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ANALYSIS OF ELECTROMECHANICAL BUCKLING OF A PRESTRESSED MICROBEAM THAT IS BONDED TO AN ELASTIC FOUNDATION

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The electromechanical buckling of a prestressed microbeam bonded to a dielectric elastic foundation is analyzed. It is shown that electrostatic forces can precipitately instigate buckling even when the prestress in the microbeam is lower than the critical value that would cause mechanical buckling. We show that electrostatic potential can be used to achieve on/off switching of surface flexures. An analytic solution of the critical electromechanical state is derived. In addition, an analytic approximation of the initial postbuckling state is also presented, and is validated numerically.

1. Introduction

Mechanical buckling is a well known phenomenon that occurs in thin elastic structures subjected to compressive loads. Mechanical buckling develops only if the compressive loads are larger than a critical value. In most buckled structures, reduction of the compressive load to a subcritical level will eliminate the buckling deformation. In thin sheet-like elastic solids that are bonded to an elastic foundation, a compressive stress can cause a dense occurrence of buckling flexures [Hetenyi 1946]. The buckling phenomenon is a *bifurcation* transition because the postbuckling state may arbitrarily develop into one of several modes [Gilmore 1981; Godoy 1999; Nguyen 2000].

A different instability that is prevalent in the field of microelectromechanical systems (MEMS) is the pull-in phenomenon in electrostatic actuators [Elata et al. 2003; Pelesko and Bernstein 2003]. This inherent instability is due to the nonlinear nature of electrostatic forces. This instability is known as a *limit point* or a *fold* in the equilibrium state of the electrostatic actuator [Elata et al. 2003; Gilmore 1981; Godoy 1999; Nguyen 2000], and is not a bifurcation.

However, a bifurcation transition is also possible in electrostatic actuators. A well known example is the *side pull-in* in electrostatic comb-drives [Elata et al. 2003; Elata and Leus 2005; Legtenberg et al. 1996]. This instability response is an electromechanical bifurcation because the comb-drive may collapse in more than one direction.

Recently, the bifurcation response of a clamped-clamped beam that is subjected to both compressive stress and an electrostatic field was investigated. This bifurcation instability was termed *electromechanical buckling* (EMB) [Elata and Abu-Salih 2005], because it is a true coupling between mechanical buckling and electromechanical bifurcation.

In the present study, the EMB response of a prestressed infinite beam, bonded to an elastic foundation, is analyzed. Specifically it is shown that buckling in this system can be reversibly switched on and off by application and elimination of a driving voltage.

Keywords: electromechanical buckling, electromechanical instability.

In the next section the governing equation of the problem is presented. In Section 3 the parameters of the critical electromechanical buckling state are analytically derived. The initial postbuckling analysis is presented in Section 4 and includes an approximate analytic solution which is numerically validated.

2. Formulation

In this study we analyze the EMB of an infinite beam. To facilitate the analysis we use the following strategy. First, we assume that the critical and postbuckling states are both periodic. This assumption will be later confirmed by numerical simulations. Second, we derive the wavelength of the periodic solution. For the critical state the wavelength is derived analytically. For the postbuckled state the wavelength is computed in the following way: we first consider the postbuckled state of a finite beam with periodic boundary conditions. The length of the finite beam which is associated with the minimal energy per unit length of the postbuckled state is the correct postbuckling wavelength in the infinite beam. Finally, we use numerical simulations to confirm that the critical and postbuckling solutions indeed converge to a periodic waveform.

Consider an infinite beam which is bonded to an elastic foundation as illustrated in Figure 1a. The beam is conductive, the elastic foundation is an isolating dielectric, and the bottom substrate is a fixed rigid conductor. The beam is prestressed and is subjected to a potential V, and the substrate is grounded.

The field equation that governs the electromechanical response of the beam is given by

$$Db\frac{d^4y}{dx^4} - \sigma bh\frac{d^2y}{dx^2} - Ebh\left(\frac{1}{L}\int_L \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx\right)\frac{d^2y}{dx^2} + k_f by = \frac{\varepsilon\, b\, V^2}{2(g-y)^2}. \tag{1}$$

The four terms on the left hand side of Equation (1) are the distributed load due to bending, residual stress, membrane stiffening, and elastic foundation. The term on the right hand side is the electrostatic distributed force.

Here y is the deflection as a function of location x along the beam, $D = E^*h^3/12$ is the bending rigidity of the beam, b is the beam width, h is the beam thickness, and $E^* = E/(1-\nu^2)$ is the effective bending modulus assuming that the beam is wide, that is, $b \gg h$, where E is Young's modulus and ν is the Poisson ratio. The variable σ is the prestress (positive in tension), L is the length of the beam, k_f is the elastic modulus of the foundation measured in N/m^3, ε is the permittivity of the dielectric elastic foundation and g is its nominal thickness.

The third term in (1), which includes the beam length L, is correct for $L \to \infty$. The equilibrium equation also holds for a beam of finite length L with periodic boundary conditions. In this case, if L is equal to an integer number of wavelengths of the periodic solution of the infinite beam, then the solution of the finite beam is identical to the solution of the infinite beam.

In the third term of (1), it is assumed that the elastic foundation does not constrain the transverse strain ε_{zz} when the beam is in tension. This may be expected when the thickness of the elastic foundation is greater than the beam width. If, however, the thickness of the elastic foundation is much smaller than the beam width, then the elastic foundation may constrain the transverse strain ε_{zz} when the beam is subjected to axial tension. In this case the effective elastic modulus in the third term should be replaced with E^*.

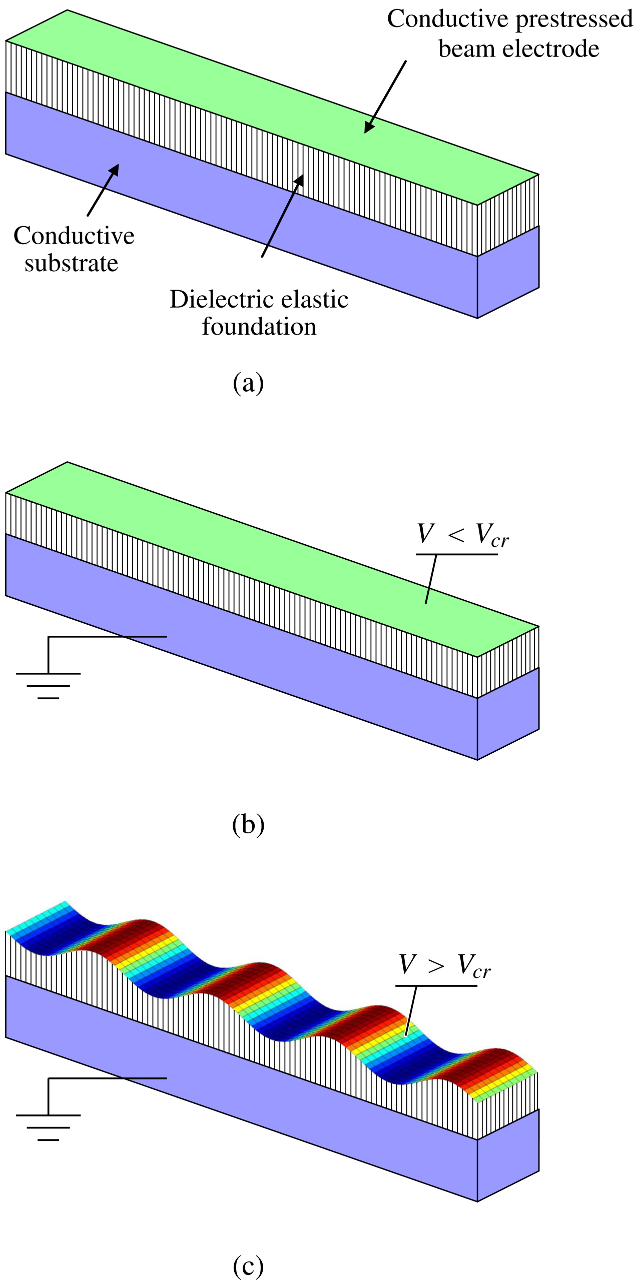

Figure 1. A prestressed beam bonded to an elastic foundation. (a) When the prestress in the conductive beam is below the critical value, and no voltage is applied, the beam remains flat; (b) when a subcritical voltage is applied, the beam deflection is uniform; (c) when a supercritical voltage is applied, electromechanical buckling occurs.

The effect of fringing fields is not considered in this work though simplified approximations of this effect may be added to the analysis [Leus and Elata 2004]. Also, the equilibrium equation (1) is valid for small rotations, that is, $|dy/dx| \ll 1$. If this condition does not hold, additional nonlinear terms must be included in the governing equation [Brush and Almaroth 1975].

The electrostatic distributed force is approximated by the local parallel-plates model which is valid for small rotations.

The governing equation may be written in the following normalized form

$$\frac{1}{(2\pi)^4}\frac{d^4\tilde{y}}{d\tilde{x}^4}+\frac{2\beta}{(2\pi)^2}\frac{d^2\tilde{y}}{d\tilde{x}^2}-\frac{\tilde{g}^2}{\alpha}\left[\int_0^\alpha \frac{1}{2}\left(\frac{d\,\tilde{y}}{d\tilde{x}}\right)^2 d\tilde{x}\right]\frac{1}{(2\pi)^4}\frac{d^2\tilde{y}}{d\tilde{x}^2}+\tilde{y}=\frac{\tilde{V}^2}{(1-\tilde{y})^2}, \tag{2}$$

where

$$\tilde{x}=\frac{x}{\Lambda_{cr}},\qquad \tilde{y}=\frac{y}{g},\qquad \beta=\frac{\sigma}{\sigma_{cr}},\qquad \sigma_{cr}=-\frac{2\sqrt{k_f D}}{h},$$

$$\tilde{g}=\sqrt{12\frac{E}{E^*}\frac{g}{h}},\qquad \alpha=\frac{L}{\Lambda_{cr}},\qquad \tilde{V}^2=\frac{\varepsilon V^2}{2k_f g^3},\qquad \Lambda_{cr}=2\pi\left(\frac{D}{k_f}\right)^{1/4}.$$

Here Λ_{cr} and σ_{cr} are the wavelength and the stress at the critical buckling state when no voltage is applied [Abu-Salih and Elata 2005; Hetenyi 1946], β is the normalized prestress parameter and α is the normalized wavelength [Abu-Salih and Elata 2005].

3. Critical state

When no electrostatic forces are applied, the prestressed beam will not buckle if the prestress is lower than the critical stress σ_{cr}. In this section we investigate the effect of electrostatic forces on the critical buckling state of the prestressed beam. Specifically, we show that electrostatic forces can induce buckling in a beam in which the prestress is subcritical (in such a beam buckling will not occur if electrostatic forces are not applied).

When a voltage is applied to the beam, the beam initially deflects uniformly, similar to a parallel-plates actuator [Pelesko and Bernstein 2003] (Figure 1b). In this case the dielectric elastic foundation constitutes both a dielectric substance between the electrodes and the elastic spring of the parallel-plates actuator.

When the deflection is nearly uniform, the nonlinear term (in square brackets) may be omitted from the governing equation (2), which reduces to

$$\frac{1}{(2\pi)^4}\frac{d^4\tilde{y}}{d\tilde{x}^4}+\frac{2\beta}{(2\pi)^2}\frac{d^2\tilde{y}}{d\tilde{x}^2}+\tilde{y}=\frac{\tilde{V}^2}{(1-\tilde{y})^2}. \tag{3}$$

The deflection of the infinite beam is assumed to be periodic and is postulated in the form

$$\tilde{y}=\tilde{y}_0+B\sin\frac{2\pi\tilde{x}}{\alpha}, \tag{4}$$

where $\tilde{y}_0$ is an average value which is equal to the uniform displacement when no buckling occurs, and B is the amplitude of the structural waves that develop due to the electromechanical buckling response.

At the verge of buckling, B is small, and therefore the electrostatic force may be approximated by the following Taylor expansion

$$\frac{\tilde{V}^2}{(1-\tilde{y})^2} \approx \tilde{V}^2\left[\frac{1}{(1-\tilde{y}_0)^2} + \frac{2}{(1-\tilde{y}_0)^3} B \sin\frac{2\pi\,\tilde{x}}{\alpha} + O(B^2)\right]. \tag{5}$$

Now, substituting the postulated deflection (4) and the approximated electrostatic force (5) into the linear governing equation (3), yields

$$B\frac{\sin(2\pi\,\tilde{x}/\alpha)}{\alpha^4}\left(\alpha^4(1-\delta) - 2\alpha^2\beta + 1\right) = \frac{\tilde{V}^2}{(1-\tilde{y}_0)^2} - \tilde{y}_0. \tag{6}$$

Here δ is the normalized electrostatic stiffness that is given by

$$\delta = \frac{2\tilde{V}^2}{(1-\tilde{y}_0)^3}. \tag{7}$$

On the verge of buckling, where $B = 0$, the solution of (6) is

$$\tilde{V}^2 = \tilde{y}_0(1-\tilde{y}_0)^2. \tag{8}$$

The deflection of the beam in this case is uniform ($\tilde{y} = \tilde{y}_0$), and the force applied to the beam by the elastic foundation balances the electrostatic force. For incipient buckling, (8) holds and can be subtracted from (6) to yield

$$B\frac{\sin(2\pi\,\tilde{x}/\alpha)}{\alpha^4}\left(\alpha^4(1-\delta) - 2\alpha^2\beta + 1\right) = 0. \tag{9}$$

The last equation can be solved for the critical buckling parameters (β, α), and the amplitude B remains arbitrary, as is usual in linear buckling analysis [Godoy 1999; Timoshenko 1936]. The stability equation of the beam is given by

$$\alpha^4(1-\delta) - 2\alpha^2\beta + 1 = 0.$$

The solution of the this equation is

$$\alpha = \sqrt{\frac{\beta \pm \sqrt{\beta^2 - (1-\delta)}}{1-\delta}}. \tag{10}$$

The normalized wavelength α must be a real positive value ($\alpha > 0$), therefore β is restricted by

$$\beta \geq \sqrt{1-\delta}. \tag{11}$$

Substituting the critical load $\beta_{cr} = \sqrt{1-\delta}$ into (10), the critical value of the wavelength is found

$$\alpha_{cr} = \frac{1}{\sqrt{\beta_{cr}}} = \frac{1}{(1-\delta)^{1/4}} = \left(1 - \frac{2\tilde{V}^2}{(1-\tilde{y}_0)^3}\right)^{-1/4}.$$

From this it is clear that when no electrostatic forces are applied (that is, $\tilde{V} = 0$) buckling cannot occur for $\beta < 1$ (that is, subcritical stress). The critical normalized wavelength in the case of $\tilde{V} = 0$ is $\alpha_{cr(\tilde{V}=0)} = 1$, which proves that Λ_{cr} is indeed the wavelength of the critical buckling state [Abu-Salih and Elata 2005].

From (11) it is clear that due to the destabilizing effect of the electrostatic forces, buckling can occur for $\beta < 1$ whenever $\delta > 0$. The buckling deflection (4) is therefore due to the combined effect of subcritical prestress and electrostatic forces. This justifies the term electromechanical buckling.

A schematic illustration of a prestressed beam that is bonded to an elastic foundation is presented in Figure 1a. When the prestress in the conductive beam is below the critical value, and no voltage is applied, the beam remains flat (Figure 1b). For the same level of prestress, a sufficient voltage will precipitately instigate buckling (Figure 1c).

For a given normalized prestress β, the critical switching voltage $\tilde{V}_{cr}$ that instigates buckling is extracted by solving (7), (8) and $\beta_{cr} = \sqrt{1-\delta}$, to yield

$$\tilde{V}_{cr}^2 = 4\frac{1-\beta_{cr}^2}{\left(3-\beta_{cr}^2\right)^3}.$$

Figure 2 presents the normalized voltage square $\tilde{V}_{cr}^2$, wavelength α_{cr}, and deflection $\tilde{y}_0$, at the critical states, for various values of normalized prestress β. At the limit of zero prestress, the critical wavelength becomes infinite and the voltage approaches an asymptotic value. This state is in essence the pull-in state of an infinite parallel-plates actuator. At this limit the deflection at the critical state is $\tilde{y}_0 = 1/3$. For such a large reduction of the elastic foundation thickness, the linear model of the foundation may be unrealistic.

4. Initial postbuckling state

In the postbuckled state, considerable membrane stresses develop and the mechanical response is governed by the *nonlinear* equilibrium Equation (2), including the term in square brackets. The periodic problem is solved for beams with various finite normalized lengths, α. In each of these lengths, it is assumed that the solution includes one period. The average total potential density (energy per unit length) associated with each length α is then computed. The solution of the infinite problem is identified with the periodic solution of the finite-length problem, for which the average total potential density is minimal.

The periodic boundary conditions of the finite beam are

$$\tilde{y}(\tilde{x}=0) = \tilde{y}(\tilde{x}=\alpha), \qquad \left.\frac{d\tilde{y}}{d\tilde{x}}\right|_{\tilde{x}=0} = \left.\frac{d\tilde{y}}{d\tilde{x}}\right|_{\tilde{x}=\alpha}. \tag{12}$$

4.1. ***Approximate analytic solution.*** As in the previous linear analysis, it is postulated that the postbuckling deflection is of the form

$$\tilde{y} = \tilde{y}_0' + B' \sin\frac{2\pi\tilde{x}}{\alpha}, \tag{13}$$

where B' is the normalized amplitude of the postbuckling deflection. In initial postbuckled states we assume that B' is small. As in the preceding critical state analysis, for small values of B' the electrostatic force may be approximated by the following Taylor expansion

$$\frac{\tilde{V}^2}{(1-\tilde{y})^2} \approx \tilde{V}^2\left(\frac{1}{(1-\tilde{y}_0')^2} + \frac{2}{(1-\tilde{y}_0')^3}B'\sin\frac{2\pi\tilde{x}}{\alpha} + O(B'^2)\right). \tag{14}$$

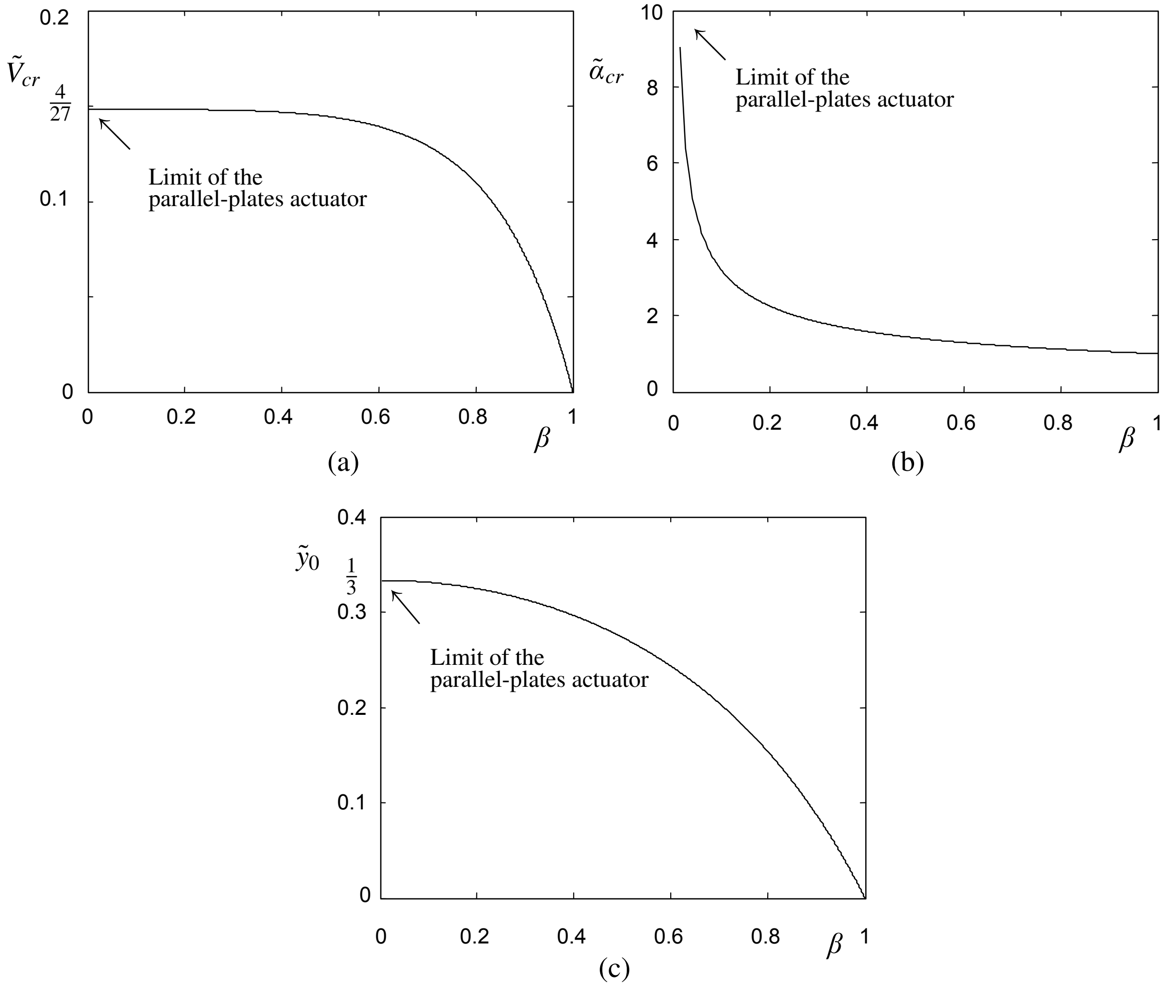

Figure 2. The critical electromechanical buckling state for different values of prestress β. (a) Normalized critical voltage squared $\tilde{V}_{cr}^2$. At zero prestress the critical voltage is the same as for a parallel-plates actuator. (b) Normalized critical wavelength α_{cr}. At zero prestress the normalized wavelength is infinite and the beam remains flat. (c) Normalized deflection $\tilde{y}_0$.

Substituting (13) and (14) into the nonlinear equation (2), and considering only small values of B', we further assume that the equilibrium equation holds separately for the average and the periodic parts of the postulated deflection.

With this assumption, the equilibrium equation (2) reduces to the following

$$\tilde{V}^2 = \tilde{y}_0'(1-\tilde{y}_0')^2, \tag{15}$$

$$B'\frac{\sin(2\pi\tilde{x}/\alpha)}{\alpha^4}\big(\alpha^4(1-\delta)-2\beta\alpha^2+\tfrac{1}{4}\tilde{g}^2B'^2+1\big)=0. \tag{16}$$

The two assumptions, namely that the postbuckling deflection is of the form (13), and that when B' is small the solution of (2) may be substituted by the simultaneous solution of (15) and (16), will be validated numerically in the next subsection.

The nontrivial solution of (16) is given by

$$B' = \frac{2}{\tilde{g}}\sqrt{2\beta\alpha^2 - \alpha^4(1-\delta) - 1}\,.$$

In an infinite beam, the normalized wavelength associated with given values of β and δ is the one that minimizes the total potential per unit length, of the system. The total potential $\psi = U_B + U_A + U_{k_f} - U_E^*$ is the sum of four energy components associated with bending U_B, axial deformation U_A, deformation of the elastic foundation U_{k_f}, and complementary electrostatic energy U_E^* [Elata and Abu-Salih 2005]. Normalizing the strain energy components by the (axial) strain energy at the verge of buckling ($U_{cr} = \sigma_{cr}^2 A/2E$), yields

$$\tilde{U}_B = \frac{\tilde{g}^2}{4(2\pi)^4\alpha}\int_0^\alpha \left(\frac{d^2\tilde{y}}{d\tilde{x}^2}\right)^2 d\tilde{x},$$

$$\tilde{U}_A = \frac{\tilde{g}^4}{4\alpha^2}\left(-\frac{2(2\pi)^2\alpha\beta}{\tilde{g}^2} + \frac{1}{2}\int_0^\alpha \left(\frac{d\tilde{y}}{d\tilde{x}}\right)^2 d\tilde{x}\right)^2,$$

$$\tilde{U}_{k_f} = \frac{\tilde{g}^2}{4\alpha}\int_0^\alpha \tilde{y}^2 d\tilde{x},$$

$$\tilde{U}_E^* = \frac{\tilde{V}^2\tilde{g}^2}{2\alpha}\int_0^\alpha \frac{d\tilde{x}}{(1-\tilde{y})}.$$

The normalized total potential is accordingly given by

$$\tilde{\psi} = \tilde{U}_B + \tilde{U}_A + \tilde{U}_{k_f} - \tilde{U}_E^*. \tag{17}$$

Substituting the postulated deflection (13) into the normalized total potential, differentiating the total potential with respect to α, and setting this derivative to zero, yields a nonlinear equation for the postbuckling wavelength α. This equation was numerically solved and the results are presented by the *dashed lines* in Figure 3, showing α, B' and $\tilde{y}_0'$ as function of $\tilde{V}^2/\tilde{V}_{cr}$ for the parameters $\tilde{y}_0 = 0.1$ (that is, $\beta = 0.88$), $g/h = 10$ and $\upsilon = 0.25$.

To clarify the notion of minimal energy, the total potential of the system (17) is presented in Figure 4 as a function of voltage and wavelength (for $\tilde{y}_0 = 0.1$ (that is, $\beta = 0.88$), $g/h = 10$ and $\upsilon = 0.25$). In this figure the flat slopes are the regions in which no buckling occurs (the total energy is only a function of V). The first valley describes buckling with a single wavelength in a beam of finite length with periodic boundary conditions. The second and third valleys are repetitions of this first valley, and describe buckling into a double and a triple flexure waves for a beam with double and triple length, respectively (and periodic boundary conditions). The center lines running through the routes of the valleys describe the same postbuckling wavelength as a function of applied voltage (that is, repetitions of the same solution). The critical stability states are marked by the solid line on the rims of the valleys [Abu-Salih and Elata 2005].

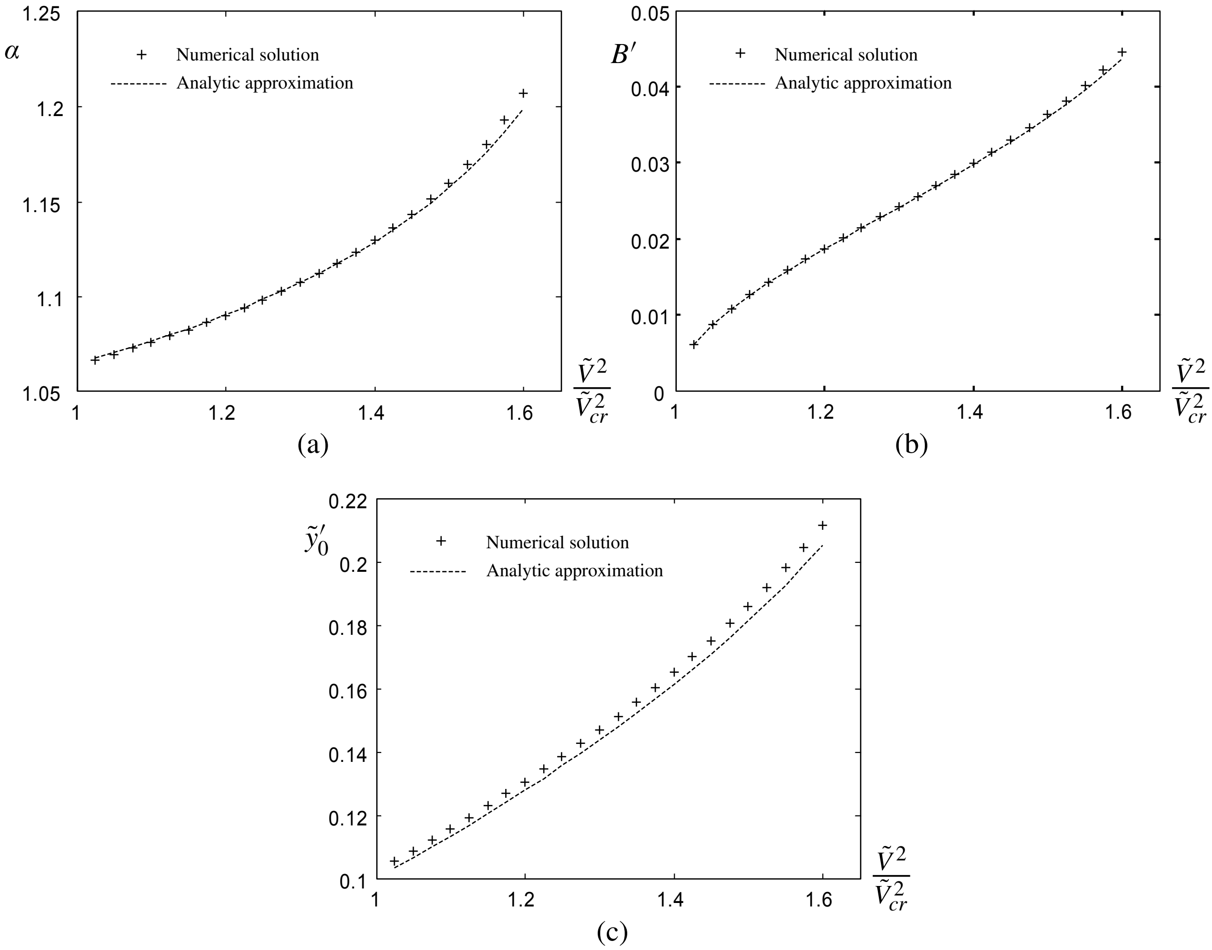

Figure 3. Postbuckling state as a function of applied voltage for $\tilde{y}_0 = 0.1$ (that is, $\beta = 0.88$), $g/h = 10$ and $\upsilon = 0.25$. (a) The normalized wavelength; (b) normalized amplitude of the sinusoidal component of the deflection; (c) the normalized deflection average.

4.2. ***Numerical validation.*** To validate the analytic solution presented in the previous subsection, a finite-difference numerical code for solving Equation (2) with periodic boundary conditions (12) was implemented in MATLAB®. For given supercritical voltages (that is, $\tilde{V}^2 > \tilde{V}_{cr}^2$), the code numerically computed the total potential of the system (17) and numerically found the normalized wavelength that minimizes this potential per unit length.

Figure 5 presents the convergence of the postbuckling deflection parameters as a function of the number of nodes n in the finite-difference mesh for the conditions $\beta = 0.9$, $g/h = 10$, $\upsilon = 0.25$, and $\tilde{V}^2/\tilde{V}_{cr}^2 = 1.35$. For these conditions the postbuckling deflection parameters are $\alpha = 1.092$, $\tilde{y}_0' = 0.1288$ and $B' = 0.02771$. To verify that the postbuckling deflection is of the functional form (13), the deflection average was subtracted from the deflection which was then normalized such that its amplitude was in the

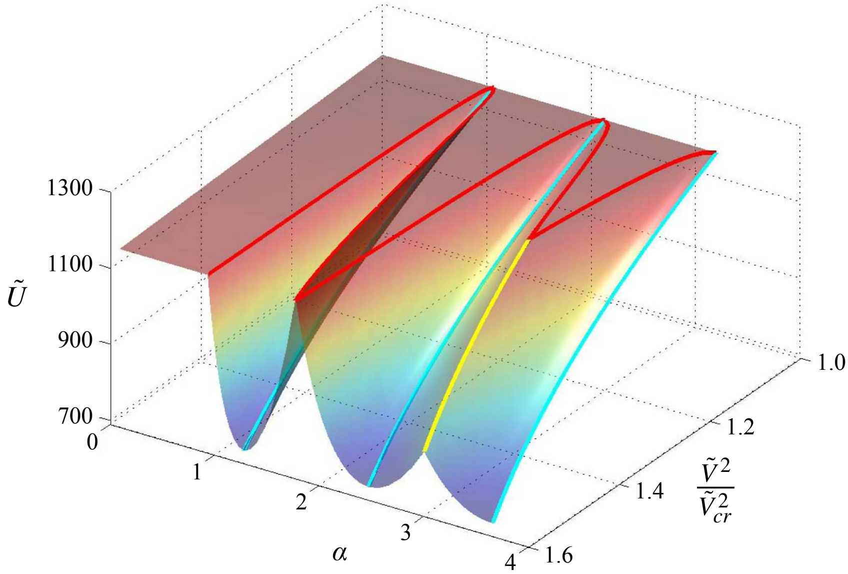

Figure 4. The total potential energy of a finite beam with periodic boundary conditions and $\tilde{y}_0 = 0.1$ (that is, $\beta = 0.88$), $g/h = 10$ and $\upsilon = 0.25$, as a function of the normalized applied voltage and the postulated wavelength. The flat regions describe states in which the applied voltage is insufficient to induce buckling for prescribed values of the postulated wavelength.

range $-1 \le \tilde{y} - \tilde{y}_0' \le 1$. The root-mean-square of the error from a perfect sine wave was then computed and found to be $rms = 6.5 \cdot 10^{-4}$.

Similar convergence properties and compliance with the postulated waveform (13) were found when the response of a beam with lengths $L/\Lambda_{cr} = 2\alpha$ and $L/\Lambda_{cr} = 3\alpha$ was simulated. In this case the solution was a double and triple repetition of the solution obtained for a beam of length α.

Last, it was assumed that the postbuckling deflection parameter B' is small and that the solution of (2) may be substituted by the simultaneous solution of (15) and (16). Figure 3 presents the numerically computed values of α, B' and $\tilde{y}_0'$ ('+' marks) as functions of $\tilde{V}^2/\tilde{V}_{cr}$ for the parameters $\tilde{y}_0 = 0.1$ (that is, $\beta = 0.88$), $g/h = 10$ and $\upsilon = 0.25$. These results are based on a finite-difference solution of (2) with boundary conditions (12). In these numerical simulations, the functional waveform of the deflection was not a priori constrained in any way (except for the periodic boundary conditions). The agreement between the numerically computed values and the analytic approximation verify that for small values of $\tilde{V}^2/\tilde{V}_{cr}^2$ these assumptions are valid.

Figure 6 presents the value of $\tilde{V}^2/\tilde{V}_{cr}^2$ for which B' is 1%, 3%, and 5%. It is clear that for small values of β the voltage range for which the analysis is applicable is rather limited. However, for $\beta \ge 0.85$, B' remains small for a considerable range of $\tilde{V}^2/\tilde{V}_{cr}^2$.

In a series of numerical simulations (not presented in this manuscript), for different values of prestress β, the applied voltage was increased up to the pull-in point [Elata and Abu-Salih 2005]. At these simulated pull-in states, not only was the average deflection $\tilde{y}_0'$ considerable (for example, $\tilde{y}_0' \approx 1/3$), but also

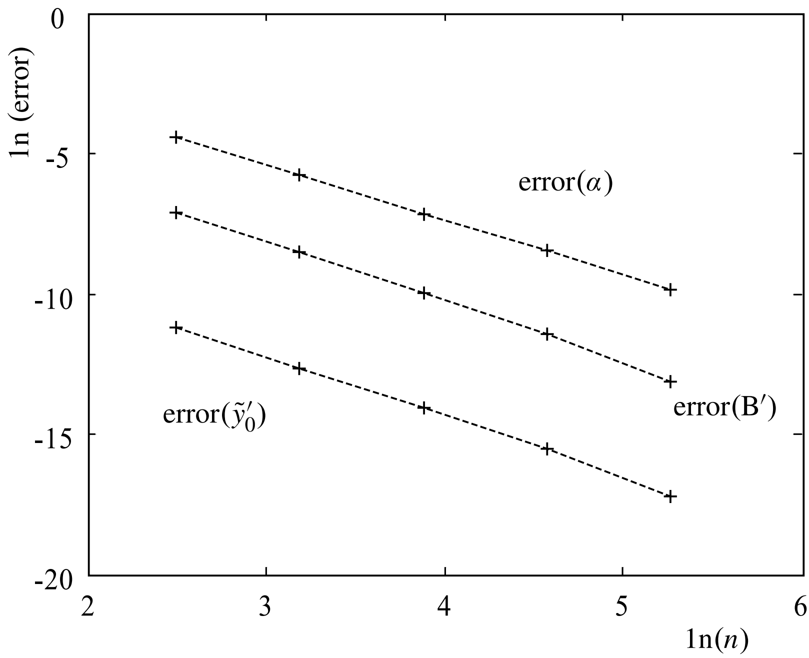

Figure 5. Convergence of the postbuckling deflection parameters as a function of the number of nodes n (for $\beta = 0.9$, $\alpha = 1.092$, $\tilde{V}^2/\tilde{V}_{cr}^2 = 1.35$).

the amplitude of the sinusoidal deflection component, B', became rather large such that the deflection form (13) is no longer a valid approximation. Under these conditions the equilibrium equation becomes invalid due to the large rotations, and in any case, the high reduction of the elastic foundation thickness would suggest that a nonlinear foundation response should be considered.

5. Discussion

In this work the electromechanical buckling of a prestressed beam which is bonded to a dielectric elastic foundation is analyzed. An analytic solution of the critical electromechanical state is derived, and it is shown that electrostatic forces can precipitately instigate buckling even when the prestress in the beam is lower than the critical value that would cause mechanical buckling. An analytic approximation of the initial postbuckling state is also presented, and is validated numerically. The numerical simulations show that a stable, initial postbuckling state exists.

The analysis presented in this study considers perfect elastic response with no residual strains, for example, plastic deformation. The work done by the voltage source is invested in elastic strain energy (of the beam and foundation) and electrostatic potential between the beam and substrate. When the voltage source is turned off (effectively setting $V = 0$) the deformed system will return to the original state with a flat prestressed beam and an unloaded elastic foundation. This suggests that electrostatic potential can be used to achieve on/off switching of flexure waves in a prestressed beam. Such on/off

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