S_e^{(II)} &= B\big(\operatorname{sgn}(\dot{\gamma}_1^p) - \operatorname{sgn}(\dot{\gamma}_2^p)\big), \\ S_e^{(II)} - S_e^{(III)} &= B\big(\operatorname{sgn}(\dot{\gamma}_2^p) - \operatorname{sgn}(\dot{\gamma}_3^p)\big), \\ S_e^{(I)} - S_e^{(III)} &= B\big(\operatorname{sgn}(\dot{\gamma}_1^p) - \operatorname{sgn}(\dot{\gamma}_3^p)\big). \end{aligned}$$

The yield surface thus defined is obviously a generalized Tresca Hexagon. If B again serves as a generalized yield stress in tension, in the π-plane defined by $S_e^{(I)}$, $S_e^{(II)}$ and $S_e^{(III)}$, this hexagon circumscribes the generalized von Mises circle as shown in Figure 1. In this case B is twice the generalized yield stress in pure shear, just as in the conventional Tresca condition.

Given the J_2 flow version of strain gradient theory as listed above, the deformation version of strain gradient theory of Fleck and Hutchinson [1993] may also be easily attained. In fact, if we assume that the equivalent plastic strain has the form

$$\bar{\varepsilon}_e = \sqrt{\tfrac{2}{3}\varepsilon_{ij}^p\varepsilon_{ij}^p + \tfrac{2}{3}l_2\chi_{ij}^p\chi_{ij}^p}, \tag{33}$$

the rate of dissipation function has the expression

$$\hat{\Phi} = B\dot{\bar{\varepsilon}}_e.$$

One can easily verify that

$$s_{ij} = \frac{\partial \hat{\Phi}}{\partial \dot{\varepsilon}_{ij}^p} = \frac{\partial \hat{\Phi}}{\partial \dot{\bar{\varepsilon}}_e} \frac{\partial \bar{\varepsilon}_e}{\partial \varepsilon_{ij}^p} = \tfrac{3}{2}\frac{B}{\bar{\varepsilon}_e}\varepsilon_{ij}^p,$$

$$m_{ij} = \frac{\partial \hat{\Phi}}{\partial \dot{\chi}_{ij}^p} = \frac{\partial \hat{\Phi}}{\partial \dot{\bar{\varepsilon}}_e} \frac{\partial \bar{\varepsilon}_e}{\partial \chi_{ij}^p} = \tfrac{3}{2}\frac{B}{\bar{\varepsilon}_e}l^2\chi_{ij}^p,$$

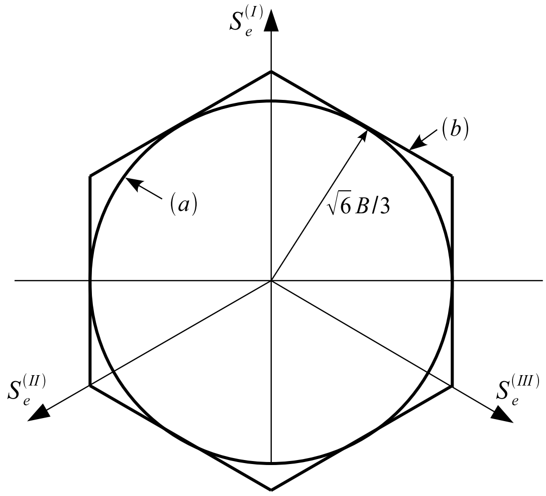

Figure 1. Yield loci of (a) the generalized von Mises and (b) generalized Tresca gradient-dependent materials in the π-plane.

where the coefficient B is actually equal to the generalized effective stress defined in Equation (30). Thus the deformation version of strain gradient theory proposed by Fleck and Hutchinson [1993] is again recovered.

For the J_2 flow version of strain gradient theory just obtained, a generalized form of Drucker's stability postulate [Drucker 1951] has been suggested by Fleck and Hutchinson when $h > 0$:

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{p} + \dot{m}_{ij}\dot{\chi}_{ij}^{p} \geq 0.$$

This result was formerly developed by Koiter [1960] for phenomenological plasticity theories with multiple yield functions, and by Hill [1966] in a more general form for a metal crystal deforming in multislip. From a thermomechanical view, the above inequality actually implies the change of plastic dissipative work is always pointing to the dissipating direction and this dissipative process is irreversible. While the widely used normality law is justified by Drucker's postulate, a more general stability postulate that may cover the nonnormality case has been proposed by Ziegler and Wehrli [1987], which, if extended to strain gradient theory, has the formulation

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{i} + \dot{\tau}_{ijk}\dot{\eta}_{ijk}^{i} \geq 0, \tag{34}$$

where

$$\dot{\varepsilon}_{ij}^{i} = \left(\frac{\partial^2\Omega}{\partial\sigma_{ij}\partial\varepsilon_{kl}^{p}}\right)\dot{\varepsilon}_{kl}^{p},$$

$$\dot{\eta}_{ijk}^{i} = \left(\frac{\partial^2\Omega}{\partial\tau_{ijk}\partial\eta_{lmn}^{p}}\right)\dot{\eta}_{lmn}^{p}$$

may be termed as the irreversible strain rate and the irreversible strain gradient rate, respectively. Equation (34) applies to coupled materials as well as decoupled materials.

Case 2. $B = 0$.

As an extreme instance of Case 1, this case implies zero dissipation in the material. Consequently, no plastic strains and strain gradients occur in the deformation, and

$$l = l_e, \qquad \varepsilon_{ij} = \varepsilon^e_{ij}, \qquad \chi_{ij} = \chi^e_{ij}.$$

The total constitutive behavior may be determined by the elastic relations in (24) only. This is similar to the linear elastic strain gradient theory of [Mindlin 1964], where a higher-order elastic compliance tensor of the sixth order, M_{ijklmn}, was defined to relate the higher-order elastic stress τ_{lmn} with elastic strain gradient η_{ijk}, in the form $\eta_{ijk} = M_{ijklmn}\tau_{lmn}$. Equation (24) in this condition may be regarded as a special case of couple stress theory. For a general isotropic elastic case, the tensor M_{ijklmn} is dependent on five material constants and has general symmetry of $M_{ijklmn} = M_{lmnijk}$ [Mindlin 1964].

Case 3. $A_1 = 2$, $A_2 = 0$, and B is a nonnegative function to be specified.

In this case, ε^p_{ij} is regarded as the only internal variable. Incompressible solids are considered to have only the deviatoric stress s_{ij} contributing to the plastic work. χ^p_{ij} does not explicitly enter the rate of dissipation function as an independent internal variable. Instead, it is assumed to enter the coefficient function B in accounting for the accumulated plastic strain gradient effects. In other words, though the strain gradient may appear in the final constitutive formulations, no work-conjugate force is specified for it. Consequently, the terms containing gradients in the free energy function will also be dropped and the free-energy function in Equation (23) turns out to be

$$\Omega = \tfrac{1}{2} C_{ijkl}\sigma_{ij}\sigma_{kl}.$$

Only the first equation in (24) remains valid, and it can be recast as

$$\dot{\sigma}_{kl} = C^{-1}_{ijkl_{ijkl}}\dot{\varepsilon}^e_{ij}.$$

The rate of dissipation function defined in (25) becomes $\hat{\Phi} = B\big(A_1(\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij})\big)^{1/2}$. And the dissipative stress conjugate with $\dot{\varepsilon}^p_{ij}$ is

$$\sigma^d_{ij} = \frac{\partial\hat{\Phi}}{\partial\dot{\varepsilon}^p_{ij}} = \frac{A_1 B^2}{\hat{\Phi}}\dot{\varepsilon}^p_{ij}. \tag{35}$$

Hence the yield function in the dissipation stress space becomes

$$F = \sqrt{(\sigma^d_{ij}\sigma^d_{ij})/A_1} - B = 0.$$

Here we have assumed that B is a nonnegative function related with the history of plastic strain and plastic strain gradient. However, no rates of plastic strain and strain gradient are involved in B, so that Equation (35) always holds true.

In this case, the dissipative stress σ^d_{ij} coincides with the deviatoric stress s_{ij}. If the effective stress is defined as

$$\tau = \sqrt{s_{ij}s_{ij}/2},$$

the yield function in the true stress presents the form

$$F = \tau - B = 0. \tag{36}$$

Here B is further assumed to be of the form

$$B = \tau_0\big(f(\gamma^p)^{r_1} + g(\ell^n \eta_n)^{r_2} + k(\gamma_0)^{r_3}\big)^{1/mr_4}, \tag{37}$$

where τ_0 is a measure of the hardening modulus in simple shear. f is a function of the effective plastic strain γ^p, and g is the measure of the effective plastic strain gradient η of any order, while k denotes a function of the initial yield strain in shear γ_0. The power n relates to the order of the gradient used to represent η. If the first order gradient is used then $n = 1$. r_1, r_2, r_3 and r_4 are assumed as phenomenological material constants (or interaction coefficients). This expression in (37) actually corresponds to an extending form of the hardening law proposed by Voyiadjis and Abu Al-Rub [2005], where a function for the initial yield stress is expressed in terms of the initial yield shear strain, which enters the bracket as a function k. These authors have demonstrated this hardening law is not phenomenological but physically based and is derived from a set of dislocation mechanics-based considerations. Variable internal length scales instead of constant ones have also been used in the same reference in developing a strain gradient plasticity model. However, most of the current strain gradient theories still retain the method of using constant length scales, due to the simplicity and convenience that it affords.

Consequently, if we choose

$$\begin{gathered} r_1 = r_2 = r_3 = 1, \quad r_4 = 1/m, \quad f(\gamma^p) = c_0(\gamma^p)^{1/m}, \\ g(\ell^n \eta_n) = c_1 \ell \eta + c_2 \ell^2 \eta_2, \quad \text{with } n = 2, \end{gathered}$$

where c_0 is a constant but c_1 and c_2 may depend on γ^p, the gradient terms are expressed as

$$c_1 \ell \eta = c_1 \ell \big|\nabla \gamma^p\big| = c_1 \ell \sqrt{\nabla \gamma^p \cdot \nabla \gamma^p} \qquad \text{and} \qquad c_2 \ell^2 \eta_2 = c_2 \ell^2 \nabla^2 \gamma^p,$$

where ∇ and ∇^2 are the forward gradient and Laplacian operators respectively. As a result, the yield function of (36) can be rewritten as

$$F = \tau - \tau_0\big(h(\gamma_0) + c_0(\gamma^p)^{1/m} + c_1 \ell \big|\nabla \gamma^p\big| + c_2 \ell^2 \nabla^2 \gamma^p\big) = 0. \tag{38}$$

It is easy to prove that a yield function presenting a form of (38) may cover a family of Aifantis and his coworkers' gradient plasticity models, by appropriately simplifying the coefficients c_0, c_1 and c_2 [Aifantis 1984; 1987; Zbib and Aifantis 1988b; 1988a; Mühlhaus and Aifantis 1991 and references therein].

Otherwise, if we adopt the parameters

$$\begin{gathered} r_1 = r_2 = r_3 = 2, \qquad n = 1, \qquad r_4 = 2/(3m), \qquad f(\gamma^p) = c_0(\gamma^p)^{1/m}, \\ g(\ell^n \eta_n) = c_1 \ell \alpha, \qquad h(\gamma_0) = \gamma_0, \end{gathered}$$

where $\alpha = \sqrt{2\chi_{ij}^p \chi_{ij}^p} = \sqrt{2}\eta$, and c_0 and c_1 are constants, we obtained the yield surface

$$F = \tau - \tau_0\big((\gamma_0)^2 + c_1(\gamma^p)^2 + c_1(\ell \alpha)^2\big)^{3/2} = 0. \tag{39}$$

If the differential of strain gradients over γ^p can be neglected, the consistency condition of Equation (39) results in the instantaneous hardening rate:

$$h = \frac{\dot{\tau}}{\dot{\gamma}^p} = 3\tau_0 c_0 \gamma^p \big((\gamma_0)^2 + c_0(\gamma^p)^2 + c_1(\ell\alpha)^2\big)^{1/2}. \tag{40}$$

Or alternatively

$$h = \frac{\dot{\tau}}{\dot{\gamma}^p} = G\xi\left(1 + \frac{c_1(\ell\alpha/\gamma_0)^2}{1 + c_0(\gamma^p/\gamma_0)^2}\right)^{1/2}, \tag{41}$$

where $G = \tau_0/\gamma_0$ is the elastic shear modulus, and

$$\xi(\gamma^p) = 3c_0\gamma^p\sqrt{1 + c_0(\gamma^p/\gamma_0)^2}.$$

The hardening rate presented in Equation (41) shares a similarity with that proposed in [Acharya and Bassani 1996; 2001, (13)]. Given the function $\xi(\gamma^p)$ in (41) is further generalized such that the power-law based hardening rate in the latter reference may be included as a special case, in conjunction with the elastic relations in (34b), the J_2-flow theory of strain gradient plasticity formulated in both references may be immediately recovered. Note that for crystal metals undergoing incompatible lattice distortion, as stated in [Bassani 2001], the physical implications of the hardening rate present in (41) may be interpreted as follows: the term in the numerator of the square bracket may be regarded to reflect the increase in hardening due to incompatibility induced by Geometrically Necessary Dislocations (GNDs), while the term in the denominator gives rise to a greater effect of incompatibility in the early stages of deformation and approximately account for a significant initial density of GNDs.

A common difference between the gradient plasticity theories pioneered by Aifantis and coworkers and other gradient theories such as those based on micromorphic, micropolar or second gradient continua is that the description of the kinematics of the deformation at a given material point by the latter theories is generally enriched by the introduction of the microdeformations along with a work-conjugate thermodynamic quantity playing the role of a higher-order stress tensor. The introduction of these extra terms into the equilibrium equations requires additional boundary conditions to be specified in the solution of practical boundary value problems. While in the models based on gradient plasticity of Aifantis and coworkers, the yield function, the flow rule and/or the dilatancy condition are assumed to depend on the Laplacian of some suitable scalar measure of the accumulated plastic strain or a set of internal variables. No higher-order stress tensors are introduced into the constitutive description. Comparisons of these theories have been addressed in [Chambon et al. 2004].

4. Derivations of constitutive models for isotropic pressure-sensitive gradient-dependent geomaterials

4.1. ***Yield functions in the dissipative stress space.*** As is well known, geomaterials are generally sensitive to hydrostatic pressure. The failure modes of these materials are quite different for small stress level and high mean compressive stresses. Under high hydrostatic pressure, they can yield and flow like metal in such a way that if gradient effects are considered, the aforementioned generalized J_2-flow theories are capable of addressing them. However, in the low and intermediate compressive stress stage, the failure criteria of geomaterials are sensitive to hydrostatic states of stress. The yield criteria described above are no longer adequate to address the plastic deformation process in this case, and thus pressure-dependent failure models are required. Typical pressure-dependent conventional plasticity models are the Mohr–Coulomb, Drucker–Prager and the critical-state Cam–Clay failure criteria. In considering pressure-dependence, it is appropriate to let pressure-terms (typically the first invariants of stresses) enter the dissipation function. As for gradient-dependent materials, here we only consider the first invariants of Cauchy stress, neglecting the first invariant of higher-order stresses. This assumption is generally acceptable and convenient for most geomaterials since, in most cases, the hydrostatic part for the Cauchy stress is easily physically interpreted while that of the higher-order stress is not. In gradient theories, the higher-order stress is frequently related with higher-order boundary traction. However, except in such special cases as pure bending where only momentum force boundary conditions are imposed (in this case the first invariant of couple stress is zero), physical meanings of other higher-order boundary tractions are yet to be comprehended. Thus in the following formulations, only the first invariant of Cauchy stress is considered whenever the explicit dependence of dissipation function on the true stresses is addressed. We note that in this subsection, the influence of intermediate stress on the failure behavior or, alternatively, on the Lode-dependence, is not considered, but will be addressed in next subsection.

In view of these assumptions, we consider the following general form for the pressure-dependent dissipation function rate for gradient materials:

$$\hat{\Phi} = (B_0 + B_1 I_1 + B_2 I_1^2)^{1/2} \big(A_1 (\dot{\varepsilon}^p_{ij} \dot{\varepsilon}^p_{ij}) + A_2 (l^2 \dot{\eta}^p_{ijk} \dot{\eta}^p_{ijk}) \big)^{1/2}, \tag{42}$$

where $I_1 = \sigma_{kk}$. A_1, A_2, B_0, B_1 and B_2 are constants, where

$$A_1 \geq 0, \quad A_2 \geq 0, \quad B_0 \geq 0, \quad B_1 \leq 0, \quad B_2 \geq 0, \quad |B_1| \geq 2\sqrt{B_0 B_2}.$$

It should also be noted that if $A_2 = 0$ in (42), it is equivalent to the expression proposed by [Ziegler and Wehrli 1987, Equation 7.11] for conventional pressure-sensitive materials without gradient effects. The dissipative stresses may thus be obtained from (42) by

$$\begin{aligned} \sigma^d_{ij} &= \frac{\partial \hat{\Phi}}{\partial \dot{\varepsilon}^p_{ij}} = \frac{A_1 (B_0 + B_1 I_1 + B_2 I_1^2)}{\hat{\Phi}} \, \dot{\varepsilon}^p_{ij}, \\ \tau^d_{ijk} &= \frac{\partial \hat{\Phi}}{\partial \dot{\eta}^p_{ijk}} = \frac{A_2 (B_0 + B_1 I_1 + B_2 I_1^2) l^2}{\hat{\Phi}} \, \dot{\eta}^p_{ijk}. \end{aligned} \tag{43}$$

Consequently, the following yield function in the dissipative stress space is obtained

$$\tilde{F} = \frac{(\sigma^d_{ij} \sigma^d_{ij})}{A_1} + \frac{(l^{-2} \tau^d_{ijk} \tau^d_{ijk})}{A_2} - (B_0 + B_1 I_1 + B_2 I_1^2) = 0. \tag{44}$$

For convenience, we assume further that $A_1 = A_2 = 2/3$, and we define the following generalized J_2 and principal invariants:

$$\tilde{J}_2 = \tfrac{3}{2} (\sigma^d_{ij} \sigma^d_{ij} + l^{-2} \tau^d_{ijk} \tau^d_{ijk}), \qquad t_{II} = \sqrt{\tfrac{3}{2} \big((\sigma^d_{II})^2 + l^{-2} \tau^2_{II} \big)},$$

$$t_I = \sqrt{\tfrac{3}{2} \big((\sigma^d_I)^2 + l^{-2} \tau^2_I \big)}, \qquad t_{III} = \sqrt{\tfrac{3}{2} \big((\sigma^d_{III})^2 + l^{-2} \tau^2_{III} \big)}.$$

The following relations hold for the most general case.

$$I_1 \le \frac{-B_1 - \sqrt{B_1^2 - 4B_0B_2}}{2B_2}. \tag{45}$$

In the compound dissipative stress, Equation (44) describes a hyperboloid shape, whose axis is

$$t_I = t_{II} = t_{III}, \qquad I_1 = \frac{-\sqrt{B_1^2 - 4B_0B_2} - B_1}{2B_2}. \tag{46}$$

The intersection of this cone with the π-plane defined by t_I, t_{II} and t_{III} is a circle with radius

$$\sqrt{B_0 + B_1 I_1 + B_2 I_1^2},$$

as shown in Figure 2, left. The longitudinal section of this hyperboloid cone, defined by $I_1/\sqrt{3}$ and $\sqrt{2\tilde{J}_2}$, by analogy to the traditional meridian plane for conventional models, is illustrated in Figure 2, right. As a special case, if

$$B_1 = -2\sqrt{B_0B_2},$$

the yield surface (44) may be further rewritten as

$$\tilde{F} = \frac{3(\sigma_{ij}^d\sigma_{ij}^d)}{2} + \frac{3(l^{-2}\tau_{ijk}^d\tau_{ijk}^d)}{2} - \left(\sqrt{B_0} - \sqrt{B_2}I_1\right)^2 = 0, \tag{47}$$

with $I_1 \le \sqrt{B_0/B_2}$. Equation (47) is actually a generalized Drucker–Prager criterion for the pressure-sensitive gradient-dependent materials. It denotes a conoid-shaped cone in the compound dissipative stress space, with a circle of radius

$$\sqrt{B_0} - \sqrt{B_1}I_1$$

in the intersected π-plane defined by t_I, t_{II} and t_{III}. The section in the meridian plane defined by $I_1/\sqrt{3}$ and $\sqrt{2\tilde{J}_2}$ is two lines (Figure 2, right), which are intersected with the curves defined by (46) at $\tilde{J}_2 = B_0$.

Based on the generalized Drucker–Prager criterion in Equation (47), if it is further assumed that $B_2 = 0$, the generalized von Mises case as summarized in Section 3 will be recovered. This is a natural consequence, since in conventional plasticity theories, the Drucker–Prager criterion is an extension of von Mises generalized to the pressure-sensitive case. Based on (47), further simplifications may be made by considering that $B_0 = 0$, so the yield surface becomes

$$\tilde{F} = \sqrt{\tilde{J}_2} \pm \sqrt{B_2}I_1 = 0. \tag{48}$$

As a result, a generalized form of the cohesionless Coulomb criterion that cannot sustain tension is obtained, which has a similar cone in the π-plane defined by t_I, t_{II} and t_{III} as the generalized Drucker–Prager criterion and also two lines in the meridian plane, except that the corner of the two lines is located at O' for (48) (Figure 2). Here $\sqrt{B_2}$ acts as a frictional coefficient and may be regarded as $\sqrt{B_2} = \tan\phi_0$, where ϕ_0 is the frictional angle of the material.

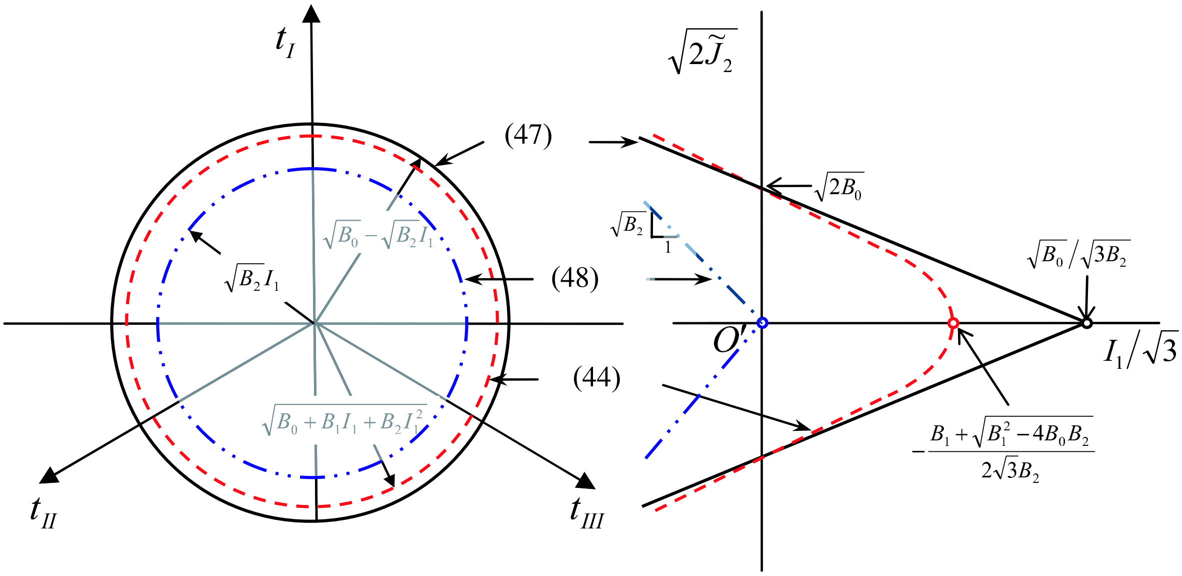

Figure 2. Yield loci for the generalized pressure-sensitive gradient materials defined by (47) (solid line), (48) (double dashed line) and (44) (dashed line) respectively. Left: view in the π-plane defined by t_I, t_{II} and t_{III}. Right: view in the meridian plane.

4.2. ***Lode-angle-dependence of isotropic pressure-sensitive gradient materials.*** Experimental investigations indicate that the third stress invariant of Cauchy stress affects significantly the behavior of pressure sensitive materials, such that the mechanical behavior of geomaterials may be better described by a Lode-angle-dependence relation. In conventional soil mechanics, this is attained by adding a function of the Lode angle into the yield functions. To achieve this aim for pressure-sensitive gradient materials, we will demonstrate in this section that Lode-angle-dependence may be easily manipulated by including the Lode angle in the expression for the rate of dissipation function. Bardet [1990] has tried to generalize the lode dependence for pressure-sensitive materials without gradient effects. In conventional soil mechanics, the Lode angle ϑ is defined by

$$\vartheta = -\frac{1}{3} \arcsin\left(\frac{3\sqrt{3}}{2} \frac{s^3}{J^3}\right), \tag{49}$$

where $s = \left(s_{ij} s_{jk} s_{ki}/3\right)^{1/3}$, $J = \left(s_{ij} s_{ij}/2\right)^{1/2}$. ϑ varies between $-\pi/6$ and $\pi/6$. In line with the experimental origin of Lode angle, including the influence of the deviatoric parts of higher-order stresses (e.g., incorporating the second invariant of higher-order stresses) into the definition of ϑ will be somewhat misleading. Thus even in the presence of strain gradients, the definition for ϑ will not be modified, and remains in form of (49). Accordingly, we use the general function of ϑ proposed by Eekelen [1980]

$$g(\vartheta) = a(1 + b \sin 3\vartheta)^{-c},$$

where a, b and c are constants. By appropriately selecting values for a, b and c, a large variety of failure criteria such as von Mises, Tresca, Mohr–Coulomb, Matsuoka and Nakai [1974] and Lade and Duncan [1975] may be approximated.

Starting from (42), it is assumed the rate of dissipation function is

$$\hat{\Phi} = (B_0 + B_1 I_1 + B_2 I_1^2)^{1/2} g(\vartheta)\big(A_1(\dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p) + A_2(l^2 \dot{\eta}_{ijk}^p \dot{\eta}_{ijk}^p)\big)^{1/2}. \tag{50}$$

Repeating the derivation in last subsection one readily obtains various Lode-dependence criteria. As an illustrative example, when $B_1 = -2\sqrt{B_0 B_2}$, a Lode-dependence generalized Drucker–Prager yield surface in the compound dissipative stress space may be obtained, yielding

$$\tilde{F} = \sqrt{\tilde{J}_2} - \big(\sqrt{B_0} - \sqrt{B_2} I_1\big) g(\vartheta) = 0. \tag{51}$$

In the π-plane defined by t_I, t_{II} and t_{III}, the deviatoric curve for Equation (51) will depend on the selection of $g(\vartheta)$. For example, when

$$a = \sin\phi_0 \quad (\text{for } \phi_0 \text{ the frictional angle}),$$
$$b = 4a/3\sqrt{3},$$
$$c = 0.25,$$

a generalized deviatoric curve like that of [Lade and Duncan 1975] is obtained. If, instead,

$$a = \sin\phi_0,$$
$$b = 2a(3-a^2)/3\sqrt{3},$$
$$c = 0.25,$$

for Equation (45), we get a generalized deviatoric curve [Matsuoka and Nakai 1974]. When $g(\vartheta) = 1$, the generalized compression Drucker–Prager curve is obtained. These curves are compared in Figure 3.

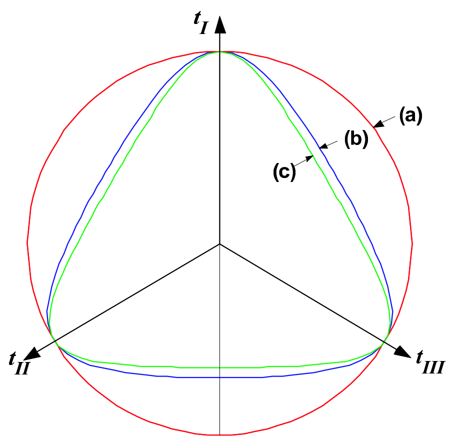

Figure 3. Deviatoric curves for various generalized criteria for pressure-sensitive gradient geomaterials in the π-plane defined by t_I, t_{II} and t_{III}. (a) Generalized compression Drucker–Prager criterion. (b) Generalized Lade–Duncan criterion. (c) Generalized Matsuoka and Nakai criterion.

4.3. ***Example of a constitutive relation derivation for a particular case.*** To derive the constitutive relations, yield function, and flow rule in the true stress space for pressure-sensitive gradient geomaterials, the free energy function must be specified. Without loss of generality, it is assumed here that the free-energy function may be decoupled into an elastic part and a plastic part as in Equation (9). In particular, we assume for the two parts the expressions

$$\Psi^e = \tfrac{1}{2} D^e_{ijkl} \varepsilon^e_{ij} \varepsilon^e_{kl} + \tfrac{1}{2} N^e_{ijklmn} l^{-2} \eta^e_{ijm} \eta^e_{lkn},$$
$$\Psi^p = \tfrac{1}{2} D^p_{ijkl} \varepsilon^p_{ij} \varepsilon^p_{kl} + \tfrac{1}{2} N^p_{ijklmn} l^{-2} \eta^p_{ijm} \eta^p_{lkn},$$

where

$$D^e_{ijkl} \quad \text{and} \quad D^p_{ijkl}$$

are elastic and plasticity stiffness, while

$$N^e_{ijklmn} \quad \text{and} \quad N^p_{ijklmn}$$

are higher-order elastic and plasticity stiffness. The rate of dissipation function uses the special case of (50) when $B_1 = -2\sqrt{B_0 B_2}$ so that the Lode-dependent generalized Drucker–Prager surface in the dissipative stress space in (51) may be attained. In view of (11), it is readily found that the elastic behavior of the material is governed by

$$\sigma_{ij} = \frac{\partial \Psi^e}{\partial \varepsilon^e_{ij}} = D^e_{ijkl} \varepsilon^e_{kl}, \qquad \tau_{ijk} = \frac{\partial \Psi^e}{\partial \eta^e_{ijk}} = N^e_{ijmlkn} l^{-2} \eta^e_{lmn},$$

while the back stresses defined in (13) are

$$\rho_{ij} = \frac{\partial \Psi^p}{\partial \varepsilon^p_{ij}} = D^p_{ijkl} \varepsilon^p_{kl}, \qquad \pi_{ijk} = \frac{\partial \Psi^p}{\partial \eta^p_{ijk}} = N^p_{ijmlkn} l^{-2} \eta^p_{lkn}.$$

In connection with Equations (12) and (51), the yield function in the true compound stress space of σ_{ij} and τ_{ijk} is

$$F = \sqrt{\bar{J}_2} - \big(\sqrt{B_0} - \sqrt{B_2} I_1\big) g(\vartheta) = 0,$$

where

$$\bar{J}_2 = \tfrac{3}{2}\big((\sigma_{ij} - \rho_{ij})(\sigma_{ij} - \rho_{ij}) + l^{-2}(\tau_{ijk} - \pi_{ijk})(\tau_{ijk} - \pi_{ijk})\big).$$

By analogy with Equation (43), normality relations hold in the dissipative stress space:

$$\sigma^d_{ij} = \frac{A_1\big(\sqrt{B_0} - \sqrt{B_2} I_1\big)^2 \big(g(\vartheta)\big)^2}{\hat{\Phi}} \dot{\varepsilon}^p_{ij},$$

$$\tau^d_{ijk} = \frac{A_2\big(\sqrt{B_0} - \sqrt{B_2} I_1\big)^2 \big(g(\vartheta)\big)^2 l^2}{\hat{\Phi}} \dot{\eta}^p_{ijk},$$

where $A_1 = A_2 = 2/3$. Therefore, in considering (13), the plastic strain and plastic strain gradient increments satisfy

$$\dot{\varepsilon}^p_{ij} = \hat{\Phi}\frac{\sigma_{ij}-\rho_{ij}}{A_1\big(\sqrt{B_0}-\sqrt{B_2}I_1\big)^2\big(g(\vartheta)\big)^2},$$

$$\dot{\eta}^p_{ijk} = \hat{\Phi}\frac{l^{-2}(\tau_{ijk}-\pi_{ijk})}{A_2\big(\sqrt{B_0}-\sqrt{B_2}I_1\big)^2\big(g(\vartheta)\big)^2}.$$

In combination with (21), the plastic multiplier may be determined, and thus the plastic incremental relations are obtained. While differentiating the yield surface equation, attention should be paid to the dependence of the Lode angle function $g(\vartheta)$ on the stresses. Thus far, the total constitutive relations for the Lode-angle-dependence pressure-sensitive gradient material are derived from the free-energy function and the rate of dissipation function.

5. Conclusions

Strain gradient plasticity constitutive models for gradient-dependent geomaterials can be constructed systematically from two thermodynamical potentials via the thermomechanical approach. In this way, the first and second thermodynamic laws are satisfied simultaneously, and yield surface in the dissipative stress space as well as the true stress space may be attained without difficulty. Both associated and nonassociated flow rule can be achieved. Appropriate selection of free-energy function and dissipation rate function makes it possible to recover a large range of strain gradient plasticity models proposed in the literature. In the framework of strain gradient theory, the flexibility of defining dissipation rate functions makes it easy to account for special features associated with geomaterials, such as pressure-sensitivity and Lode-angle dependency. Various failure criteria in geomechanics, such as those of von Mises, Tresca, Drucker–Prager, Coulomb, Lade–Duncan and Matsuoka–Nakai, have been generalized to include the strain gradient effects. Further extension of the obtained formulation to other criteria, such as critical state models, may also be obtained with relative ease. Further investigation will be directed towards incorporating the coupling between plasticity and damage for a range of geomaterials, like quasibrittle rocks, within the framework of strain gradient theory and thermomechanics. It is also noted that, regarding the numerical implementations of the gradient-dependent models developed from the current framework, some of the existing finite element methods proposed for strain gradient plasticity can be used and there is no need to develop new algorithms [Shu et al. 1999; Zervos et al. 2001; Abu Al-Rub and Voyiadjis 2005].

Acknowledgments

The authors thank the two anonymous reviewers for their helpful comments.

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Received 8 Dec 2005. Revised 17 Jan 2006. Accepted 16 Feb 2006.

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JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

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JoMMS (ISSN 1559-3959) is published in 10 issues a year. The subscription price for 2006 is US $400/year for the electronic version, and $500/year for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720–3840.

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Typeset in LaTeX

Journal of Mechanics of Materials and Structures

Volume 1, Nº 5 May 2006

1559-3959(2006)1:5;1-E