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AXIALLY SYMMETRIC CONTACT PROBLEM OF FINITE ELASTICITY AND ITS APPLICATION TO ESTIMATING RESIDUAL STRESSES BY CONE INDENTATION

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We discuss the axially symmetric contact problem of finite elasticity (theory of small deformation on initial stress body) and its application to estimating residual stresses by cone indentation. In particular, we determine the relation among the penetration depth, the contact radius and the residual stress.

1. Introduction

The residual stress problem is very important in engineering. Suresh and Giannakopoulos [1998] have pointed out that in the classical theory of elasticity the penetration depth, contact radius, and contact pressure are all independent of the residual stress and, thus, cannot be used to determine it. This independence can be explained as follows. In the linear theory of elasticity two independent solutions can be superposed to form a new solution according to the principle of superposition. Therefore, the cone indentation stress solution without residual stress and the residual (homogeneous) stress solution can be superposed to form a new solution. This new superposed solution is for the cone indentation stress field with residual stress and is unique due to the uniqueness of the linear theory of elasticity. It is apparent that the cone indentation stress field is independent of the residual stress in this solution. Therefore, the residual stress cannot be determined from it. In order to avoid this independence one has to deviate from the classical theory of elasticity. The theory of finite elasticity, that is, the theory of small deformations with initial stress body, is nonlinear so that two independent stress fields cannot be superposed. On the basis of it, one can deal with the residual stress. Naturally, the theory of plasticity is also nonlinear. However, according to it one can deal with the problem of unloading, that is, tension residual stress and indentation, which is difficult to discuss. Therefore, the results in this paper are only an initial effort in studying the plasticity behavior. Further developments will be discussed in another paper.

Here we address the theory of small deformation with initial stress body, which has been studied for a long time [Southwell 1913; Green and Shield 1951; Ericksen 1953; Bernstein and Toupin 1960; Payne and Weinberger 1961; Truesdell 1961; Hayes and Rivlin 1961; Pearson 1950; Holden 1964; Beatty 1971; Savwers and Rivlin 1973; 1977; 1978; Lurie 1990, Chapter 8; 1986; Hwang 1989]. Lurie [1990] and Hwang [1989] summarize the known results prior to 1989. On the basis of the theory of finite elasticity, that is, the theory of small deformations with initial stress body, we discuss in this paper the axially symmetric contact problem. Its application to estimating residual stresses by cone indentation

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is considered. Specifically, we have determined the relation among the penetration depth, the contact radius, and the residual stress.

2. Axially symmetric deformation and strain energy function

For the axially symmetric case let (x_1, x_2) be the position of a point before the deformation, where $x_1 = z$, $x_2 = r$, and (y_1, y_2) after the deformation. $\lambda_1, \lambda_2, \lambda_\theta$ denote the initial stretches due to residual stress with $y_j = \lambda_j x_j + u_j$, for $j = 1, 2$. For comparison, in the classical theory, $y_j = x_j + u_j$. Then,

$$F_{ij} = \frac{\partial y_i}{\partial x_j} = \lambda_i \delta_{ji} + u_{i,j}, \qquad (u_\theta = 0),$$

where there is no sum over i, $u_{i,j} = \partial u_i / \partial x_j$ and $|u_{i,j}| = O(\varepsilon) \ll 1$.

Consider the components λ_θ and $F_{\theta\theta}$. Since λ_θ is the length after the initial deformation (residual stress) divided by the length before the deformation, and $F_{\theta\theta}$ is the same ratio after the deformation, one has

$$\lambda_\theta = \frac{\lambda_2 x_2 \theta}{x_2 \theta} = \lambda_2, \qquad F_{\theta\theta} = \frac{(\lambda_2 x_2 + u_2)\theta}{x_2 \theta} = \frac{\lambda_2 x_2 + u_2}{x_2} = \lambda_2 + \frac{u_2}{x_2}.$$

For the deformation tensor components C_{ij} we have

$$C_{ij} = \frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j} = \lambda_j u_{j,i} + \lambda_i u_{i,j} + \lambda_i \lambda_j \delta_{ij} + O(\varepsilon^2),$$

no sum over i or j implied.

Expanding $C_{ij} = C_{ij\,0} + \delta C_{ij} + O(\varepsilon^2)$ one has

$$C_{ij\,0} = \lambda_i \lambda_j \delta_{ij} \qquad \text{and} \qquad \delta C_{ij} = \lambda_j u_{j,i} + \lambda_i u_{i,j}.$$

For the deformation tensor component $C_{\theta\theta}$,

$$\begin{aligned}
C_{\theta\theta} &= C_{\theta\theta\,0} + \delta C_{\theta\theta} = \lambda_2^2 + 2\lambda_2 \frac{u_2}{x_2},\\
C_{\theta\theta\,0} &= \lambda_2^2,\\
\delta C_{\theta\theta} &= 2\lambda_2 \frac{u_2}{x_2},\\
\delta C_{\theta k} &= 0 \quad (\theta \neq k),\\
C_{kk} &= \lambda_r^2 + \lambda_1^2 + \lambda_\theta^2 + \delta C_{11} + \delta C_{22} + \delta C_{\theta\theta},\\
C_{kk\,0} &= \lambda_r^2 + \lambda_1^2 + \lambda_\theta^2 = 2\lambda_r^2 + \lambda_1^2.
\end{aligned}$$

According to Lurie [1990], the strain energy function can utilize a variety of materials, for example, the Money material, the Monahan material, the Blats–Ko material, semi-linear material, the neo-Hook material, and others. It must be pointed out that Ranht et al. [1978] used the neo-Hook material for the incompressible case. This result can only be considered as a preliminary attempt to deal with the plasticity behavior, which we will discuss in another paper. Here, for convenience, we use the semi-linear material as follows:

$$W = \tfrac{1}{8}\lambda(C_{kk}-3)^2 + \tfrac{1}{4}\mu(C_{ij}-\delta_{ij})(C_{ij}-\delta_{ij}), \qquad W(\lambda_i = 1, u_j = 0) = 0,$$

where λ and μ are material constants to be discussed later.

Setting $\lambda_1 = \lambda_2 = \lambda_\theta = 1$, one has

$$\begin{aligned}
\delta C_{ij} &= \lambda_i u_{i,j} + \lambda_j u_{j,i} + O(\varepsilon^2) = u_{i,j} + u_{j,i} + O(\varepsilon^2) = 2\varepsilon_{ij},\\
\delta C_{\theta\theta} &= 2\lambda_\theta \frac{u_r}{x_2} = 2\frac{u_r}{x_2} = 2\varepsilon_{\theta\theta},\\
C_{ij\,0} &= \delta_{ij},\\
C_{\theta\theta\,0} &= 1,\\
(C_{kk}-3) &= \delta_{kk} + 2\varepsilon_{kk} - 3 = 2\varepsilon_{kk} = 2(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{\theta\theta}),\\
(C_{ij}-\delta_{ij}) &= \delta_{ij} + 2\varepsilon_{ij} - \delta_{ij} = 2\varepsilon_{ij},\\
W &= \tfrac{1}{8}\lambda(C_{kk}-3)^2 + \mu\tfrac{1}{4}(C_{ij}-\delta_{ij})(C_{ij}-\delta_{ij})\\
&= \tfrac{1}{2}\lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{\theta\theta})^2 + \mu(\varepsilon_{11}^2+\varepsilon_{22}^2+\varepsilon_{\theta\theta}^2+\varepsilon_{12}\varepsilon_{12}+\varepsilon_{21}\varepsilon_{21}).
\end{aligned}$$

This coincides with the classical theory. In it μ is the shear modulus and $\lambda = E\nu/\big((1+\nu)(1-2\nu)\big)$ is Lamé's constant, where E is the Young's modulus and ν is the Poisson ratio.

3. Stresses and equilibrium

The Piola stress is given by

$$\sigma_{ij} = 2\frac{\partial y_i}{\partial x_k}\frac{\partial W}{\partial C_{kj}}, \qquad \sigma_{\theta\theta} = 2\left(\lambda_\theta + \frac{u_r}{x_2}\right)\frac{\partial W}{\partial C_{\theta\theta}}.$$

It is not symmetric, but on the basis of moment equilibrium satisfies the identity

$$\sigma_{k2}\frac{\partial x_1}{\partial y_k} = \sigma_{k1}\frac{\partial x_2}{\partial y_k} \quad (k = 1, 2).$$

Using the Taylor series expansion $f(a+\delta) = f(a) + f'(a)\delta + O(\delta^2)$, one has

$$\frac{\partial W}{\partial C_{kj}} = W_{kj} + W_{kjlm}\delta C_{lm} + O(\varepsilon^2),$$

where $W_{ij} = (\partial W/\partial C_{ij})_0$, $W_{ijkl} = (\partial^2 W/\partial C_{ij}\partial C_{kl})_0$. Here the notation $(\ )_0$ means that W is a function of the initial stretches λ_i due to residual stress, but not a function of δC_{lm}.

The expansion of σ_{ij} is given by

$$\begin{aligned}
\sigma_{ij} &= \sigma_{ij\,0} + \delta\sigma_{ij}, &\quad \sigma_{ij\,0} &= 2\lambda_i W_{ij}, &\quad \delta\sigma_{ij} &= 2W_{kj}u_{i,k} + 2\lambda_i W_{ijkl}\delta C_{kl},\\
\sigma_{\theta\theta} &= \sigma_{\theta\theta\,0} + \delta\sigma_{\theta\theta}, &\quad \sigma_{\theta\theta\,0} &= 2\lambda_\theta W_{\theta\theta}, &\quad \delta\sigma_{\theta\theta} &= 2W_{\theta\theta}u_{2,2} + 2\lambda_\theta W_{\theta\theta kl}\delta C_{kl}.
\end{aligned}$$

In order to clarify the stress components, we now turn to the coefficients W_{ij} and W_{ijkl}. Since we are interested in the axially symmetric case, $\lambda_\theta = \lambda_2$, $u_\theta = 0$, and u_1, u_2 are functions of x_1, x_2. Altogether, the coefficients are

$$W_{ij} = \left(\frac{\partial W}{\partial C_{ij}}\right)_0, \qquad W_{ijkl} = \left(\frac{\partial^2 W}{\partial C_{ij} \partial C_{kl}}\right)_0,$$

$$W_{11} = \tfrac{1}{4}\lambda(\lambda_2^2 + \lambda_1^2 + \lambda_\theta^2 - 3) + \tfrac{1}{2}\mu(\lambda_1^2 - 1),$$

$$W_{22} = W_{\theta\theta} = \tfrac{1}{4}\lambda(\lambda_2^2 + \lambda_1^2 + \lambda_\theta^2 - 3) + \tfrac{1}{2}\mu(\lambda_2^2 - 1),$$

$$W_{12} = W_{21} = W_{1\theta} = W_{\theta 1} = W_{2\theta} = W_{\theta 2} = 0,$$

$$W_{1111} = W_{2222} = W_{\theta\theta\theta\theta} = \tfrac{1}{4}\lambda + \tfrac{1}{2}\mu, \qquad W_{2211} = W_{\theta\theta 11} = W_{22\theta\theta} = \tfrac{1}{4}\lambda,$$

$$W_{1212} = W_{2121} = W_{1\theta 1\theta} = W_{\theta 1\theta 1} = W_{2\theta 2\theta} = W_{\theta 2\theta 2} = \tfrac{1}{2}\mu.$$

Since the surface of the half plane is free of traction before the press of the cone, the stress component $\sigma_{11\,0}$ must be zero. Therefore, one has

$$\sigma_{11\,0} = 2\lambda_1 W_{11} = \tfrac{1}{2}\lambda_1\big(\lambda(2\lambda_2^2 + \lambda_1^2 - 3) + 2\mu(\lambda_1^2 - 1)\big) = 0.$$

Setting $W_{11} = 0$ and solving for λ_1 we get

$$\lambda_1^2 = 1 - 2\lambda\left(\frac{\lambda_2^2 - 1}{\lambda + 2\mu}\right). \tag{1}$$

Similarly, one can obtain the relation between λ_2 and the residual stress $\sigma_R = \sigma_{22\,0}$:

$$\sigma_R = \sigma_{22\,0} = \lambda_2\mu(\lambda_2^2 - 1)\left(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\right) \quad \text{or} \quad \lambda_2^3 - \lambda_2 = \sigma_{22\,0}\left(\frac{\lambda + 2\mu}{3\lambda\mu + 2\mu^2}\right). \tag{2}$$

The homogeneous residual stress σ_R leads to residual stresses $\sigma_x = \sigma_y = \sigma_R$ when the xy plane is parallel to the surface.

Let us now look at the stress components σ_{ij}, $\sigma_{\theta\theta}$. Using $\lambda_2 = \lambda_\theta$, $u_\theta = 0$ and the fact that u_1, u_2 are functions of x_1, x_2, one obtains

$$\sigma_{\theta\theta} = \lambda_2\mu(\lambda_2^2 - 1)\left(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\right) + 2\frac{u_2}{x_2}\left(\frac{\lambda}{4}(\lambda_2^2 + \lambda_1^2 + \lambda_\theta^2 - 3) + \frac{\mu}{2}(\lambda_\theta^2 - 1)\right) + 2\lambda_\theta\left(\left(\frac{\lambda}{2} + \mu\right)\lambda_\theta\frac{u_2}{x_2} + \frac{\lambda}{4}(2\lambda_2 u_{2,2} + 2\lambda_1 u_{1,1})\right),$$

$$\sigma_{\theta\theta\,0} = \lambda_2\mu(\lambda_2^2 - 1)\left(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\right),$$

$$\delta\sigma_{\theta\theta} = \left(\frac{\lambda}{2}(2\lambda_2^2+\lambda_1^2-3)+\mu(\lambda_2^2-1)+\lambda_2^2(\lambda+2\mu)\right)\frac{u_2}{x_2}+\lambda_2\lambda(\lambda_2 u_{2,2}+\lambda_1 u_{1,1});$$

$$\sigma_{22} = \lambda_2\left(\frac{\lambda}{2}(2\lambda_2^2+\lambda_1^2-3)+\mu(\lambda_2^2-1)\right)+u_{2,2}\left(\frac{\lambda}{2}(2\lambda_2^2+\lambda_1^2-3)+\mu(\lambda_2^2-1)\right)$$
$$+\lambda_2\left((\lambda+2\mu)\lambda_2 u_{2,2}+\lambda\left(\lambda_1 u_{1,1}+\lambda_2\frac{u_2}{x_2}\right)\right),$$

$$\sigma_{22\,0} = \lambda_2\left(\frac{\lambda}{2}(2\lambda_2^2+\lambda_1^2-3)+\mu(\lambda_2^2-1)\right),$$

$$\delta\sigma_{22} = \left(\frac{\lambda}{2}(2\lambda_2^2+\lambda_1^2-3)+\mu(\lambda_2^2-1)+\lambda_2^2(\lambda+2\mu)\right)u_{2,2}+\lambda\lambda_2\left(\lambda_1 u_{1,1}+\lambda_2\frac{u_2}{x_2}\right),$$

$$\sigma_{11} = \left(\frac{\lambda}{2}(2\lambda_2^2+\lambda_1^2-3)+\mu(\lambda_1^2-1)+\lambda_1^2(\lambda+2\mu)\right)u_{1,1}+\lambda\lambda_1\left(\lambda_2 u_{2,2}+\lambda_2\frac{u_2}{x_2}\right),$$

$$\sigma_{11\,0} = 0,$$

$$\delta\sigma_{11} = \left(\frac{\lambda}{2}(2\lambda_2^2+\lambda_1^2-3)+\mu(\lambda_1^2-1)+\lambda_1^2(\lambda+2\mu)\right)u_{1},_{1}+\lambda\lambda_1\lambda_2\left(u_{2,2}+\frac{u_2}{x_2}\right);$$

$$\sigma_{21} = 2\lambda_2 W_{21}+2u_{2,1}W_{zz}+2\lambda_2 W_{2121}\delta C_{21} = \lambda_2\mu(\lambda_2 u_{2},_{1}+\lambda_1 u_{1,2}),$$

$$\sigma_{21\,0} = 0, \qquad \delta\sigma_{21} = \lambda_2\mu(\lambda_2 u_{2,1}+\lambda_1 u_{1,2});$$

$$\sigma_{12} = 2u_{1,2}W_{22}+\left(\frac{\lambda_1}{\lambda_2}\right)\sigma_{21}, \quad \sigma_{12\,0} = 0, \quad \delta\sigma_{12} = 2u_{1,2}W_{22}+\left(\frac{\lambda_1}{\lambda_2}\right)\delta\sigma_{21}. \tag{3}$$

We also have the equilibrium equations

$$\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\sigma_{22}-\sigma_{\theta\theta}}{x_2} = 0 \qquad \text{and} \qquad \frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\sigma_{12}}{x_2} = 0 \quad \text{for } \sigma_{12}\neq\sigma_{21}.$$

Using the equalities $\sigma_{22\,0} = \sigma_{\theta\theta\,0}$ and $\sigma_{12\,0} = \sigma_{21\,0} = 0$, and the fact that the components $\sigma_{ij\,0}$ are not functions of x_1 and x_2, we have

$$\frac{\partial\delta\sigma_{22}}{\partial x_2}+\frac{\partial\delta\sigma_{21}}{\partial x_1}+\frac{\delta\sigma_{22}-\delta\sigma_{\theta\theta}}{x_2} = 0 \qquad \text{and} \qquad \frac{\partial\delta\sigma_{12}}{\partial x_2}+\frac{\partial\delta\sigma_{11}}{\partial x_1}+\frac{\delta\sigma_{12}}{x_2} = 0 \quad \text{for } \delta\sigma_{12}\neq\delta\sigma_{21}.$$

Substituting the expressions for $\sigma_{\theta\theta 0}$, σ_{220} and σ_{21} into the first equilibrium equation, we obtain

$$A\nabla^2 u_2-\left(A-\lambda_2^2\mu\right)u_{2},_{11}+(\lambda+\mu)\lambda_2\lambda_1 u_{1},_{12}-A\left(\frac{u_2}{x_2^2}\right) = 0, \tag{4}$$

where $A = (2\lambda+3\mu)(\lambda_2^2-1)+\dfrac{\lambda}{2}(\lambda_1^2-1)+\lambda+2\mu$ and $\nabla^2 f = f,_{22}+f,_{2}/x_2+f,_{11}$. Introduce the function Φ such that

$$u_2 = -C\Phi_{21} \qquad \text{and} \qquad u_1 = A\nabla^2\Phi-B\Phi_{11}, \tag{5}$$

where $B = A-\lambda_2^2\mu$ and $C = (\lambda+\mu)\lambda_2\lambda_1$. Then (4) is satisfied automatically.

Using the equalities

$$\delta\sigma_{12} = 2u_{1,2}W_{22} + (\lambda_1/\lambda_2)\delta\sigma_{21},$$

$$\delta\sigma_{21} = \lambda_2\mu(\lambda_2 u_{2,1} + \lambda_1 u_{1,2}),$$

$$\delta\sigma_{11} = \big(2W_{11} + \lambda_1^2(\lambda + 2\mu)\big)u_{1,1} + \lambda_z\lambda(\lambda_2 u_{2,2} + \lambda_2 u_2/x_2),$$

the second equilibrium equation becomes

$$D\nabla^2 u_1 + C\Big(u_{2,12} + \frac{u_{2,1}}{x_2}\Big) + \big(2W_{11} - 2W_{22} + (\lambda+\mu)\lambda_1^2\big)u_{1,11} = 0, \tag{6}$$

where $D = (\lambda + 2\mu)(\lambda_1^2 - 1)/2 + (\lambda + \mu)(\lambda_2^2 - 1) + \mu$.

We can rewrite the above equation substituting (5) into (6) and considering Equations (4)–(6) with the result

$$DA\nabla^4\Phi - DB\nabla^2\Phi_{11} - C^2\nabla^2\Phi_{11} + C^2\Phi_{1111} + EA\nabla^2\Phi_{11} - EB\Phi_{1111} = 0, \tag{7}$$

where $E = 2W_{11} - 2W_{22} + (\lambda + \mu)\lambda_1^2$.

Using the Hankel transform of zeroth order, one obtains

$$DA\xi^4 G(\xi, x_1) + (DB - EA - 2DA + C^2)\xi^2 G(\xi, x_1)_{11} + (-EB + DA - DB + EA)G(\xi, x_1)_{1111} = 0,$$

where $G(\xi, x_1) = \int_0^\infty x_2 J_0(\xi x_2)\Phi dx_2$ and $J_0(\xi x_2)$ is the zeroth order Bessel function. The subscript notation used here means $G_{11} = d^2G/dx_1^2$.

Letting $G(\xi, x_1) = F(\xi)e^{H(\xi)x_1}$ we can rewrite this as a quadratic equation in H^2/ξ^2:

$$p_1 + \frac{p_2 H^2}{\xi^2} + \frac{p_3 H^4}{\xi^4} = 0, \tag{8}$$

where $p_1 = DA$, $p_2 = (DB - EA - 2DA + C^2)$, and $p_3 = (-EB + DA - DB + EA)$.

For the classical case $\lambda_j = 1$, the characteristic equation becomes

$$\frac{H^4}{\xi^4} - \frac{2H^2}{\xi^2} + 1 = 0. \tag{9}$$

When the determinant $\Delta = p_2^2 - 4p_1p_3$ of (8) vanishes, the equation has two equal positive roots. One can consider that $\lambda_j = 1 + \delta_j$, $\Delta \to 0$ but $\Delta < 0$ or $\Delta > 0$ and the equation has two complex roots ($\Delta < 0$) or two positive real roots ($\Delta > 0$). Now one only deals with the case with two different real positive roots r_1^2 and r_2^2, where $r_1 > 0$ and $r_2 > 0$. The other cases will be discussed in detail in Section A1, page 1376. In the limit $x_1 \to \infty$, $G \to 0$, we have

$$G(\xi, x_1) = N_1(\xi)e^{-r_1\xi x_1} + N_2(\xi)e^{-r_2\xi x_1}. \tag{10}$$

In what follows we write $N_1(\xi)$ and $N_2(\xi)$ simply as N_1 and N_2, respectively.

Now we turn our attention to the stress component $\delta\sigma_{21}$. Using Equations (27) and (30) from the Appendix as well as (10), one has

$$\delta\sigma_{21} = \lambda_2\mu P \int_0^\infty \xi^4\big(N_1(r_1^2 + w)e^{-r_1\xi x_1} + N_2(r_2^2 + w)e^{-r_2\xi x_1}\big)J_1(\xi x_2)\, d\xi, \tag{11}$$

where

$$P = \lambda_2 C + \lambda_1 B - \lambda_1 A \quad \text{and} \quad w = \lambda_1 A/(\lambda_2 C + \lambda_1 B - \lambda_1 A).$$

With $x_1 = 0$ and $\delta\sigma_{21} = 0$, the above equation simplifies to

$$\delta\sigma_{21} = \lambda_2 \mu P \int_0^\infty \xi^4 \big(N_1(r_1^2 + w) + N_2(r_2^2 + w)\big) J_1(\xi x_2)\, d\xi = 0, \tag{12}$$

which gives

$$N_2 = -\left(\frac{r_1^2 + w}{r_2^2 + w}\right) N_1. \tag{13}$$

Substituting (13) into (10), we get

$$G(\xi, x_1) = N_1(e^{-r_1 \xi x_1} - Q e^{-r_2 \xi x_1}), \tag{14}$$

where

$$Q = \frac{r_1^2 + \lambda_1 (A/P)}{r_2^2 + \lambda_1 (A/P)}.$$

Let us now discuss the stress component $\delta\sigma_{11}$ and displacement component u_1. Substituting (14) into (10), one obtains

$$\delta\sigma_{11} = R(A\nabla^2\Phi_1 - B\Phi_{111}) - C\lambda\lambda_2\lambda_1(\nabla^2\Phi_1 - \Phi_{111}), \tag{15}$$

where

$$R = \tfrac{1}{2}\lambda(2\lambda_2^2 + \lambda_1^2 - 3) + \mu(\lambda_1^2 - 1) + \lambda_1^2(\lambda + 2\mu) = 2W_{11} + \lambda_1^2(\lambda + 2\mu).$$

The zeroth order Hankel transform of (15) is

$$\int_0^\infty x_2 J_0(\xi x_2) \delta\sigma_{11} dx_2 = (RA - RB)G_{111} + (C\lambda\lambda_2\lambda_1 - RA)\xi^2 G_1. \tag{16}$$

Similarly, the zeroth order Hankel transform of u_1 in (9) is

$$\int_0^\infty x_2 J_0(\xi x_2) u_1 dx_2 = (A - B)G_{11} - A\xi^2 G. \tag{17}$$

Setting $x_1 = 0$ and using Equations (14), (16) and (17), one has

$$\delta\sigma_{11} = \big(-(RA - RB)(r_1^3 - Q r_2^3) - (C\lambda\lambda_2\lambda_1 - RA)(r_1 - Q r_2)\big) \int_0^\infty \xi^4 J_0(\xi x_2) N_1 d\xi,$$

$$u_1 = \big((A - B)(r_1^2 - Q r_2^2) - A(1 - Q)\big) \int_0^\infty \xi^3 J_0(\xi x_2) N_1 d\xi.$$

The boundary conditions are

$$\big((A-B)(r_1^2 - Q r_2^2) - A(1-Q)\big) \int_0^\infty \xi^3 J_0(\xi x_2) N_1 d\xi = [u_1(x_2)]_{x_1=0}, \quad x_2 \le a,$$

$$\int_0^\infty \xi^4 J_0(\xi x_2) N_1 d\xi = 0, \qquad x_2 > a,$$

where a is the radius of contact area, which will be discussed in detail later.

Setting $\xi a = p$, $x_2 = a\rho$, $p^3 N_1 = f(p)$, and

$$a^4[u_1(x_2)]_{x_1=0} = \big((A-B)(r_1^2 - Qr_2^2) - A(1-Q)\big)g(\rho),$$

one has

$$\begin{aligned}\int_0^\infty f(p) J_0(p\rho)\,dp &= g(\rho), \quad 0 \le \rho \le 1,\\ \int_0^\infty f(p)\,p J_0(p\rho)\,dp &= 0, \qquad \rho > 1.\end{aligned} \tag{18}$$

Let $g(\rho) = \sum_{n=0}^{\infty} A_n \rho^n (0 \le \rho \le 1)$. Then from [Sneddon 1951] the solution of (18) is

$$f(p) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} A_n \left(\cos p + p \int_0^1 u^{n+1} \sin(pu)\,du\right) \frac{\Gamma(1+n/2)}{\Gamma(3/2+n/2)}, \tag{19}$$

where Γ is the gamma function (recall that $\Gamma(1) = 1$ and $\Gamma(3/2) = \sqrt{\pi}/2$).

We can write

$$[u_1(\rho)]_{x_1=0} = b + a\cot\alpha(1-\rho)$$

for $0 \le \rho \le 1$ and $g(\rho) = A_0 + A_1\rho$, where α is the angle of the circular cone (the angle between the asymmetric axis $Ox_1(Oz)$ and the mother line of the surface of the circular cone). Then, one has

$$a^4[u_1(x_2)]_{x_1=0} = \big((A-B)(r_1^2 - Qr_2^2) - A(1-Q)\big)(A_0 + A_1\rho),$$

where

$$A_0 = \frac{(b+a\cot\alpha)a^4}{(A-B)(r_1^2-Qr_2^2)-A(1-Q)}, \qquad A_1 = \frac{-a^5\cot\alpha}{(A-B)(r_1^2-Qr_2^2)-A(1-Q)}.$$

Considering that $A_n = 0$ for $n \ge 2$, from Equation (19), we get

$$f(p) = 2\left(\frac{A_0}{\pi} + \frac{A_1}{2}\right)\frac{\sin p}{p} + A_1 \frac{(\cos p - 1)}{p^2},$$

$$\begin{aligned}\delta\sigma_{11} &= \big(-(RA-RB)\big(r_1^3 - Qr_2^3\big) - \big(C\lambda\lambda_2\lambda_1 - RA\big)\big(r_1 - Qr_2\big)\big)\\ &\quad\times\left(\left(\frac{2A_0}{\pi} + A_1\right)a^{-5}\int_0^\infty J_0(p\rho)\sin p\,dp + A_1 a^{-5}\int_0^\infty J_0(p\rho)\frac{(\cos p - 1)}{p}dp\right).\end{aligned}$$

Since the integral $\int_0^\infty J_0(p)\sin p\,dp$ is divergent, to make sure the stress component $\delta\sigma_{11}$ is finite at the edge of the punch we require $(2A_0/\pi + A_1) = 0$, which means $b = a\cot\alpha\,(\pi/2 - 1)$ and

$$[u(z, x_2)]_{x_1=0, x_2=0} = \frac{\pi}{2} a\cot\alpha, \qquad f(p) = A_1 \frac{\cos p - 1}{p^2}. \tag{20}$$

Noting that

$$\int_0^\infty J_0(p\rho)\frac{\cos p - 1}{p}\,dp = -\cosh^{-1}(1/\rho),$$

we get

$$\delta\sigma_{11} = \big((RA - RB)(r_1^3 - Qr_2^3) + (C\lambda\lambda_2\lambda_1 - RA)(r_1 - Qr_2)\big)\frac{A_1}{a^5}\cosh^{-1}(1/\rho). \tag{21}$$

We prove in the Appendix (see (59)) that the compressive force T is given by

$$T = a^2\,\frac{\pi(RA-RB)\big(p_1 + w\big(-p_2 + \sqrt{p_1p_3}\big)\big) - \pi(C\lambda\lambda_2\lambda_1 - RA)\big(\sqrt{p_1p_3} - wp_3\big)}{\big((A-B)w + A\big)\big(-p_2p_3 + 2p_1^{1/2}p_3^{3/2}\big)^{1/2}}, \tag{22}$$

where the p_j are the coefficients in the characteristic equation (8).

The contact radius a is therefore

$$a = \left(\frac{T\tan\alpha\,\big((A-B)w + A\big)\big(-p_2p_3 + 2p_1^{1/2}p_3^{3/2}\big)^{1/2}}{\pi(RA-RB)p_1 + \pi(RA-RB)w\big(-p_2 + \sqrt{p_1p_3}\big) - \pi(C\lambda\lambda_2\lambda_1 - RA)\big(\sqrt{p_1p_3} - wp_3\big)}\right)^{1/2}. \tag{23}$$

By (20), the penetration depth $[u_1(x_1, x_2)]_{x_1=x_2=0}$ equals $(\pi/2)a\cot\alpha$, that is,

$$u_1(x_1,x_2)\big|_{x_1=x_2=0} = \frac{\pi}{2}\left(\frac{T\cot\alpha\,\big((A-B)w + A\big)\big(-p_2p_3 + 2p_1^{1/2}p_3^{3/2}\big)^{1/2}}{\pi(RA-RB)p_1 + \pi(RA-RB)w\big(-p_2 + \sqrt{p_1p_3}\big) - \pi(C\lambda\lambda_2\lambda_1 - RA)\big(\sqrt{p_1p_3} - wp_3\big)}\right)^{1/2}. \tag{24}$$

Using Equations (1) and (2), we can solve for λ_1 and λ_2 given the constants μ, λ and the residual stress $\sigma_{22\,0}$. We can then find the values of coefficients A, B, C, D, E, p_1, p_2, p_3, P, w, Q, R using Equations (4)–(8), (11), (14) and (15). We can find the contact radius a and penetration depth $[u(x_1, x_2)]_{x_1=0,x_2=0}$ with the help of (23) and (24).

We have obtained the relation between the residual stress, the contact radius and the penetration depth. As a result, we can determine the residual stress from the contact radius or the penetration depth.

We now look at a numerical example. The result for $\lambda = 30$–50 GPa, $\mu = 60$–80 GPa, $T = 0.23$ kg and $\alpha = \pi/12$ according to (23), is plotted in Figure 1.

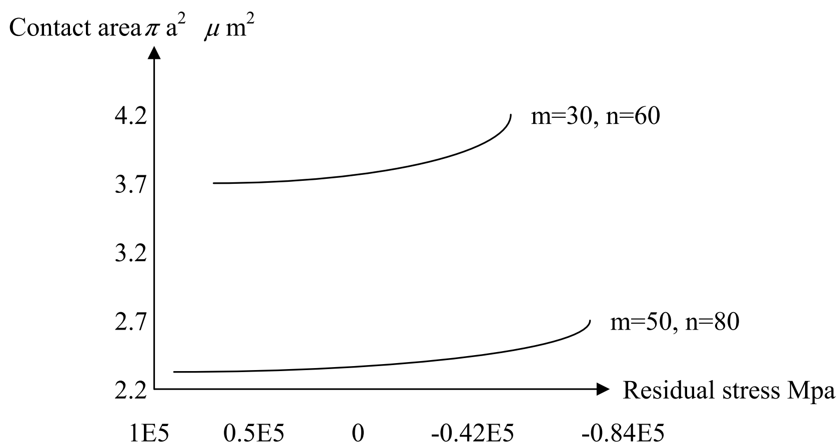

Figure 1. Relation between the area and residual stress: $\lambda = m$ GPa and $\mu = n$ GPa. The Poisson ratio is $\nu = m/(2n + 2m)$ and Young's modulus is $E = (2n + 3m)\mu/(n + m)$.

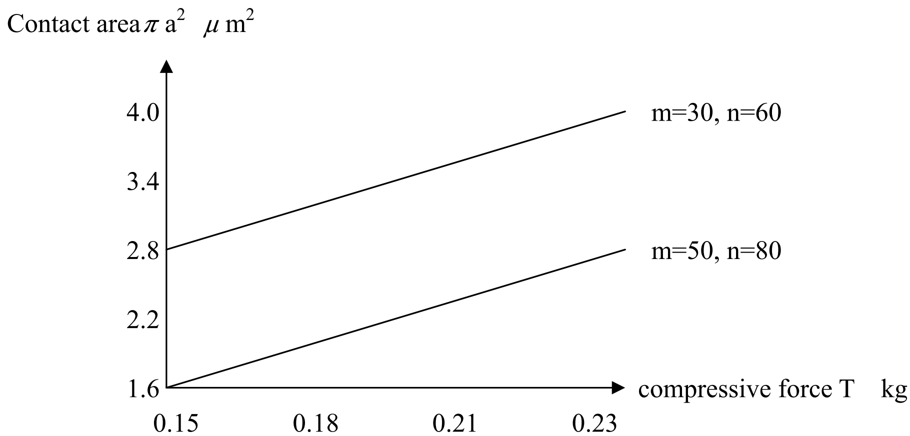

Figure 2. Relation between contact area and compressive force for zero residual stress: $\lambda = m\,\text{GPa}$, $\mu = n\,\text{GPa}$, $\nu = m/(2n+2m)$, $E = (2n+3m)\mu/(n+m)$.

We see that a body under tensile residual stress behaves like a string under tension. That leads to a decrease of the contact area so that it is smaller than without the stress. However, for compressive residual stress the opposite effect is obtained. These results coincide with those obtained in [Hao 1986].

To check the numerical results, consider the case of zero residual stress. For $\lambda = 30$–$50\,\text{GPa}$, $\mu = 60$–$80\,\text{GPa}$, and $\alpha = \pi/12$ the relation between the contact area and the compressive force T for zero residual stress is given in Figure 2. We see that it agrees with the numerical results in Figure 1.

4. Concluding remarks

We have studied the axially symmetric contact problem in the framework of the theory of finite elasticity, that is, the theory of small deformation on initial stress body. We have also considered its application to estimating residual stresses by cone indentation. In particular, we have been able to determine the relation among the penetration depth, the contact radius and the residual stress. Further study must focus on the more general method to solve the residual stress problems and consider the plasticity behavior.

Appendix

A1. ***The complex root case.*** We consider the complex root case, where

$$H^2 = \xi^2(r \pm is) = \xi^2 \eta e^{\pm i\vartheta}, \qquad = \pm\xi\eta^{1/2}(\cos\vartheta/2 \pm i\sin\vartheta/2) - \pi \le \vartheta \le \pi \quad \text{with } \cos\vartheta/2 > 0,$$

$$G(\xi, x_1) = K(\xi)e\xi^{\eta^{1/2}(\cos\vartheta/2 + i\sin\vartheta/2)x_1} + L(\xi)e^{\xi\eta^{1/2}(\cos\vartheta/2 - i\sin\vartheta/2)x_1} + M(\xi)e^{-\xi\eta^{1/2}(\cos\vartheta/2 + i\sin\vartheta/2)x_1} + N(\xi)e^{-\xi\eta^{1/2}(\cos\vartheta/2 - i\sin\vartheta/2)x_1}.$$

The case $x_1 \ge 0$ is considered, where $x_1 \to \infty$ and $u, \sigma_{ij} \to 0$:

$$G(\xi, x_1) = Me^{-\xi\eta^{1/2}(\cos\vartheta/2 + i\sin\vartheta/2)x_1} + Ne^{-\xi\eta^{1/2}(\cos\vartheta/2 - i\sin\vartheta/2)x_1}. \tag{25}$$

For convenience, $M(\xi)$ and $N(\xi)$ are replaced by M and N but they are functions of ξ.

It is apparent that $G(\xi, x_1)$ is real; therefore,

$$G(\xi, x_1) = Me^{-\xi d x_1} + \overline{M} e^{-\xi \bar{d} x_1}, \tag{26}$$

where $d = \eta^{1/2}(\cos\vartheta/2 + i\sin\vartheta/2)$ and $\bar{d} = \eta^{1/2}(\cos\vartheta/2 - i\sin\vartheta/2)$.

Next we deal with the stress component $\delta\sigma_{21}$:

$$\begin{aligned}\sigma_{21} &= \lambda_2\mu(\lambda_2 u_{r},_1 + \lambda_1 u_{1},_2) = \lambda_2\mu\big(-\lambda_2 C\Phi_{211} + \lambda_1(A\nabla^2\Phi - B\Phi_{11})_2\big)\\ &= \lambda_2\mu\big(-(\lambda_2 C + \lambda_1 B)\Phi_{211} + \lambda_1(A\nabla^2\Phi)_2\big)\end{aligned} \tag{27}$$

From Equation (54) in the Appendix, one knows that

$$\begin{aligned}&-(\lambda_2 C + \lambda_z B)\lambda_2\mu\int_0^\infty x_2 J_1(\xi x_2)\,d\Phi_{11} + \lambda_1 A\lambda_2\mu\int_0^\infty x_2 J_1(\xi x_2)\,d\nabla^2\Phi\\ &\quad = (\lambda_2 C + \lambda_z B)\lambda_2\mu\xi\int_0^\infty \Phi_{11} x_2 J_0(\xi x_2)\,dx_2 - \lambda_1 A\lambda_2\mu\xi\int_0^\infty \nabla^2\Phi x_2 J_0(\xi x_2)\,dx_2.\end{aligned} \tag{28}$$

Hence

$$\begin{aligned}\int_0^\infty x_2 J_1(\xi x_2)\delta\sigma_{21} dx_2 &= \lambda_2\mu\int_0^\infty x_2 J_1(\xi x_2)\big(-(\lambda_2 C + \lambda_z B)\Phi_{211} + \lambda_1(A\nabla^2\Phi)_2\big)\,dx_2\\ &= -(\lambda_2 C + \lambda_z B)\lambda_2\mu\int_0^\infty x_2 J_1(\xi x_2)\Phi_{211} dx_2 + \lambda_1 A\lambda_2\mu\int_0^\infty x_2 J_1(\xi x_2)(\nabla^2\Phi)_2\,dx_2\\ &= -(\lambda_2 C + \lambda_z B)\lambda_2\mu\int_0^\infty x_2 J_1(\xi x_2)\,d\Phi_{11} + \lambda_1 A\lambda_2\mu\int_0^\infty x_2 J_1(\xi x_2)\,d\nabla^2\Phi\\ &= (\lambda_2 C + \lambda_z B)\lambda_2\mu\xi\int_0^\infty \Phi_{11} x_2 J_0(\xi x_2)\,dx_2 - \lambda_1 A\lambda_2\mu\xi\int_0^\infty \nabla^2\Phi x_2 J_0(\xi x_2)\,dx_2\\ &= \lambda_2\mu(\lambda_2 C + \lambda_1 B)\xi(d^2/dx_1^2)G(\xi, x_1) - \lambda_2\mu\lambda_1 A\xi(d^2/dx_1^2 - \xi^2)G(\xi, x_1)\\ &= \big(\lambda_2\mu(\lambda_2 C + \lambda_1 B) - \lambda_2\mu\lambda_1 A\big)\xi G(\xi, x_1)_{11} + \lambda_2\mu\lambda_1 A\xi^3 G(\xi, x_1),\end{aligned} \tag{29}$$

which leads to

$$\begin{aligned}\delta\sigma_{21} &= \int_0^\infty \xi^2\big(\lambda_2\mu P G(\xi, x_1)_{11} + \lambda_2\mu\lambda_1 A\xi^2 G(\xi, x_1)\big)J_1(\xi x_2)\,d\xi,\\ G(\xi, x_1) = G(\xi, x_1) &= Me^{-\xi d x_1} + \overline{M}e^{-\xi\bar{d}x_1}, \quad G(\xi, x_1)_{11} = \xi^2(Md^2 e^{-\xi d x_1} + \overline{M}\bar{d}^2 e^{-\xi\bar{d}x_1}).\end{aligned} \tag{30}$$

When $x_1 = 0$, one obtains, denoting by M_{re} the real part of M,

$$\begin{aligned}G(\xi, 0) &= M + \overline{M} = 2M_{\text{re}}, \qquad [G(\xi, z)_{11}]_{x_1=0} = \xi^2(Md^2 + \overline{M}\bar{d}^2),\\ \delta\sigma_{21} &= \int_0^\infty \xi^2\big(\lambda_2\mu P\xi^2(Md^2 + \overline{M}\bar{d}^2) + 2\lambda_2\mu\lambda_1 A\xi^2 M_{\text{re}}\big)J_1(\xi x_2)\,d\xi.\end{aligned}$$

When $x_1 = 0$, $\delta\sigma_{21} = 0$ $(x_1 = z, x_2 = r)$, one obtains $(Md^2 + \overline{M}\bar{d}^2) + 2wM_{\mathrm{re}} = 0$, leading to

$$M_{\mathrm{re}}(d^2 + 2w + \bar{d}^2) + i M_{\mathrm{im}}(d^2 - \bar{d}^2) = 0,$$

$$M = M_{\mathrm{re}}\left(\frac{1 - (d^2 + 2w + \bar{d}^2)}{(d^2 - \bar{d}^2)}\right) = \frac{-2M_{\mathrm{re}}(\bar{d}^2 + w)}{(d^2 - \bar{d}^2)},$$

$$\overline{M} = M_{\mathrm{re}}\left(\frac{1 + (d^2 + 2w + \bar{d}^2)}{(d^2 - \bar{d}^2)}\right) = \frac{2M_{\mathrm{re}}(d^2 + w)}{(d^2 - \bar{d}^2)}; \tag{31}$$

$$G(\xi, x_1) = \frac{2M_{\mathrm{re}}\big(-(\bar{d}^2 + w)e^{-\xi d x_1} + (d^2 + w)e^{-\xi \bar{d} x_1}\big)}{(d^2 - \bar{d}^2)},$$

$$G_1 = \frac{-2M_{\mathrm{re}}\xi\big(-(\bar{d}^2 + w)\, d e^{-\xi d x_1} + (d^2 + w)\bar{d} e^{-\xi \bar{d} x_1}\big)}{(d^2 - \bar{d}^2)},$$

$$G_{11} = \frac{2M_{\mathrm{re}}\xi^2\big(-(\bar{d}^2 + w)\, d^2 e^{-\xi d x_1} + (d^2 + w)\bar{d}^2 e^{-\xi \bar{d} x_1}\big)}{(d^2 - \bar{d}^2)},$$

$$G_{111} = \frac{-2M_{\mathrm{re}}\xi^3\big(-(\bar{d}^2 + w)\, d^3 e^{-\xi d x_1} + (d^2 + w)\bar{d}^3 e^{-\xi \bar{d} x_1}\big)}{(d^2 - \bar{d}^2)}. \tag{32}$$

Now the stress component $\delta\sigma_{11}$ and displacement component u_1 are discussed. On the basis of equations (16)–(17), one obtains

$$\delta\sigma_{11} = \int_0^\infty \xi J_0(\xi x_2)\big((RA - RB)G_{111} + (C\lambda\lambda_2\lambda_1 - RA)\xi^2 G_1\big)\, d\xi, \tag{33}$$

$$u_1 = \int_0^\infty \xi J_0(\xi x_2)\big((A - B)G_{11} - A\xi^2 G\big)\, d\xi. \tag{34}$$

For $x_1 = 0$, one has

$$\begin{aligned}
&G(\xi, x_1) = 2M_{\mathrm{re}},\\
&\quad G_1 = -2M_{\mathrm{re}}\xi\frac{-(\bar{d}^2 + w)d + (d^2 + w)\bar{d}}{d^2 - \bar{d}^2} = 2M_{\mathrm{re}}\xi\frac{(\bar{d}d - w)(d - \bar{d})}{d^2 - \bar{d}^2} = 2M_{\mathrm{re}}\xi\frac{\bar{d}d - w}{d + \bar{d}},\\
&\quad G_{11} = 2M_{\mathrm{re}}\xi^2\frac{-(\bar{d}^2 + w)d^2 + (d^2 + w)\bar{d}^2}{d^2 - \bar{d}^2} = -2M_{\mathrm{re}}\xi^2 w,\\
&\quad G_{111} = -2M_{\mathrm{re}}\xi^3\frac{-(\bar{d}^2 + w)d^3 + (d^2 + w)\bar{d}^3}{d^2 - \bar{d}^2} = -2M_{\mathrm{re}}\xi^3\frac{\bar{d}^2 d^2(d - \bar{d}) + w(d^3 - \bar{d}^3)}{d^2 - \bar{d}^2}\\
&\qquad\qquad = -2M_{\mathrm{re}}\xi^3\frac{\bar{d}^2 d^2 + w(d^2 + \bar{d}d + \bar{d}^2)}{d + \bar{d}}.
\end{aligned} \tag{35}$$

Substituting these into Equations (33)–(34), one obtains

$$\delta\sigma_{11} = \int_0^\infty \xi J_0(\xi x_2) \frac{-(RA-RB)2M_{\mathrm{re}}\xi^3\big(\bar{d}^2d^2 + w(d^2+\bar{d}d+\bar{d}^2)\big)}{d+\bar{d}} + \frac{(C\lambda\lambda_2\lambda_1-RA)\xi^2 2M_{\mathrm{re}}\xi(\bar{d}d-w)}{d+\bar{d}}\,d\xi$$

$$= -2(RA-RB)\frac{\bar{d}^2d^2 + w(d^2+\bar{d}d+\bar{d}^2)}{d+\bar{d}}\int_0^\infty \xi M_{\mathrm{re}} J_0(\xi x_2)\xi^3 d\xi$$

$$+ 2(C\lambda\lambda_2\lambda_1-RA)\frac{\bar{d}d-w}{d+\bar{d}}\int_0^\infty \xi M_{\mathrm{re}} J_0(\xi x_2)\xi^3 d\xi$$

$$= \frac{-2(RA-RB)\big(\bar{d}^2d^2 + w(d^2+\bar{d}d+\bar{d}^2)\big) + 2(C\lambda\lambda_2\lambda_1-RA)(\bar{d}d-w)}{d+\bar{d}}\int_0^\infty \xi M_{\mathrm{re}} J_0(\xi x_2)\xi^3 d\xi; \quad (36)$$

$$u_1 = \int_0^\infty \xi J_0(\xi x_2)\big(-(A-B)2M_{\mathrm{re}}\xi^2 w - 2A\xi^2 M_{\mathrm{re}}\big)\,d\xi = \big(-(A-B)2w - 2A\big)\int_0^\infty \xi J_0(\xi x_2) M_{\mathrm{re}}\xi^2 d\xi. \quad (37)$$

The boundary conditions are

$$\begin{aligned} &\big(-(A-B)2w-2A\big)\int_0^\infty \xi J_0(\xi x_2)M_{\mathrm{re}}\xi^2 d\xi = [u_1(x_2)]_{x_1=0}, && x_1=0,\ 0\le x_2\le a,\\ &\int_0^\infty \xi^4 J_0(\xi x_2) M_{\mathrm{re}} d\xi = 0, && x_1=0,\ x_2>a, \end{aligned} \quad (38)$$

where a is the radius of the contact area, which will be discussed later.

Let $\xi a = p,\ x_2 = a\rho,\ a^4[u_1(x_2)]_{x_1=0} = -2\big((A-B)w+A\big)g(\rho),\ p^3 M_{\mathrm{re}} = f(p)$. Then

$$\begin{aligned} &\int_0^\infty f(p)J_0(p\rho)\,dp = g(\rho), && 0\le\rho\le 1,\\ &\int_0^\infty f(p)\,pJ_0(p\rho)\,dp = 0, && \rho>1. \end{aligned} \quad (39)$$

Let $g(\rho) = \sum_{n=0}^\infty A_n\rho^n$, with $0\le\rho\le 1$; by Sneddon 1951, the solution of the equations is

$$f(p) = \pi^{-1/2}\sum_{n=0}^\infty A_n\left(\cos p + p\int_0^1 u^{n+1}\sin(pu)\,du\right)\frac{\Gamma(1+n/2)}{\Gamma(3/2+n/2)}. \quad (40)$$

Let $[u_1(\rho)]_{x_1=0} = b + a\cot\alpha(1-\rho)$ with $0\le\rho\le 1$; that is, $g(\rho) = A_0 + A_1\rho$, so

$$-2\big((A-B)w+A\big)\big(A_0+A_1\rho\big) = a^4[u_1(\rho)]_{x_1=0} = a^4\big(b+a\cot\alpha(1-\rho)\big),$$

$$A_0 = -a^4(b+a\cot\alpha)/2\big((A-B)w+A\big),$$

$$A_1 = a^4 a\cot\alpha/2\big((A-B)w+A\big). \quad (41)$$

On the basis of equation (40), one obtains

$$f(p) = 2(A_0/\pi + A_1/2)\frac{\sin p}{p} + A_1\frac{\cos p - 1}{p^2}; \quad (42)$$

hence

$$\delta\sigma_{11} = \frac{-2(RA-RB)\big(\bar{d}^2d^2+w(d^2+\bar{d}d+\bar{d}^2)\big)+2(C\lambda\lambda_2\lambda_1-RA)(\bar{d}d-w)}{d+\bar{d}}\int_0^\infty \xi^4 M_{\mathrm{re}} J_0(\xi x_2)\,d\xi$$

$$= \frac{-2(RA-RB)\big(\bar{d}^2d^2+w(d^2+\bar{d}d+\bar{d}^2)\big)+2(C\lambda\lambda_2\lambda_1-RA)(\bar{d}d-w)}{a^5(d+\bar{d})}\int_0^\infty pf(p)J_0(p\rho)\,dp$$

$$= \frac{-2(RA-RB)\big(\bar{d}^2d^2+w(d^2+\bar{d}d+\bar{d}^2)\big)+2(C\lambda\lambda_2\lambda_1-RA)(\bar{d}d-w)}{a^5(d+\bar{d})}$$
$$\times\int_0^\infty p\bigg((2A_0/\pi+A_1)\frac{\sin p}{p}+A_1\frac{\cos p-1}{p^2}\bigg)J_0(p\rho)\,dp$$

$$= \frac{-2(RA-RB)\big(\bar{d}^2d^2+w(d^2+\bar{d}d+\bar{d}^2)\big)+2(C\lambda\lambda_2\lambda_1-RA)(\bar{d}d-w)}{d+\bar{d}}$$
$$\times\bigg(\frac{2A_0/\pi+A_1}{a^5}\int_0^\infty J_0(p\rho)\sin p\,dp+\frac{A_1}{a^5}\int_0^\infty J_0(p\rho)\frac{\cos p-1}{p}\,dp\bigg). \quad (43)$$

As the integral $\int_0^\infty J_0(p)\sin p\,dp$ is divergent, for the finiteness of stress component $\delta\sigma_{11}$ at the edge of the punch, we have $(2A_0/\pi+A_1)=0$, that is, $b=a\cot\alpha(\pi/2-1)$. Hence

$$u(x_1,x_2)_{x_1=0,x_2=0}=b+a\cot\alpha=\frac{\pi}{2}a\cot\alpha,\qquad f(p)=A_1\frac{\cos p-1}{p^2}, \quad (44)$$

$$\delta\sigma_{11} = \frac{-2(RA-RB)\big(\bar{d}^2d^2+w(d^2+\bar{d}d+\bar{d}^2)\big)+2(C\lambda\lambda_2\lambda_1-RA)(\bar{d}d-w)}{a^5(d+\bar{d})}A_1\int_0^\infty J_0(p\rho)\frac{\cos p-1}{p}dp$$
$$= \frac{-2(RA-RB)\big(\bar{d}^2d^2+w(d^2+\bar{d}d+\bar{d}^2)\big)+2(C\lambda\lambda_2\lambda_1-RA)(\bar{d}d-w)}{a^5(d+\bar{d})}A_1\cosh^{-1}(1/\rho), \quad (45)$$

$$T=-2\pi\int_0^a[\delta\sigma_{11}]_{x_1=0}x_2dx_2$$
$$=-2\pi\frac{-2(RA-RB)\big(\bar{d}^2d^2+w(d^2+\bar{d}d+\bar{d}^2)\big)+2(C\lambda\lambda_2\lambda_1-RA)(\bar{d}d-w)}{a^5(d+\bar{d})}A_1$$
$$\int_0^a\big(\cosh^{-1}(a/x_2)x_2\big)dx_2. \quad (46)$$

The integral on the right is equal to

$$\int_{1/a}^\infty\frac{\cosh^{-1}(av)}{v^3}dv=a^2\int_1^\infty\frac{\cosh^{-1}u}{u^3}du=a^2\int_0^\infty\frac{w\sin hw}{(\cosh w)^3}\,dw=\frac{a^2}{2};$$

therefore

$$T=-\pi a^2\frac{-2(RA-RB)\big(\bar{d}^2d^2+w(d^2+\bar{d}d+\bar{d}^2)\big)+2(C\lambda\lambda_2\lambda_1-RA)(\bar{d}d-w)}{a^5(d+\bar{d})}A_1. \quad (47)$$

Substituting the value of A_1 from (41), we obtain

$$T = \pi a^2 \frac{(RA-RB)\big(\bar{d}^2 d^2 + w(d^2+\bar{d}d+\bar{d}^2)\big) - (C\lambda\lambda_2\lambda_1 - RA)(\bar{d}d - w)}{d+\bar{d}} \, \frac{\cot\alpha}{(A-B)w+A}. \tag{48}$$

The contact radius a is thus

$$a = \left(\frac{P\tan\alpha\big((A-B)w+A\big)(d+\bar{d})}{\pi(RA-RB)\big(\bar{d}^2 d^2 + w(d^2+\bar{d}d+\bar{d}^2)\big) - \pi\big(C\lambda\lambda_2\lambda_1 - RA\big)(\bar{d}d - w)} \right)^{1/2}. \tag{49}$$

The penetration depth is

$$[u(x_1,x_2)]_{x_1=0,x_2=0} = \frac{\pi}{2} a \cot\alpha, \tag{50}$$

where a is given by the previous equation.

Using the equalities $\bar{d}^2 d^2 = p_1/p_3$ and $\bar{d}^2 + d^2 = p_2/p_3$, we obtain successively

$$\bar{d}d = \sqrt{p_1/p_3}, \qquad \bar{d}+d = \sqrt{-p_2/p_3 + 2\sqrt{p_1/p_3}}, \qquad d^2 + \bar{d}d + \bar{d}^2 = -p_2/p_3 + \sqrt{p_1/p_3}. \tag{51}$$

With this one can find the value of a in (49).

A2. ***About Q.*** It is known that $Q = ({r_1}^2 + w)/({r_2}^2 + w)$, where $w = \lambda_1 A/(\lambda_2 C + \lambda_1 B - \lambda_1 A)$. Therefore,

$$\begin{aligned}
1 - Q &= 1 - \frac{{r_1}^2 + w}{r_2^2 + w} = \frac{{r_2}^2 - {r_1}^2}{{r_2}^2 + w}, \\
r_1 - Q r_2 &= r_1 - r_2 \frac{{r_1}^2 + w}{{r_2}^2 + w} = \frac{(r_2 - r_1)(r_1 r_2 - w)}{{r_2}^2 + w}, \\
{r_1}^2 - Q{r_2}^2 &= \frac{({r_2}^2{r_1}^2 + w{r_1}^2) - ({r_1}^2{r_2}^2 + w{r_2}^2)}{{r_2}^2 + w} = w\frac{{r_1}^2 - {r_2}^2}{{r_2}^2 + w}, \\
{r_1}^3 - Q{r_2}^3 &= \frac{{r_1}^3({r_2}^2 + w) - {r_2}^3({r_1}^2 + w)}{{r_2}^2 + w} = \frac{(r_1 - r_2){r_1}^2{r_2}^2 + w({r_1}^3 - {r_2}^3)}{{r_2}^2 + w} \\
&= (r_1 - r_2)\frac{{r_1}^2{r_2}^2 + w({r_1}^2 + r_1 r_2 + {r_2}^2)}{{r_2}^2 + w},
\end{aligned} \tag{52}$$

A3. ***Research on some integrals.*** When the integrals

$$\int_0^\infty x_2 J_1(\xi x_2)\, d\Phi_{11} \quad \text{and} \quad \int_0^\infty x_2 J_1(\xi x_2)\, d\nabla^2\Phi$$

are calculated, it is supposed that

$$x_2 J_1(\xi x_2)\Phi_{11} \to 0 \quad \text{and} \quad x_2 J_1(\xi x_2)\nabla^2\Phi \to 0$$

by letting $({x_1}^2 + {x_2}^2)^{1/2} = r \to \infty$.

It is known that

$$x \gg 1 \quad J_n(r) \to O(r^{-1/2}) \quad \text{and} \quad x_2 J_1(\xi x_2) \to O(r^{1/2}).$$

In our problem, u_1 and $u_2 \to 0$ when $r \to \infty$. In [Sneddon 1951], it is found that u_1 and $u_2 \to O(1/r)$ when $r \to \infty$.

On view of the relation between the displacements u_j and the function Φ,

$$u_2 = -C\Phi_{21},$$
$$u_1 = A\nabla^2\Phi - B\Phi_{11},$$

one can deem that $\Phi \to O(r)$; that is to say,

$$\Phi_{11} \to O(1/r) \quad \text{and} \quad \nabla^2\Phi \to O(1/r).$$

From this, one obtains

$$x_2 J_1(\xi x_2)\Phi_{11} \to O(1/r^{-1/2}) \quad \text{and} \quad x_2 J_1(\xi x_2)\nabla^2\Phi \to O(1/r^{-1/2}). \tag{53}$$

Therefore

$$\begin{aligned}
-(\lambda_2 C + \lambda_z B)\lambda_2\mu \int_0^\infty & x_2 J_1(\xi x_2)\, d\Phi_{11} + \lambda_1 A\lambda_2\mu \int_0^\infty x_2 J_1(\xi x_2)\, d\nabla^2\Phi \\
&= -(\lambda_2 C + \lambda_z B)\lambda_2\mu \Big(\int_0^\infty dx_2 J_1(\xi x_2)\Phi_{11} - \frac{1}{\xi}\int_0^\infty \Phi_{11}\, d\xi x_2 J_1(\xi x_2) \Big) \\
&\qquad + \lambda_1 A\lambda_2\mu \Big(\int_0^\infty dx_2 J_1(\xi x_2)\nabla^2\Phi - \frac{1}{\xi}\int_0^\infty \nabla^2\Phi\, d\xi J_1(\xi x_2) x_2 \Big) \\
&= (\lambda_2 C + \lambda_z B)\lambda_2 \frac{\mu}{\xi}\int_0^\infty \Phi_{11}\, d\xi x_2 J_1(\xi x_2) - \lambda_1 A\lambda_2 \frac{\mu}{\xi}\int_0^\infty \nabla^2\Phi\, d\xi J_1(\xi x_2) x_2 \\
&= (\lambda_2 C + \lambda_z B)\lambda_2\mu\xi \int_0^\infty \Phi_{11} x_2 J_0(\xi x_2)\, dx_2 - \lambda_1 A\lambda_2\mu\xi \int_0^\infty \nabla^2\Phi x_2 J_0(\xi x_2)\, dx_2,
\end{aligned} \tag{54}$$

because $d\big(vJ_1(v)\big) = vJ_0(v)\,dv$.

A4. *Force on the cone, contact radius and penetration depth.* For the compressive force T on the cone, from (21), one obtains

$$\begin{aligned}
T &= -2\pi \int_0^a [\delta\sigma_{11}]_{x_1=0} x_2\, dx_2 \\
&= -2\pi \big((RA - RB)(r_1{}^3 - Qr_2{}^3) + \big(C\lambda\lambda_2\lambda_1 - RA\big)(r_1 - Qr_2) \big) \frac{A_1}{a^5} \int_0^a (\cosh^{-1}(a/x_2) x_2)\, dx_2 \\
&= -\pi a^2 \big((RA - RB)(r_1{}^3 - Qr_2{}^3) + \big(C\lambda\lambda_2\lambda_1 - RA\big)(r_1 - Qr_2) \big) \frac{A_1}{a^5} \\
&= \pi a^2 \frac{(RA - RB)\big(r_1{}^3 - Qr_2{}^3\big) + \big(C\lambda\lambda_2\lambda_1 - RA\big)(r_1 - Qr_2)}{(A - B)(r_1{}^2 - Qr_2{}^2) - A(1 - Q)} \cot\alpha.
\end{aligned} \tag{55}$$

The contact radius a is thus

$$a = \left(\frac{T\tan\alpha\, \big((A - B)(r_1{}^2 - Qr_2{}^2) - A(1 - Q)\big)}{\pi(RA - RB)(r_1{}^3 - Qr_2{}^3) + \pi(C\lambda\lambda_2\lambda_1 - RA)(r_1 - Qr_2)} \right)^{1/2}, \tag{56}$$

and we know from (20) that the penetration depth is given by

$$[u_1(x_1, x_2)]_{x_1=0,x_2=0} = \frac{\pi}{2}\, a \cot\alpha. \tag{57}$$

For convenience, r_1, r_2 and Q will be replaced by the coefficients of the characteristic equation. First the expressions in Q are replaced by by expressions in r_1, r_2, using (52). We obtain

$$\begin{aligned} T &= a^2\, \frac{\pi(RA-RB)\big({r_1}^2{r_2}^2 + w({r_1}^2+r_1r_2+{r_2}^2)\big) - \pi(C\lambda\lambda_2\lambda_1-RA)(r_1r_2-w)}{\big((A-B)w+A\big)(r_2+r_1)}\cot\alpha,\\ a &= \left(\frac{T\tan\alpha\,\big((A-B)w+A\big)(r_2+r_1)}{\pi(RA-RB)\big({r_1}^2{r_2}^2 + w({r_1}^2+r_1r_2+{r_2}^2)\big) - \pi(C\lambda\lambda_2\lambda_1-RA)(r_1r_2-w)}\right)^{1/2}, \end{aligned} \tag{58}$$

from which we also obtain the penetration depth via (57).

From (8), one knows that p_j are the coefficients of characteristic equation; therefore, the relations between p_j and r_1, r_2 are exactly as in Equation (51), with r_1, r_2 replacing d, $\bar{d}$.

Substituting this into (58), one obtains

$$\begin{aligned} T &= a^2\, \frac{\pi(RA-RB)\big(p_1/p_3 + w\big(-p_2/p_3 + \sqrt{p_1/p_3}\big)\big) - \pi(C\lambda\lambda_2\lambda_1 - RA)\big(\sqrt{p_1/p_3} - w\big)}{((A-B)w+A\big)\big(-p_2/p_3 + 2\sqrt{p_1/p_3}\big)^{1/2}}\cot\alpha\\ &= a^2\, \frac{\pi(RA-RB)\big(p_1 + w\big(-p_2 + \sqrt{p_1p_3}\big)\big) - \pi(C\lambda\lambda_2\lambda_1-RA)\big(\sqrt{p_1p_3} - wp_3\big)}{\big((A-B)w+A\big)\big(-p_2p_3 + 2p_1^{1/2}p_3^{3/2}\big)^{1/2}},\\ a &= \left(\frac{T\tan\alpha\,\big((A-B)w+A\big)\big(-p_2p_3+2p_1^{1/2}p_3^{3/2}\big)^{1/2}}{\pi(RA-RB)\big(p_1 + w\big(-p_2 + \sqrt{p_1p_3}\big)\big) - \pi(C\lambda\lambda_2\lambda_1-RA)\big(\sqrt{p_1p_3} - wp_3\big)}\right)^{1/2} \end{aligned} \tag{59}$$

and

$$[u(x_1, x_2)]_{x_1=0,x_2=0} = \frac{\pi}{2}\, a\cot\alpha.$$

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