Journal of
Mechanics of
Materials and Structures

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Volume 1, № 7 September 2006

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 1, No. 7, 2006
dx.doi.org/10.2140/jomms.2006.1.1257

TRANSIENT ANALYSIS OF A SUDDENLY-OPENING CRACK IN A COUPLED THERMOELASTIC SOLID WITH THERMAL RELAXATION

LOUIS MILTON BROCK AND MARK TODD HANSON

For a semiinfinite crack that opens in an unbounded thermoelastic solid initially at rest under uniform plane-strain tension at uniform temperature, the governing equations contain as special cases the Fourier model, and two thermal relaxation models with, respectively, one and two relaxation times. Integral transforms reduce the initial/mixed boundary value problem to a Wiener–Hopf equation. Its solution produces analytical expressions for temporal transforms of normal stress and temperature change near the crack edge. For 4340 steel, numerical inversions allow comparisons of the crack edge stress for the three thermoelastic models with the isothermal result, and temperature change at the crack edge for the two thermal relaxation models with the Fourier model result. Calculations indicate that thermoelasticity has a mild relaxation effect on the stress, and that temperature changes for the thermal relaxation model are much larger than those that arise for the Fourier model just after the crack opens. After a time interval in the order of a nanosecond, however, the Fourier changes are larger, although the deviation is minuscule.

Introduction

Thermal waves appear as a feature of heat conduction when thermal relaxation times are introduced into the classical Fourier law [Joseph and Preziosi 1989]. The modified heat equation is in turn incorporated in the coupled equations of thermoelasticity [Chandrasekharia 1986; Sharma and Sharma 2002] in a fashion similar to that used for the Fourier model [Chadwick 1960].

Fracture analysis, however, is often based on the Fourier model with partial or complete uncoupling of the heat and momentum balance equations, for example, [Rice and Levy 1969; Sumi and Katayama 1980; Noda et al. 1989; Rizk and Radwan 1993]. Such models can be justified on various grounds, as follows:

- For equilibrium or slow-moving cracks under thermal loading only or static mechanical loading, inertial effects may well be negligible.
- Heat production near a crack edge is controlled by plastic energy when yield occurs [Freund and Hutchinson 1985].
- The Fourier law is adequate for describing temperature fields measured near a crack edge [Mason and Rosakis 1993].
- Calculations for fluids [Fan and Lu 2002] suggest that thermal relaxation effects are highly transient.

Keywords: transient analysis, thermoplastic crack, thermal relaxation, dynamic stress intensity.

- Asymptotic analyses of dynamic steady-state fracture [Brock 2004; 2006] show that, even for high crack speeds, solution behavior can be described by the Fourier model, except very near the crack edge.

For more insight into thermal effects in dynamic fracture, in this paper we consider the canonical situation of a semiinfinite crack that opens suddenly in an unbounded solid. The solid is initially at rest at a uniform temperature in a uniform plane-strain tension field, and plastic yield is ignored. To ascertain the importance of thermal relaxation effects, we investigated a general form [Sharma and Sharma 2002] of the coupled thermoelastic equations which contains as special cases thermal relaxation models with one [Lord and Shulman 1967] and two [Green and Lindsay 1972] relaxation times, as well as the Fourier model [Chadwick 1960].

We use multiple integral transforms and a Wiener–Hopf technique, and analytically obtain the temporal transforms of normal stress and temperature change in the crack plane. By numerical inversion we then obtain calculations for 4340 steel that allow us to gauge thermal effects on stress at the crack edge and thermal relaxation effects on temperature change at the crack edge.

Governing equations

Consider governing equations for an isotropic, homogeneous linear coupled thermoelastic solid initially at uniform (absolute) temperature T_0 of the form

$$\nabla^2 \boldsymbol{u} + m\nabla\Delta - \boldsymbol{u}_{ss} - \alpha_v \nabla D_2 \theta = 0, \tag{1a}$$

$$h\nabla^2\theta - \frac{\varepsilon}{\alpha_v} D\Delta_s - D_1\theta_s = 0, \tag{1b}$$

$$\frac{1}{\mu}\boldsymbol{S} = \big[(m-1)\Delta - \alpha_v D_2\theta\big]\boldsymbol{I} + \nabla\boldsymbol{u} + \boldsymbol{u}\nabla, \tag{1c}$$

where $(\boldsymbol{u}, \Delta, \theta, \boldsymbol{S}, \boldsymbol{I})$ are, respectively, the displacement vector, dilatation, temperature change, and stress and identity tensors. In terms of $(\Delta, \theta, \boldsymbol{r})$, decomposition of Equation (1a) gives

$$a\nabla^2\Delta - \Delta_{ss} - \alpha_v D_2 \nabla^2\theta = 0, \qquad \nabla^2\boldsymbol{r} - \boldsymbol{r}_{ss} = 0, \tag{2}$$

where (∇^2, ∇) are, respectively, the Laplacian and gradient operators, where $\boldsymbol{r}$ is the rotation tensor.

For the single- and double-relaxation time model 1 [Lord and Shulman 1967] and 2 [Green and Lindsay 1972], and Fourier model F [Chadwick 1960], respectively, it is understood that

$$(1): D_2 = 1,\ D = D_1 \qquad (2): D = D_2 \qquad (F): D = D_1 = D_2 = 1, \tag{3}$$

where the thermal relaxation operators are

$$D_1 = 1 + h_1()_s \qquad D_2 = 1 + h_2()_s. \tag{4}$$

The operator $()_s$ signifies differentiation with respect to temporal variable $s = v_r \times$ (time). Note that the additional relaxation time in model 2 serves to introduce thermal relaxation explicitly into constitutive

Equation (1c). In (1)–(4) dimensionless parameters are defined as

$$m = \frac{1}{1-2\nu}, \qquad a = 2\left(\frac{1-\nu}{1-2\nu}\right), \qquad \varepsilon = \frac{\mu T_0}{\rho c_v}\alpha_v^2, \tag{5}$$

and thermoelastic characteristic lengths (h, h_1, h_2) and rotational wave speed v_r are defined as

$$h = \frac{K}{c_v\sqrt{\mu\rho}}, \qquad h_1 = v_r t_1, \qquad h_2 = v_r t_2, \qquad v_r = \sqrt{\frac{\mu}{\rho}}, \tag{6}$$

where material constants $t_1 > t_2 \geq 0$ are thermal relaxation times, and $(\nu, \mu, \rho, \alpha_v, c_v, K)$ are, respectively, Poisson's ratio, shear modulus, mass density, coefficient of volumetric thermal expansion, specific heat at constant volume, and thermal conductivity.

For many materials [Chadwick 1960; Achenbach 1973; Sharma and Sharma 2002]

$$v_r \approx O(10^3)\ \text{m/s}, \qquad m \geq 2, \qquad \varepsilon \approx O(10^{-2}), \tag{7a}$$

$$h \approx O(10^{-9})\ \text{m}, \qquad (t_1, t_2), \approx O(10^{-13})\ \text{s}. \tag{7b}$$

In view of (6), these values suggest that $h \gg h_1 > h_2$.

Crack problem

When at $s = 0$ a semiinfinite crack $(y = 0, x < 0)$ forms, an unbounded solid is motionless at uniform temperature T_0 in a state of plane strain generated by the biaxial tension field

$$\left(\sigma_x^0, \sigma_{xy}^0, \sigma_{yz}^0, \sigma_{zx}^0\right) = 0, \qquad \left(\sigma_y^0, \sigma_z^0\right) = (\sigma, \nu\sigma), \tag{8}$$

where (x, y, z) are Cartesian coordinates and σ is a positive constant. We invoke symmetry to study the crack problem in half-space $y > 0$ as the superposition

$$\boldsymbol{u} = \boldsymbol{u}^C + \boldsymbol{u}^0, \qquad \theta = \theta^C, \tag{9}$$

where $\boldsymbol{u}^0$ corresponds to (8) and $(\boldsymbol{u}^C, \boldsymbol{S}^C, \theta^C)$ are governed by (1)–(6) in the plane-strain limit, that is,

$$\boldsymbol{u}^C = \boldsymbol{u}^C(x, y, s) = (u_x^C, u_y^C), \qquad \boldsymbol{S}^C = \boldsymbol{S}^C(x, y, s) = (\sigma_x^C, \sigma_y^C, \sigma_z^C, \sigma_{xy}^C), \qquad \theta^C = \theta^C(x, y, s).$$

In addition, for $y > 0$ initial conditions are

$$s \leq 0 : \left(\boldsymbol{u}^C, \boldsymbol{S}^C, \theta^C\right) \equiv 0. \tag{10}$$

For $s > 0$, boundary conditions are

$$y = 0 : \sigma_y^C = -\sigma\ (x < 0), \qquad u_y^C = 0\ (x > 0), \qquad \left(\sigma_{xy}^C, \frac{\partial\theta^C}{\partial y}\right) = 0. \tag{11}$$

For $y > 0$ and finite $s \geq 0$, $(\boldsymbol{u}^C, S^C, \theta^C)$ must also be bounded above as $\sqrt{x^2 + y^2} \to \infty$. The condition on θ^C in (11) means that no heat is assumed to flow across the crack plane.

Initial/mixed boundary value problem

We adopt a Wiener–Hopf technique [Noble 1958] whereby the mixed conditions in (11) are replaced for $s > 0$ by the unmixed set

$$y = 0 : \sigma_y^C = -\sigma H(-x) + \Sigma_+(x, s)H(x), \qquad u_y^C = V_-(x, s)H(-x), \tag{12}$$

where $H()$ is the Heaviside function, (Σ_+) is the the unknown traction ahead of the crack edge, and (V_-) is the unknown crack opening displacement. The function $\Sigma_+(x, s)$ must be integrable for $x \geq 0$ and bounded and continuous for $x > 0$, while $V_-(x, s)$ must be bounded and continuous for $x < 0$, and vanish as $x \to 0$.

Unilateral [Sneddon 1972] and bilateral [Van der Pol and Bremmer 1950] Laplace transforms over temporal and spatial variables (s, x) are defined, respectively, as

$$\hat{f}(x) = \int_0^\infty f(x, s)e^{-ps}ds, \qquad \tilde{f} = \int_{-\infty}^\infty \hat{f}(x)e^{-pqx}dx, \tag{13}$$

where the transform variables (p, q) are real and imaginary, respectively, with $p > 0$. Application of (13) to (1)–(6) in view of the boundary conditions in (10), (12), the unmixed conditions in (11), the boundedness of (Σ_+, V_-) and $(\boldsymbol{u}^C, \boldsymbol{S}^C, \theta^C)$ produces a coupled set of linear ordinary differential equations. For $y > 0$, these yield the transforms

$$\begin{bmatrix} \tilde{u}_x^C \\ \tilde{u}_y^C \\ \dfrac{ap}{\alpha_v d_2}\tilde{\theta}^C \end{bmatrix} = \begin{bmatrix} q & q & B \\ -A_+ & -A_- & q \\ k_0^2 - k_+^2 & k_0^2 - k_-^2 & 0 \end{bmatrix} \begin{bmatrix} U_+e^{-pA_+y} \\ U_-e^{-pA_-y} \\ U_Be^{-pBy} \end{bmatrix}, \tag{14a}$$

$$\begin{bmatrix} \tilde{\sigma}_x^C \\ \tilde{\sigma}_y^C \\ \tilde{\sigma}_z^C \\ \tilde{\sigma}_{xy}^C \end{bmatrix} = \mu p \begin{bmatrix} 1 - 2A_+^2 & 1 - 2A_-^2 & 2qB \\ T & T & -2qB \\ T - 2A_+^2 & T - 2A_-^2 & 0 \\ -2qA_+ & -2qA_- & -T \end{bmatrix} \begin{bmatrix} U_+e^{-pA_+y} \\ U_-e^{-pA_-y} \\ U_Be^{-pBy} \end{bmatrix}, \tag{14b}$$

where $U_\pm$ and U_B are arbitrary functions of (p, q), and

$$A_\pm = \sqrt{k_\pm^2 - q^2}, \qquad B = \sqrt{1 - q^2}, \qquad T = 1 - 2q^2 \tag{15a}$$

$$k_\pm = \frac{k_0}{2\sqrt{hp}}\left(\sqrt{(\sqrt{hp} + \sqrt{ad_1})^2 + \varepsilon d} \pm \sqrt{(\sqrt{hp} - \sqrt{ad_1})^2 + \varepsilon d}\right), \quad k_0 = \frac{1}{\sqrt{a}}, \tag{15b}$$

$$(1) : d = d_1, \qquad (2) : d = d_2, \qquad (F) : d = d_1 = 1. \tag{15c}$$

The thermal relaxation factors (d_1, d_2) in (15c) are defined by

$$\left(k_0^2 - k_+^2\right)(k_0^2 - k_-^2) = -k_0^4\frac{\varepsilon d}{hp}, \qquad d_1 = 1 + h_1 p, \qquad d_2 = 1 + h_2 p. \tag{16}$$

For positive real p, (14a) and (14b) are bounded above for $y \to \infty$ only when branches $\mathrm{Im}(q) = 0$, $|\mathrm{Re}(q)| > (k_\pm, 1)$ are introduced, so that $\mathrm{Re}(A_\pm, B) \geq 0$ in the corresponding cut planes. For positive real p,

$$k_+ > k_0 > k_-, \; k_- < 1 \tag{17a}$$

$$hp \leq \frac{m+\varepsilon}{m(1-l_1)-\varepsilon l} : \; k_+ \geq 1 \tag{17b}$$

$$hp \geq \frac{m+\varepsilon}{m(1-l_1)-\varepsilon l} : \; k_+ \leq 1 \tag{17c}$$

$$(1) : l = l_1 = \frac{h_1}{h} \qquad (2) : l = \frac{h_2}{h}, \; l_1 = \frac{h_1}{h} \qquad (F) : l = l_1 = 1. \tag{17d}$$

Thus, the value of positive real p defines the relative position of the branch points of (A_+, B). Using (12) and the unmixed conditions in (11) in view of (14a) and (14b) gives

$$(U_+, U_-) = \frac{T}{\mu R}(A_-, -A_+)\Big(\tilde{\Sigma}_+ - \frac{\sigma}{p^2 q}\Big), \tag{18a}$$

$$U_B = \frac{2qA_+A_-}{\mu R}(k_- - k_+)\Big(\tilde{\Sigma}_+ - \frac{\sigma}{p^2 q}\Big), \tag{18b}$$

$$R = \big(k_0^2 - k_-^2\big)A_-R_+ - (k_0^2 - k_+^2)A_+R_-, \qquad R_\pm = 4q^2 A_\pm B^2 + T^2, \tag{18c}$$

subject to the constraint

$$\frac{A_+A_-}{\mu p R}(k_-^2 - k_+^2)\Big(\tilde{\Sigma}_+ - \frac{\sigma}{p^2 q}\Big) = \tilde{V}_-. \tag{19}$$

We solve Equation (19), which is of the Wiener–Hopf type, using (13) such that transforms $(\tilde{\Sigma}_+, \tilde{V}_-)$ are analytic in the overlapping regions $\mathrm{Re}(q) > 0-$ and $\mathrm{Re}(q) < 0+$, respectively, of the q-plane for positive real p.

Wiener–Hopf solution

In light of (15a) and (17), the term A_+A_- can be written as the product of factors $\sqrt{k_+ + q}\sqrt{k_- + q}$ and $\sqrt{k_+ - q}\sqrt{k_- q}$ that are analytic for positive real p in the overlapping regions $\mathrm{Re}(q) > -k_-$ and $\mathrm{Re}(q) < k_-$. Equation (18c) gives a Rayleigh function that for positive real p has branch cuts $\mathrm{Im}(q) = 0$, $|\,\mathrm{Re}(q)| > k_-$ and isolated real roots at $q = \pm k_R$ $(0 < k_R < k_-)$, and is given as

$$\frac{R}{q^2 A_-} \approx 2(k_+^2 - k_-^2)(k_0^2 - 1) \; (|q| \to \infty). \tag{20}$$

It follows that the function

$$G = \frac{R}{2(k_+^2 - k_-^2)\,(k_0^2 - 1)\,A_-(q^2 - k_R^2)} \tag{21}$$

has branch cuts $\mathrm{Im}(q) = 0$, $k_- < |\,\mathrm{Re}(q)| < \max(1, k_+)$ but no isolated roots or poles, is integrable at $q = \pm k_-$, and approaches unity for $|q| \to \infty$. It can therefore be written as the product G_+G_-, where

$G_\pm$ are analytic in overlapping regions $\mathrm{Re}(q) > -k_-$ and $\mathrm{Re}(q) < k_-$, respectively. For positive real p, we apply a standard technique [Noble 1958] for case (17b) and (17c) of the form, respectively,

$$\begin{aligned} \ln G_\pm(q) &= -\frac{1}{\pi}\int_{k_-}^{1}\frac{\psi_b dt}{t\pm q} + \frac{1}{\pi}\int_{1}^{k_+}\frac{\psi dt}{t\pm q}, \\ \ln G_\pm(q) &= -\frac{1}{\pi}\int_{1}^{k_-}\frac{\psi_c dt}{t\pm q} + \frac{1}{\pi}\int_{k_-}^{k_+}\frac{\psi dt}{t\pm q}, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \psi_b &= \tan^{-1}\frac{\sqrt{t^2-k_-^2}}{(k_+^2-k_0^2)T^2}\Big(4(k_+^2-k_-^2)t^2B + (k_0^2-k_-^2)\frac{T^2}{A_+}\Big), \\ \psi_c &= \tan^{-1}\frac{4(k_+^2-k_-^2)t^2A_+A_-\sqrt{t^2-1}}{T^2[(k_0^2-k_-^2)A_- - (k_0^2-k_+^2)A_+]}, \\ \psi &= \tan^{-1}\frac{T^2}{A_+}\frac{(k_0^2-k_-^2)\sqrt{t^2-k_-^2}}{4(k_+^2-k_-^2)\sqrt{t^2-1}\,\sqrt{t^2-k_-^2} + (k_0^2-k_+^2)T^2A_+}. \end{aligned} \tag{23}$$

Because $(G_\pm, G)$ are all analytic at $q = 0$, (21) and (22) also yield the Rayleigh root as

$$k_R = \frac{1}{G_0}\sqrt{\frac{k_+^2+k_-^2+k_+k_- - k_0^2}{2k_-(k_++k_-)(1-k_0^2)}}, \qquad G_0 = G_+(0) = G_-(0) \tag{24}$$

We can perform multiplication of (19) and rearrange terms to produce an equation whose sides are single-valued in overlapping regions $\mathrm{Re}(q) > 0-$ and $\mathrm{Re}(q) < 0+$. We then render the sides analytic in their respective regions by appropriate addition and subtraction of the residues of poles at $q = (0+, -\sqrt{k_R})$ that remain after multiplication and factorization. The result is

$$\begin{aligned} 2(k_0^2-1)G_-(q)\frac{q-k_R}{\sqrt{k_+-q}}\tilde{V}_- + \frac{\sigma}{\mu p^3}\frac{\sqrt{k_+}}{k_RG_+(0)q} \\ = \frac{\sigma}{\mu p^3 q}\Big(\frac{\sqrt{k_+}}{k_RG_+(0)} - \frac{\sqrt{k_++q}}{G_+(q)(q+k_R)}\Big) - \frac{\tilde{\Sigma}_+}{\mu p}\frac{\sqrt{k_++q}}{G_+(q)(q+k_R)}. \end{aligned} \tag{25}$$

Here the left and right sides are analytic for $\mathrm{Re}(q) < 0+$ and $\mathrm{Re}(q) > 0-$, respectively, and must be analytic continuations of a bounded entire function in the q-plane. That V_- must vanish continuously as $x \to 0-$ implies, in light of (13), that

$$pq\tilde{V}_- \to 0 \ (|q| \to \infty). \tag{26}$$

Thus, the left-hand side of (25) vanishes as $|q| \to \infty$, such that the entire function vanishes, and Equation (25) becomes

$$\tilde{\Sigma}_+ = \frac{\sigma}{p^2 q}\left(\frac{\sqrt{k_+}G_+(q)}{G_+(0)\sqrt{k_+ + q}}\frac{q+k_R}{k_R} - 1\right) \tag{27a}$$

$$\tilde{V}_- = \frac{\sigma}{2\mu(1-k_0^2)p^3}\,\frac{\sqrt{k_+}\sqrt{k_+ - q}}{G_+(0)G_-(q)k_R(q-k_R)q}. \tag{27b}$$

We then combine Equation (14a), (18) and (27) to obtain the temperature change transform

$$\tilde{\theta}^C = \frac{-\varepsilon\sigma}{2\mu\alpha_v m}\,\frac{d}{hp^3 d_2}\left(\frac{1}{A_+} - \frac{1}{A_-}\right)\frac{\sqrt{k_+}\sqrt{k_+ - q}}{G_+(0)G_-(q)k_R(q-k_R)}\,\frac{1}{(k_+^2 - k_-^2)q}, \tag{28a}$$

$$(1): d = d_1, d_2 = 1, \qquad (2): d = d_2, \qquad (F): d = d_2 = 1, \tag{28b}$$

for $y = 0$, where (18) holds.

Transient response ahead of a crack

In view of (14a) and (14b), (18) and (27) we can construct an exact transform solution for $\boldsymbol{u}^C$. The dynamic stress intensity factor for the total field $\boldsymbol{u}$, however, can be studied directly using (27a). The bilateral Laplace transform [Van der Pol and Bremmer 1950] has the inverse

$$\hat{f}(x) = \frac{p}{2\pi i}\int \tilde{f}e^{pqx}\,dq, \tag{29}$$

where p is real and positive, and for $\tilde{\Sigma}_+$ integration can be along the positive real side of the entire $\mathrm{Im}(q)$ axis. However the first and second terms in (27a) exhibit, respectively, only the branch cut $\mathrm{Im}(q) = 0$, $\mathrm{Re}(q) < -\sqrt{k_+}$ and the pole $q = 0$. The integrand in (29) decays exponentially as $|q| \to 0$ for $x > 0$ in the same region, so that residue theory can be used to give the temporal transform

$$\hat{\Sigma}_+ = -\frac{\sigma}{p} + \frac{\sigma}{\pi p^2}\sqrt{2(1-k_0^2)}\sqrt{\frac{k_+k_-(k_+ + k_-)}{k_+^2 + k_-^2 + k_+k_- - k_0^2}}\int_{k_+}^{\infty}\frac{G_-(u)(k_R - u)}{u\sqrt{u - k_+}}e^{-pux}\,du. \tag{30}$$

Integrating over positive real u, and using (24),we eliminate the product $k_R G_+(0)$. For $x \to 0+$ we find an asymptotic form for the integral in (30) analytically. Then, in view of (8), (9), (13), (15), (16) and (28) the temporal transform of the normal traction ahead of the crack edge ($y = 0$, $x \approx 0+$) is

$$\hat{\sigma}_y = \sqrt{\frac{2(1-k_0^2)}{\pi x}}\,\frac{\sigma}{p^{3/2}}\sqrt{\frac{k_+k_-(k_+ + k_-)}{k_+^2 + k_-^2 + k_+k_- - k_0^2}} + O(x). \tag{31}$$

Study of (28) shows that we can find a similar analytic solution for positive real p in the $q-$plane. Thus, for $y=0$, $x \approx 0+$, and using (9) and (29),

$$\hat{\theta} = \frac{-\varepsilon\sigma}{\mu\alpha_v} \frac{k_0^2 d}{hp^3 d_2} \frac{k_+ k_-}{k_+ - k_-} \frac{1}{k_+^2 + k_-^2 + k_+ k_- - k_0^2} + \frac{\varepsilon\sigma}{\mu\alpha_v} \frac{d}{hd_2(k_+ - k_-)p^{3/2}} \sqrt{\frac{k_+ k_-}{(k_+^2 + k_-^2 + k_+ k_- - k_0^2)(k_+ + k_-)}} \frac{1}{\sqrt{\pi(1-k_0^2)x}} + O(x). \quad (32)$$

In light of (22) and (23), (31) and (32) are insensitive to any formal distinction between cases (17b), and (17c). Equations (16) and (28b) again govern parameters (d, d_2). The isothermal result corresponding to (31) is extracted from work by Achenbach [1973] and inverted to give

$$\sigma_y^i = 2\sigma \sqrt{\frac{2(1-k_0^2)}{\pi\sqrt{a}}} \sqrt{\frac{s}{x}} + O(x) \quad (y=0, x \approx 0+). \quad (33)$$

We can compare transient isothermal and thermoelastic crack edge stresses by examining the dimensionless ratio of the transform inversions of the singular terms in (31) with those in (33), that is,

$$\left.\frac{\sigma_y}{\sigma_y^i}\right|_{(y=0, x\approx 0+)} = \left(\frac{K_1}{K_i}, \frac{K_2}{K_i}, \frac{K_F}{K_i}\right), \quad (34)$$

where K_i, K_1, K_2, and K_F are coefficients of the dynamic stress intensity factors for respectively, the isothermal model, the thermal relaxation models 1 and 2, and the Fourier model. The coefficient for the isothermal model is defined as

$$K_i = \frac{2}{a^{1/4}} \sqrt{\frac{s}{\pi}}. \quad (35)$$

By (15) the temporal transforms for K_1, K_2, and K_F are

$$\hat{K}_1 = \frac{\left((\sqrt{hp} + \sqrt{ad_1})^2 + \varepsilon d_1\right)^{1/4}}{p^{3/2}\sqrt{a_\varepsilon \sqrt{d_1} + \sqrt{ahp}}}, \quad a_\varepsilon = a + \varepsilon,$$

$$\hat{K}_2 = \frac{d_1^{1/4}\left((\sqrt{hp} + \sqrt{ad_1})^2 + \varepsilon d_2\right)^{1/4}}{p^{3/2}\sqrt{ad_1 + \varepsilon d_2 + \sqrt{ahpd_1}}}, \quad (36)$$

$$\hat{K}_F = \frac{\left((\sqrt{a} + \sqrt{hp})^2 + \varepsilon\right)^{1/4}}{p^{3/2}\sqrt{a_\varepsilon + \sqrt{ahp}}}.$$

Similarly, transient temperature changes at the crack edge for the thermal relaxation and Fourier models can be compared in terms of the dimensionless ratios

$$\left.\frac{\theta}{\theta_F}\right|_{(y=0, x\approx 0+)} = \left(\frac{K_1^T}{K_F^T}, \frac{K_2^T}{K_F^T}\right), \quad (37)$$

where K_1^T, K_2^T and , K_F^T are coefficients of the singular terms in (32). Their temporal transforms are

$$\hat{K}_1^T = \frac{d_1}{p^{3/2}[(\sqrt{hp}+\sqrt{ad_1})^2+\varepsilon d_1]^{1/4}} \frac{1}{\sqrt{(\sqrt{hp}-\sqrt{ad_1})^2+\varepsilon d_1}\sqrt{a_\varepsilon\sqrt{d_1}+\sqrt{ahp}}}, \tag{38a}$$

$$\hat{K}_2^T = \frac{d_1^{1/4}}{p^{3/2}[(\sqrt{hp}+\sqrt{ad_1})^2+\varepsilon d_2]^{1/4}} \frac{1}{\sqrt{(\sqrt{hp}-\sqrt{ad_1})^2+\varepsilon d_2}\sqrt{ad_1+\varepsilon d_2+\sqrt{ahpd_1}}}, \tag{38b}$$

$$\hat{K}_F^T = \frac{1}{p^{3/2}[(\sqrt{hp}+\sqrt{a})^2+\varepsilon]^{1/4}} \frac{1}{\sqrt{(\sqrt{hp}-\sqrt{a})^2+\varepsilon}\sqrt{a_\varepsilon+\sqrt{ahp}}}. \tag{38c}$$

Some numerical results

To demonstrate the behavior of (34) and (37), consider 4340 steel initially at rest at room temperature (294 K) with elastic properties [Brock 2006]

$$\nu = \frac{1}{3}, \qquad \rho = 7834\,\text{kg/m}^3, \qquad \mu = 75\,\text{GPa},$$

and thermal properties

$$c_v = 448\,\text{J/kgK}, \qquad \alpha_v = 88.2(10^{-6})\,1/\text{K}, \qquad K = 34.6\,\text{W/mK},$$
$$t_1 = 0.75(10^{-13})\,\text{s}, \qquad t_2 = 0.5(10^{-13})\,\text{s}.$$

The ratios in (36) and (38) are actually functions of dimensionless parameter s/h. In light of (6), the following correspondence holds:

$$\frac{s}{h} = 1.0: \quad 1.233(10^{-12})\,\text{s}. \tag{39}$$

we use a standard procedure [Weeks 1966] to carry out numerical inversion of (36) and (38).

For fluids, thermal relaxation effects may be most prominent near a disturbance (here, the crack edge) for extremely short times after the disturbance arises [Fan and Lu 2002]. Therefore, our calculations of (34) and (37) are given in Table 1 for values $s/h \ll 1$ and $s/h \gg 1$ that correspond to the nano-second range.

For the stress field ratios of Equation (34), all three thermoelastic models serve to relax the isothermal crack edge stress, but by margins of less than 1%. For $s/h \ll 1$ some variation exists between the thermoelastic models. The result of the Fourier model is closer to the isothermal model than either of the thermal relaxation models. In addition, for $s/h \ll 1$ the differences between results of the isothermal and thermoelastic models decrease with time. For $s/h \gg 1$, however, all three thermoelastic models produce essentially the same constant deviation from the isothermal result.

For transient temperature changes at the crack edge of (37), much more transient behavior is seen for $s/h \ll 1$. Crack edge temperature changes for models 1 and 2 exceed the Fourier model change. For model 1 the deviation is orders of magnitude larger, but for both relaxation models, the deviations themselves diminish with s/h. This domination by model 1 can be predicted by noting that transforms

(38a), (38b) are proportional to $(d_1, d_1^{1/4})$, respectively. In light of (16)

$$\frac{\hat{K}_1^T}{\hat{K}_2^T} \approx O(p^{3/4}) \quad (p \to \infty). \tag{40}$$

Therefore it is not surprising that an asymptotic analysis gives

$$\frac{K_1^T}{K_2^T} \approx O\left(\frac{h}{s}\right), \quad \left(\frac{s}{h} \to 0\right). \tag{41}$$

For $s/h \gg 1$ the crack edge temperature changes for models 1 and 2 are essentially identical. They fall below the Fourier model change, but the deviation is almost negligible and decreases with s/h. In view of (39) it is less than 0.1% at times after crack opening of nano-second order.

Discussion and conclusions

In this article we treat the transient problem of a semiinfinite crack that opens instantaneously in an isotropic, homogeneous solid that is initially at rest at uniform temperature in a state of uniform plane strain. The coupled thermoelastic governing equations for the solid included as special cases thermal models with one [Lord and Shulman 1967] and two [Green and Lindsay 1972] relaxation times, as well as the classical Fourier model [Chadwick 1960].

To solve the initial/mixed boundary value problem, we used integral transforms and a Wiener–Hopf technique. We obtained exact expressions for the temporal transforms of the normal stress and temperature change near the crack edge. Numerical inversion for 4340 steel gave dimensionless ratios of crack edge stress for the three thermoelastic models with the isothermal result, and temperature change at the crack edge for the two thermal relaxation models with the Fourier model.

Calculations for crack edge stress showed that all three thermoelastic models relaxed the isothermal crack edge stress, but by less than 1%. Moreover, within time intervals on the order of a nanosecond after a crack opens, the deviations of the three thermoelastic models were essentially the same, and constant.

Calculations for temperature change at the crack edge showed pronounced transient behavior within a time interval of orders magnitude less than a nanosecond after a crack opens. Then, temperature changes at the crack edge for both thermal relaxation models exceeded values for the Fourier model. Indeed, deviation for the single-relaxation time model was by orders of magnitude. The deviation of both models did decrease with time. Within times on the order of a nanosecond, thermal relaxation changes were essentially identical, with values less than the Fourier results. However, the deviation was by a fraction of 1% in both cases. This behavior indicates that explicit inclusion of thermal relaxation in the constitutive equation [Green and Lindsay 1972] can serve to moderate short-time temperature change.

In summary, our results indicate that thermoelasticity may have a small effect on crack edge stress, and one that quickly becomes time-invariant and insensitive to thermal relaxation.Temperature changes at the crack edge, however, are highly transient early on, at which time thermal relaxation effects are dominant. By times of nano-second order, the Fourier model gives the larger changes, but the deviation is both negligible and decreasing.

$\frac{s}{h} \ll 1$	$\frac{K_1}{K_i}$	$\frac{K_2}{K_i}$	$\frac{K_F}{K_i}$	$\frac{K_1^T}{K_F^T}$	$\frac{K_2^T}{K_F^T}$
0.0001	0.9982	0.9988	0.9998	1522	1.416
0.001	0.99818	0.99875	0.9993	151	1.401
0.01	0.99813	0.99867	0.9978	13.739	1.258
$\frac{s}{h} \gg 1$					
100	0.99687	0.99687	0.9532	0.9915	0.9915
1000	0.99688	0.99687	0.9831	0.9968	0.9968
10,000	0.9969	0.9969	0.9969	0.9989	0.9989
100,000	0.9969	0.9969	0.9969	0.9996	0.9996

Table 1. Dimensionless ratios of crack edge stress and temperature change versus dimensionless time $\big(s/h = 1 : 1.233(10^{-12})\, s\big)$.

As a dynamic fracture model, the canonical problem treated here is especially idealized because plastic yield is neglected, and the crack opens instantaneously along its entire length. Nevertheless, both the relative insensitivity of crack edge stress to thermoelasticity, and the highly transient temperature changes at the crack edge, which are sensitive to thermal relaxation seen here, may be worth noting when considering a less idealized situation.

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Received 5 Mar 2006. Accepted 12 May 2006.

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