Journal of
Mechanics of
Materials and Structures

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Volume 1, Nº 5 May 2006

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 1, No. 5, 2006

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An integral identity is constructed from properties of the energy momentum tensor and is used to demonstrate uniqueness of the displacement on star-shaped regions to the affine boundary value problem of the nonlinear homogeneous elastic dielectric. The method of proof, nontrivially adapted from that of the corresponding elastic problem, assumes the electric enthalpy function to be rank-one convex and strictly quasiconvex. Furthermore, for a given displacement gradient, the electric quantities are proved unique for specified nonaffine and nonuniform electric boundary conditions subject to the electric enthalpy and strain energy functions satisfying additional convexity conditions.

1. Introduction

This paper considers uniqueness of smooth solutions to certain simple boundary value problems for the nonlinear homogeneous elastic dielectric in equilibrium and occupying a bounded region of n-dimensional Euclidean space subject to zero body-force and electric source charges. Uniqueness in corresponding electromagnetic problems is guaranteed by strict convexity of the energy. A similar condition on the strain energy function also ensures uniqueness to boundary value problems of nonlinear elastostatics, but several well known counterexamples demonstrate that universal uniqueness is untenable. The counterexamples mean that also for the nonlinear elastic dielectric unqualified uniqueness is unacceptable and consequently the condition of strict convexity is too restrictive even for ferroelectrics and similar materials exhibiting phase transitions. But equally, we do not expect there to be universal nonuniqueness since it is intuitively evident that certain simple problems should possess a unique solution.

In elastostatics, this topic has been treated in [Knops and Stuart 1984] where the notions of strict quasiconvexity and rank-one convexity of the strain energy function are introduced to establish uniqueness of a smooth solution to affine displacement boundary value problems on star-shaped bounded regions. Pivotal to the proof is a Noetherian conservation law [Gelfand and Fomin 1963] which in the context of elasticity has been separately derived either directly or from properties of the energy-momentum (Eshelby) tensor. Contributions notably include those by Chadwick [1975], Eshelby [1975], Green [1973], Günther [1962], Gurtin [2000], Hill [1986], Knowles and Sternberg [1972]. Pohozaev [1965] and Pucci and Serrin [1986] are among those who have used a similar law in partial differential equations.

The uniqueness proof presented here for the nonlinear elastic dielectric is patterned on that described in [Knops and Stuart 1984], and therefore is likewise restricted to star-shaped regions. Another application of the basic proof is by Mareno [2004] who investigated uniqueness in the second order theory of nonlinear elasticity. As a consequence of these previous studies, the intrinsic mathematical interest of the present paper is seen as lying not so much in the details required to extend the proof, but rather

Keywords: elastic dielectric, affine boundary values, uniqueness.

in the electromagnetic problems under consideration, for which the uniqueness results are new. In this respect, it is perhaps worth remarking that uniqueness results generally are restricted to the linear theory and comparatively little information appears available for nonlinear elastic dielectrics. Therefore the following conclusions are viewed as contributing to a basic understanding of the coupled theory of elasticity and electromagnetism, with particular relevance for numerical computation and those aspects concerned with buckling and hysteresis.

For ease of presentation, it is convenient to ignore magnetic effects, although these may also be included. It is also convenient to conduct the analysis with respect to the reference configuration which requires the electric fields and the governing Maxwell–Lorentz equations to be appropriately transformed from their usual formulation in the current configuration. For this purpose, we appeal in part to a variational procedure in which the strain energy function is replaced by an electric enthalpy regarded as a function of the deformation gradient, electric field and the polarization referred to the reference configuration. The consequent enlarged set of Euler–Lagrange equations contain the transformed Maxwell–Lorentz equations, and enable certain properties of the energy-momentum tensor to be derived. These in turn lead to an identity, analogous to the elastostatic conservation law, whose construction, while nontrivial, appears to be more direct than is usual; compare for example, [Ericksen 2006, to appear; Maugin 1993; Maugin and Trimarco 1991; 2001, Trimarco 2002; 2003; and Pack and Herrmann 1986]. The Maxwell–Lorentz equations also imply that the electric enthalpy is independent of the polarization, so that the conditions for the electric enthalpy to be quasiconvex and rank-one convex, both essential for the proof, need only be defined in terms of the deformation gradient and the electric field. As is well known, these generalized notions of convexity hold for deformation gradients and electric fields that may be discontinuous across an internal surface provided both satisfy geometrical compatibility conditions that characterize coherent phases in solids. This aspect is not developed in what follows. Furthermore, the generalized convexity notions both reduce to convexity in the usual sense for functions whose arguments are vectors or scalars. The reduction is pertinent to the concluding discussion regarding uniqueness under mixed boundary conditions of the electric constituents for a given deformation gradient, which includes application to a dielectric embedded in a capacitor.

Section 2 assembles essential preliminaries, introduces the electric enthalpy function, states, and, for completeness, proves the conservation law. The notions of rank-one convexity and quasiconvexity are introduced in Section 3 which also specifies the affine boundary conditions and constructs the uniqueness proof. The concluding remarks, given in Section 4, include alternative conditions for uniqueness in the affine boundary value problem, and a discussion of uniqueness of the electric displacement and polarization in the affine problem. For completeness, we supplement these results by demonstrating that for a given deformation gradient the electric field remains unique when the affine electric boundary conditions are replaced by those of standard nonuniform mixed type. Under these boundary conditions we also prove uniqueness of the polarization and the electric displacement subject additionally to the strain energy being rank-one convex with respect to the polarization, a condition possibly too severe for ferroelectrics but for which uniqueness of the deformation and electric field remains valid. The conclusions are, of course, not unexpected, although perhaps less so in the nonlinear theory, and merely reflect the reduction, already remarked, of generalized notions of convexity to that of standard convexity for functions of vector quantities.

The direct tensor notation is mainly employed except when greater clarity is achieved by the corresponding suffix notation. Existence of a smooth solution is assumed, but in this respect we note that under the generalized convexity conditions discussed here, existence of elastostatic weak solutions has been established by Ball [1976], while Serre [2004] has discussed the corresponding electromagnetic problem.

2. Notation and other preliminaries

A nonlinear homogeneous elastic dielectric in its reference configuration occupies the bounded region $\Omega \subset \mathbb{R}^n$ of n-dimensional Euclidean space. The piecewise continuously differentiable boundary $\partial\Omega$ of Ω has unit outward normal N and is assumed to be star-shaped with respect to an interior point. Boundary conditions, precisely stated in the next section, produce an equilibrium deformation of the dielectric in which a point $X \in \Omega$ becomes displaced to the point x, where X, x represent vectors in $\mathbb{R}^n$ whose components with respect to a Cartesian coordinate system are X_A and x_i. The deformation, supposed smooth, possesses a gradient expressed by

$$F = \frac{\partial x}{\partial X}, \qquad F_{iA} = \frac{\partial x_i}{\partial X_A}, \tag{2–1}$$

while the inverse is given by

$$F^{-1} = \frac{\partial X}{\partial x}, \qquad F^{-1}_{Ai} = \frac{\partial X_A}{\partial x_i}, \tag{2–2}$$

and the determinant associated with Equation (2–1) is $J = det\ F$. Let $M^{m\times n}$ denote the set of $m \times n$ matrices and suppose that $J \in M^{n\times n}_{+}$, the set of square matrices with positive determinant. The transpose of a tensor A is denoted by A^T; the identity tensor by I; and the tensorial trace operator by tr. Tensor and vector multiplication is indicated by juxtaposition, the precise form being clear from the particular context, while the inner product of tensors AB is given by

$$AB = tr AB^T.$$

In the deformed dielectric there is an electric field e, an electric displacement d, and a polarization p per unit deformed volume. These vector quantities satisfy the appropriate time independent Maxwell–Lorentz equations which in the assumed absence of electric source charges lead to the following the expressions $e = -\operatorname{grad} \varphi$ and $d = \epsilon_0 e + p$, where grad denotes the gradient operator with respect to the system of current coordinates x, $\varphi(x)$ is a scalar potential function of the variables x, and ϵ_0 is the *in vacuo* dielectric constant supposed positive.

It is convenient to develop the subsequent analysis with respect to the reference configuration Ω. Now, in terms of the notation

$$E = F^T e, \quad D = JF^{-1}d, \quad P = JF^{-1}p, \tag{2–3}$$

it has been shown by, for example, Walker et al. [1965] that the Maxwell–Lorentz equations in the reference configuration imply the relations

$$E = -\operatorname{Grad} \Phi, \tag{2–4}$$

$$D = \epsilon_0 J F^{-1} F^{-T} E + P, \tag{2–5}$$

where Grad represents the standard gradient operator with respect to the system of reference coordinates X, and $\Phi(X)$ is a scalar potential function of the variable X.

Next, we suppose that the dielectric possesses a strain energy function

$$W : M^{(n+1)\times n} \to \mathbb{R}$$

per unit volume of the reference configuration that depends upon both the deformation gradient and the electric polarization:

$$W = W(F, FP). \tag{2–6}$$

Furthermore, we introduce the electric enthalpy function $H(F, P, E)$ [Maugin and Trimarco 1991; Trimarco 2002] defined by

$$H(F, P, E) = W(F, FP) - \frac{1}{2}\epsilon_0 J E F^{-1} F^{-T} E - EP, \tag{2–7}$$

from which the identity immediately follows:

$$D = -\frac{\partial H}{\partial E}(F, P, E). \tag{2–8}$$

The complete set of equations governing the deformation of the dielectric may be obtained by considering the stationary points of the electric enthalpy (2–7) with respect to independent variations of Φ, F, and P. (See, for example, [Trimarco 2002; 2003; Maugin and Trimarco 1991; 2001; Maugin 1993; Pack and Herrmann 1986; Yu 1995] and the important discussion in [Ericksen 2006, to appear].) This yields the Euler–Lagrange equations:

$$\operatorname{Div} \frac{\partial H}{\partial F}(F, P, E) = 0, \tag{2–9}$$

$$\operatorname{Div} D = 0, \tag{2–10}$$

$$\frac{\partial H}{\partial P} \equiv \frac{\partial W}{\partial P} - E = 0, \tag{2–11}$$

where Div denotes the divergence operator with respect to the system of reference coordinates X.

We observe that Equations (2–10) and (2–4) are the usual electrostatics equations in the absence of electric free charge, and in fact (2–10) implies that d is solenoidal in the current configuration of the elastic dielectric. The classical electrostatics equations are consequentially recovered. Moreover, the transformation (2–3) and (2–5) together with (2–10) are consistent with the fundamental requirement that total electric charge be conserved.

From (2–11) we conclude that H is independent of P so that $H(F, E) : M^{(n+1)\times n} \to \mathbb{R}$

A crucial ingredient of the uniqueness proof described in the next section is a conservation law (or integral identity) that is stated and proved in the following lemma.

Lemma 2.1 [Ericksen 2006, to appear; Maugin 1993; Maugin and Epstein 1991; Trimarco 2002; 2003]. *Let $\Omega \subset \mathbb{R}^n$ have smooth boundary $\partial\Omega$ with unit outward normal N. Let* (2–4), (2–5) *and* (2–7) *hold, let* $(x, \operatorname{Grad} \Phi)$ *be a smooth solution to the equilibrium equations* (2–9)–(2–11), *and let the electric enthalpy* *satisfy* $H \in C^2(M^{(n+1)\times n}, \mathbb{R})$. *Then*:

$$\begin{aligned} n\int_\Omega H(F,E)\,dX \quad &= \quad \int_{\partial\Omega}\Big((NX)H(F,E) + tr\,\frac{\partial H(F,E)}{\partial F}\,\big(x-(X\,\mathrm{Grad})x\big) \\ &\qquad +ND(F,E)\,(\Phi-(X\,\mathrm{Grad})\Phi)\Big)\,dS, \end{aligned} \tag{2–12}$$

where dX and dS represent respectively the volume and surface elements of integration in the reference configuration.

Proof. The identity Equation (2–12) may be established by application of the divergence theorem either to the surface integral on the right and noting that

$$\frac{\partial H}{\partial X} = \frac{\partial H}{\partial F}\,\mathrm{Grad}\;F - (D\,\mathrm{Grad})E, \tag{2–13}$$

or, after rearrangement of the integrand and appeal to (2–9), to the integral identity

$$\int_\Omega\left(XF^T\,\mathrm{Div}\,\frac{\partial H}{\partial F}\right)\,dX = 0. \tag{2–14}$$

Instead, we prefer to employ the energy-momentum, or Eshelby, tensor B defined by

$$B = (W - EP)I - F^T\frac{\partial W}{\partial F} + E\otimes P, \tag{2–15}$$

where $E\otimes P$ denotes the tensor product of the vectors E and P. The relation

$$\frac{\partial H}{\partial F} = \frac{\partial W}{\partial F} + JTF^{-T}, \tag{2–16}$$

where T is the Maxwell stress tensor given explicitly by

$$T = \epsilon_0(e\otimes e - \frac{1}{2}eeI), \tag{2–17}$$

enables (2–15) to be alternatively expressed as

$$B = HI - F^T\frac{\partial H}{\partial F} + E\otimes D, \tag{2–18}$$

from which by appeal to (2–9) and (2–10) we may directly prove that

$$\mathrm{Div}\;B = 0. \tag{2–19}$$

On following Chadwick's [Chadwick 1975] or Hill's [Hill 1986] approach to the corresponding elastic problem, we have

$$\begin{aligned} \mathrm{Div}(XB) &= tr\;B \\ &= nH - \mathrm{Div}(x\frac{\partial H}{\partial F} - \mathrm{Div}(D\Phi), \end{aligned} \tag{2–20}$$

which by integration over Ω leads to (2–12). □

3. Uniqueness in the affine boundary value problem

In this section we prove that the affine boundary value problem has a unique smooth solution provided the electric enthalpy satisfies generalized convexity conditions. The region Ω is supposed to be star-shaped with respect to an interior point which without loss may be taken as the origin of coordinates so that

$$NX > 0, \quad X \in \partial\Omega. \tag{3–1}$$

We commence by considering two distinct smooth equilibrium solutions (x, Φ) and (y, Ψ) to the dielectric Equations (2–9)–(2–11) that satisfy the same boundary conditions in the following sense

$$x - y = 0, \qquad X \in \partial\Omega, \tag{3–2}$$

$$\Phi - \Psi = 0, \qquad X \in \partial\Omega. \tag{3–3}$$

Then by Hadamard's lemma [Hadamard 1903] it follows that

$$\operatorname{Grad}(y - x) = \lambda \otimes N, \quad \lambda = N \operatorname{Grad}(y - x), \quad X \in \partial\Omega, \tag{3–4}$$

$$\operatorname{Grad}(\Psi - \Phi) = \mu N, \quad \mu = N \operatorname{Grad}(\Psi - \Phi), \quad X \in \partial\Omega. \tag{3–5}$$

The first convexity assumption imposed on the electric enthalpy is that of rank-one convexity. The precise notion used is defined in the second of the following two related definitions.

Definition 3.1 (Rank-one convexity at a point). The function

$$H \in C(M^{(n+1)\times n}, \mathbb{R})$$

is (strictly) rank-one convex at F and E if and only if

$$H(F + ta \otimes b, E + tQ) \le tH(F + a \otimes b, E + Q) + (1 - t)H(F, E), \tag{3–6}$$

for all

$$t \in [0, 1], \quad F \in M_+^{n\times n}, \quad E \in \mathbb{R}^n, \quad Q \in \mathbb{R}^n, \quad a \in \mathbb{R}^n, \quad b \in \mathbb{R}^n,$$

such that $F + ta \otimes b \in M_+^{n\times n}$. Strict rank-one convexity at a point holds when the inequality in (3–6) is strict.

When $H \in C^1(M^{(n+1)\times n}, \mathbb{R})$, an immediate deduction from (3–6), obtained on taking the limit $t \to 0$, is the further inequality

$$H(F + a \otimes b, E + Q) \ge H(F, E) + \frac{\partial H(F, E)}{\partial F} a \otimes b + \frac{\partial H(F, E)}{\partial E} Q, \tag{3–7}$$

for all

$$F \in M_+^{n\times n}, \quad a \in \mathbb{R}^n, l \quad b \in \mathbb{R}^n, \quad E \in \mathbb{R}^n, \text{ and } Q \in \mathbb{R}^{n+},$$

such that $F + a \otimes b \in M_+^{n\times n}$.

Definition 3.2 (Rank-one convexity). The function $H(F, E)$ is (strictly) rank-one convex if and only if H is (strictly) rank-one convex at F and E for all $F \in M_+^{n\times n}$, and all $E \in \mathbb{R}^n$.

We can now state and prove the first lemma needed in the proof of uniqueness.

Lemma 3.1. *Let Ω be star-shaped with respect to the origin, and let*

$$H : M^{(n+1)\times n} \to \mathbb{R}$$

be rank-one convex. Let (x, Φ) and (y, Ψ) be distinct pairs of equilibrium smooth solutions to (2–9)–(2–11) *that satisfy the same Dirichlet boundary conditions in the sense of* (3–2) *and* (3–3). *When*

$$H \in C^1(M^{(n+1)\times n}, \mathbb{R}),$$

we have

$$\begin{aligned} n\int_{\Omega}\big(H(\text{Grad } x, -\text{Grad } \Phi) - H(\text{Grad } y, -\text{Grad } \Psi)\big)\,dX \\ \le \int_{\partial\Omega}\left\{\frac{\partial H(\text{Grad } x, -\text{Grad } \Phi)}{\partial F} - \frac{\partial H(\text{Grad } y, -\text{Grad } \Psi)}{\partial F}\right\}\times\{N\otimes(y-(X\,\text{Grad})\,y)\}\,dS \\ + \int_{\partial\Omega}\left\{N\frac{\partial H(\text{Grad } x, -\text{Grad } \Phi)}{\partial E} - N\frac{\partial H(\text{Grad } y, -\text{Grad } \Psi)}{\partial E}\right\}\times\{-\Psi + X\,\text{Grad } \Psi\}\,dS. \end{aligned} \tag{3–8}$$

Proof. The conservation law (2–12) by hypothesis is satisfied by both solutions (x, Φ) and (y, Ψ). Consequently, subtraction of the respective identities and appeal to (3–2)–(3–5) leads to the relation

$$\begin{aligned} n\int_{\Omega}\big(H(\text{Grad } x, -\text{Grad } \Phi) - H(\text{Grad } y, -\text{Grad } \Psi)\big)\,dX \\ = \int_{\partial\Omega} NX\big(H(\text{Grad } x, -\text{Grad } \Phi) - H(\text{Grad } x + \lambda\otimes N, -\text{Grad } \Phi - \mu N)\big)\,dS \\ + \int_{\partial\Omega} NX\Big(\frac{\partial H}{\partial F}(\text{Grad } x, \text{Grad } \Phi)N\otimes\lambda - \mu N\frac{\partial H}{\partial E}(\text{Grad } x, \text{Grad } \Phi)\Big)\,dS \\ + \int_{\partial\Omega}\Big(\frac{\partial H}{\partial F}(\text{Grad } x, \text{Grad } \Phi) - \frac{\partial H}{\partial F}(\text{Grad}\, y, \text{Grad } \Psi)\Big)\times\big(N\otimes(y - X, \text{Grad } y)\big)\,dS \\ + \int_{\partial\Omega}\big(ND(\text{Grad}\, x, \text{Grad } \Phi) - ND\,\text{Grad}\,(y, \text{Grad } \Psi)\big)\times(\Psi - X\,\text{Grad } \Psi)\,dS. \end{aligned} \tag{3–9}$$

The first two terms on the right are nonpositive by virtue of the star-shaped assumption (3–1) and inequality (3–7) for the rank-one convex function H. Consequently, the lemma is proved. □

Remark 3.1. It is apparent from the proof of Lemma 3.1 that rank-one convexity of H is required only on the set of surface values of $\text{Grad}(y - x)$ and $\text{Grad}(\Psi - \Phi)$.

We next restrict our attention to affine boundary conditions. We have as a corollary to Lemma 3.1 the following lemma:

Lemma 3.2. *Let Ω be star-shaped and let* (3–1) *be satisfied. Let (x, Φ) be a smooth equilibrium solution to* (2–9)–(2–11), *let the electric enthalpy H be rank-one convex, and let x and Φ satisfy the respective affine boundary conditions*

$$x = c + AX, \quad X \in \partial\Omega, \tag{3–10}$$

$$\Phi = d + bX, \quad X \in \partial\Omega, \tag{3–11}$$

where $A \in M_+^{n\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ *and* $d \in \mathbb{R}^n$ *are constant.*

Then

$$\int_\Omega H(\text{Grad } x, -\text{Grad } \Phi)\, dX \le \int_\Omega H(A, -b)\, dX. \tag{3–12}$$

Proof. Consider the affine equilibrium solution pair (y, Ψ) given by

$$y = c + AX, \qquad X \in \bar{\Omega}, \tag{3–13}$$

$$\Psi = d + bX, \qquad X \in \bar{\Omega}, \tag{3–14}$$

where as usual the overbar denotes closure; that is, $\bar{\Omega} = \Omega \cup \partial\Omega$. It easily follows that (y, Ψ) satisfies the boundary conditions (3–10) and (3–11), and also the relationships

$$\text{Grad } y = A, \quad \text{Grad } \Psi = b, \quad X \in \bar{\Omega}, \tag{3–15}$$

and

$$y - X\,\text{Grad } y = c, \qquad X \in \bar{\Omega}, \tag{3–16}$$

$$\Psi - X\,\text{Grad } \Psi = d, \qquad X \in \bar{\Omega}. \tag{3–17}$$

The proof of the Lemma is completed upon noticing that the right side of Equation (3–8) vanishes by virtue of (3–15)–(3–17), the divergence theorem, and the equilibrium equations (2–9)–(2–10). □

Uniqueness of the affine solution (3–13) and (3–14) requires the introduction of our second general convexity assumption defined as follows:

Definition 3.3 (Quasiconvexity). The function $H \in C(M^{(n+1)\times n}, \mathbb{R})$ is quasiconvex at (A, b) if and only if

$$\int_\Sigma H(A + \text{Grad } \chi, -b + \text{Grad } \theta)\, dX \ge \int_\Sigma H(A, -b)\, dX, \tag{3–18}$$

for every bounded open set Σ, and $\chi \in W_0^{1,\infty}(\Sigma, \mathbb{R})$, $\theta \in W_0^{1,\infty}(\Sigma, \mathbb{R})$.

Definition 3.4 (Strict Quasiconvexity.). The function H is strictly quasiconvex at (A, b) if and only if H is quasiconvex at (A, b) and equality holds only when $\chi = \theta = 0$.

The relation between rank-one convexity, quasiconvexity, and other notions of convexity is further discussed in, for example, [Ball 1976] and [Knops and Stuart 1984]. For present purposes, it is sufficient to note that all generalized notions of convexity reduce to the standard condition of convexity when the functions concerned are defined only on scalar and vector quantities.

We now proceed to establish uniqueness of the solution to the affine boundary problem. We have:

Proposition 3.1 (Uniqueness). *Let* Ω *be star-shaped with respect to the origin, and let the affine boundary conditions be* (3–10) *and* (3–11). *Let* $H \in C^2(M^{(n+1)\times n}, \mathbb{R})$ *be rank-one convex and strictly quasiconvex at* $(A, -b)$. *Then the unique smooth equilibrium solution is*

$$x = c + AX, \qquad X \in \bar{\Omega}, \tag{3–19}$$

$$\Phi = d + bX, \qquad X \in \bar{\Omega}. \tag{3–20}$$

Proof. Suppose that (x, Φ) and (y, Ψ) are equilibrium solutions satisfying the affine boundary conditions (3–10) and (3–11) such that $x \not\equiv y \equiv c + AX$ and $\Phi \not\equiv \Psi \equiv d + bX$ for $X \in \Omega$. Consider the volume integral on the left of Equation (2–12), which may be rewritten

$$\begin{aligned}\int_\Omega & H(\mathrm{Grad}\ x, -\mathrm{Grad}\ \Phi)\, dX \\ &= \int_\Omega H\big(\mathrm{Grad}\ y + \mathrm{Grad}(x-y), -\mathrm{Grad}\,\Psi + \mathrm{Grad}(\Psi-\Phi)\big)\, dX \\ &= \int_\Omega H\big(A + \mathrm{Grad}(x-y), -b + \mathrm{Grad}(\Psi-\Phi)\big)\, dX. \end{aligned} \tag{3–21}$$

By hypothesis, $x - y = \Psi - \Phi = 0$ for $X \in \partial\Omega$, and consequently strict quasiconvexity of H at $(A, -b)$ implies that

$$\int_\Omega H(\mathrm{Grad}\, x, -\mathrm{Grad}\,\Phi)\, dX > \int_\Omega H(A, -b)\, dX, \tag{3–22}$$

which contradicts inequality Equation (3–12) and the Proposition is proved. □

4. Concluding Remarks

This final section provides several remarks that supplement the previous results. In particular, we explore the implication of Proposition 3.1 for the uniqueness of the electric displacement and polarization, and consequently the electric free charge density on the surface $\partial\Omega$. For completeness, we also demonstrate for a given deformation gradient that the electric constituents are uniquely determined subject to mixed boundary conditions and a rank-one convex electric enthalpy. The conclusion represents a slight extension of the familiar property in electrostatics.

We commence with an observation whose validity is evident from an examination of the proof of Proposition 3.1.

Remark 4.1 (Alternate conditions). The conditions stipulated in Proposition 3.1 for H may be replaced by the alternative conditions of strict rank-one convexity and quasiconvexity at $(A, -b)$.

Remark 4.2 (Electric displacement and polarization). Suppose for simplicity that the strain energy function W is convex with respect to P so that (2–11) is invertible to give P uniquely in terms of E and F. We conclude that the conditions of Proposition 3.1 uniquely determine P to be constant. Consequently, (2–8) yields a unique constant value for D under the same conditions. Furthermore, the electric free charge surface density $\sigma(X)$ for $X \in \partial\Omega$ is given by

$$DN = \sigma. \tag{4–1}$$

where from (2–10) it necessarily follows that $\int_{\partial\Omega} \sigma\, dS = 0$. An appeal to (2–5) and (4–1) shows that σ is uniquely determined by P, E, and F, and therefore under the stipulated conditions is likewise unique.

On the other hand, when the electric potential is constant on the boundary, it follows as a special case of Proposition 3.1 that the electric field E vanishes everywhere in Ω, and by the assumed unique invertibility of (2–11), that the polarization is also identically zero. (See also Remark 4.4).

Remark 4.3 (Mixed boundary conditions: electric field). Suppose that

$$H(F, E) : M^{(n+1)\times n} \to \mathbb{R}$$

is strictly rank-one convex with respect to $E \in \mathbb{R}^n$ at each $F \in M_+^{n\times n}$ so that

$$H(F, tE + (1-t)Q) < tH(F, E) + (1-t)H(F, Q), \tag{4–2}$$

for all $E \in \mathbb{R}^n$, $Q \in \mathbb{R}^n$ and $t \in [0, 1]$. Notice, as already observed, that (4–2) is the usual condition for strict convexity as generalized definitions of convexity reduce to the corresponding standard definitions for scalar and vector quantities. Now consider the function defined by

$$I(t) = \int_\Omega \big(H(F, tE + (1-t)Q) - tH(F, E) - (1-t)H(F, Q)\big)\, dX, \tag{4–3}$$

which by inspection and (4–2) possesses the properties

$$I(0) = I(1) = 0, \tag{4–4}$$
$$I(t) < 0, \qquad t \in (0, 1). \tag{4–5}$$

Next assume that for each F there exist two distinct electric fields E, Q with potentials Φ, Ψ and corresponding electric displacements $D(F, E)$ and $D(F, Q)$. Instead of the affine boundary condition (3–11) we suppose nonaffine and nonuniform mixed boundary conditions such that for all $F \in M_+^{n\times n}$ and $E \not\equiv Q$ we have:

$$\Phi = \Psi, \qquad X \in \partial\Omega_1, \tag{4–6}$$
$$D(F, E)N = D(F, Q)N, \qquad X \in \partial\Omega_2, \tag{4–7}$$

where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, and

$$E = -\,\mathrm{Grad}\ \Phi, \qquad Q = -\,\mathrm{Grad}\ \Psi, \qquad X \in \Omega. \tag{4–8}$$

Let a superposed prime denote differentiation with respect to t. Examination of the graph of $I(t)$ immediately shows that

$$\begin{aligned} 0 > I'(0) &= \int_\Omega \Big(\frac{\partial H(F, Q)}{\partial E}(E - Q) - H(F, E) + H(F, Q)\Big)\, dX \\ &= \int_\Omega \Big(\frac{\partial H(F, Q)}{\partial E}(\mathrm{Grad}\ \Psi - \mathrm{Grad}\ \Phi) - H(F, E) + H(F, Q)\Big)\, dX \\ &= \int_\Omega \Big(-D(F, Q)(\mathrm{Grad}\ \Psi - \mathrm{Grad}\ \Phi) - H(F, E) + H(F, Q)\Big)\, dX, \end{aligned}$$

which after an integration by parts and appeal to Equation (2–10) and (4–6) gives

$$\int_\Omega H(F, E)\, dX > \int_\Omega H(F, Q)\, dX + \int_{\partial\Omega_2} D(F, Q)N(\Phi - \Psi)\, dS. \tag{4–9}$$

On noting Equation (4–7) and either by reversing the roles of E and Q, or by evaluating $I'(1) > 0$, we are led to a contradiction and consequently we conclude that

$$\Phi \equiv \Psi, \qquad X \in \bar{\Omega}, \tag{4–10}$$

and uniqueness of the electric field is established. This is not necessarily constant, unlike the case of the affine boundary value problem.

Remark 4.4 (Mixed boundary conditions: electric displacement and polarization). The conditions introduced into the previous remark are insufficient to provide uniqueness of the corresponding electric displacement and polarization vectors, which is not surprising, especially for ferroelectrics and similar materials. We emphasize, however, that for such materials the argument can easily be modified as follows to additionally obtain uniqueness of the electric displacement and polarization. Assume the conclusion is false and that P and R are the distinct respective polarizations. Let the nonaffine and nonuniform mixed boundary conditions be such that Equations (4–6) and (4–7) hold, and in addition to (rank-one) convexity of the electric enthalpy (4–2) with respect to E, suppose that the strain energy is strictly (rank-one) convex with respect to P in the sense that for each given F

$$W(F, tFP + (1-t)FR) < tW(F, FP) + (1-t)W(F, FR), \tag{4–11}$$

for $P, R \in \mathbb{R}^n$. The function $G(t)$, defined by

$$G(t) = \int_\Omega \big(W(F, tFP + (1-t)FR) - tW(F, FP) - (1-t)W(F, FR)\big)\, dX, \tag{4–12}$$

satisfies $G(0) = G(1) = 0$, and $G(t) < 0$, $0 < t < 1$, so that $G'(0) < 0$ and therefore by Equation (2–11) we have

$$\int_\Omega \big(E(P-R) - W(F, FP) + W(F, FR)\big)\, dX < 0. \tag{4–13}$$

But for each F we have shown already that E is uniquely determined, and so by interchange of P and R we are led to a contradiction and the polarization is unique. It is worth remarking that uniqueness of P is established here subject to conditions more general than those assumed in Remark 4.2. Uniqueness of the electric displacement now follows from relation (2–5).

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Received 7 Feb 2006. Accepted 21 Apr 2006.

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JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

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JoMMS (ISSN 1559-3959) is published in 10 issues a year. The subscription price for 2006 is US $400/year for the electronic version, and $500/year for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720–3840.

JoMMS peer-review and production is managed by EditFLOW™ from Mathematical Sciences Publishers.

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Typeset in LaTeX

Journal of Mechanics of Materials and Structures

Volume 1, № 5 May 2006

1559-3959(2006)1:5;1-E