The electro-elastic interaction between a piezoelectric dislocation and collinear rigid lines embedded in a piezoelectric medium is studied in the framework of linear elastic theory. The rigid lines are considered, respectively, as dielectrics or conductors. We present a general solution of the problem based on the extended Stroh’s formalism. Explicit expressions of the field intensity factors are obtained for the special case of a single rigid line. The image force acting on the piezoelectric dislocation due to the presence of a single rigid line is calculated by using the generalized Peach-Koehler formula. Numerical examples show the shielding effects of field intensity factors and image force on the dislocation. The solution we present can be served as a Green’s function for investigating the micro-crack initiation mechanism at the tip of a rigid line.

1. Introduction

Piezoelectric materials are widely used in devices such as sensors and actuators. When subjected to mechanical and electric loads, these piezoelectric materials can fail prematurely due to defects arising in the manufacturing process. It is therefore important to study how defects such as dislocations and inclusions disturb the field variables, and how stress concentration arises as a result of defects. When a flat inclusion is much harder than the matrix, it is reasonable to consider it as a rigid line. There are numerous contributions to the literature on electro-elastic coupling characteristics of piezoelectric composite materials. To name a few, Pak [1992a] studied the anti-plane problem of a piezoelectric circular inclusion; Meguid and Zhong [1997] provided a general solution for the elliptical inhomogeneity problem in piezoelectric material under anti-plane shear and an in-plane electric field; Kattis et al. [1998] investigated the electro-elastic interaction effects of a piezoelectric screw dislocation with circular inclusion in piezoelectric material; Deng and Meguid [1998; 1999] considered the interaction between the piezoelectric elliptical inhomogeneity and a screw dislocation located inside inhomogeneity and outside inhomogeneity respectively under anti-plane shear and an in-plane electric field. More recently, Huang and Kuang [2001] evaluated the generalized electro-mechanical force for dislocation located inside, outside and on the interface of elliptical inhomogeneity in an infinite piezoelectric medium.

For rigid line problems in piezoelectric media, Liang et al. [1995] derived an exact general solution for an infinite piezoelectric medium with a rigid line and a crack. Shi [1997] investigated the collinear rigid lines under anti-plane deformation and in-plane electric field in piezoelectric media. Deng and Meguid [1998] addressed the plane problem of an interfacial rigid line between dissimilar piezoelectric materials. Gao and Fan [2000] investigated the generalized plane problem of piezoelectric media with collinear rigid lines under the loads at infinity. Chen et al. [2002] studied the problem of a screw dislocation
near a semi-infinite rigid line in a piezoelectric solid. More recently, Liu and Fang [2003] dealt with the interaction problem of a piezoelectric screw dislocation with circular interfacial rigid lines.

In the present work, we address the plane problem of a dislocation interacting with collinear rigid lines in piezoelectric media. Following this brief introduction, in Section 2 we outline the basic theory of the Stroh formalism. In Section 3 we state the problem to be investigated. We solve the problem of dielectric lines in Section 4 and that of conducting lines in Section 5. We present numerical examples in Section 6, and concluding remarks in Section 7.

2. The Stroh formalism

In fixed rectangular coordinates $x_i$ ($i = 1, 2, 3$), the basic equations for linear piezoelectric materials at constant temperature can be written as

$$
\sigma_{ij,j} = 0, \quad \gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad D_{i,i} = 0, \quad E_i = -\phi_i, \quad \sigma_{ij} = c_{ijkl} \gamma_{kl} - e_{ki j} E_k, \quad D_i = e_{ikl} \gamma_{kl} + \varepsilon_{ik} E_k,
$$

(2–1) \hspace{1cm} (2–2) \hspace{1cm} (2–3) \hspace{1cm} (2–4) \hspace{1cm} (2–5) \hspace{1cm} (2–6)

where $\sigma_{ij}, \gamma_{ij}, u_i, D_i, E_i, \phi$ are stress, strain, mechanical displacement, electric displacement, electric field and electric potential, respectively. $c_{ijkl}, e_{ki j}$ and $\varepsilon_{ij}$ are the corresponding elastic, piezoelectric and dielectric constants, respectively, which satisfy the symmetric relations

$$
c_{ijkl} = c_{klij} = c_{ijlk} = c_{jikl}, \quad e_{ki j} = e_{kji}, \quad \varepsilon_{ik} = \varepsilon_{ki},
$$

(2–7)

where $i, j, k, l = 1, 2, 3$, repeated Latin indices mean summation, and a comma stands for partial differentiation.

Substitution of (2–3) and (2–4) into (2–5) and (2–6) yields

$$
\sigma_{ij} = c_{ijkl} u_{k,l} + e_{ki j} \phi_k, \quad D_i = e_{ikl} u_{k,l} - \varepsilon_{ik} \phi_k.
$$

(2–8) \hspace{1cm} (2–9)

Furthermore, substituting (2–8) and (2–9) into (2–1) and (2–2) results in

$$
(c_{ijkl} u_k + e_{ij} \phi)_l = 0, \quad (e_{ikl} u_k - \varepsilon_{ij} \phi)_l = 0.
$$

(2–10) \hspace{1cm} (2–11)

Here we only address a generalized two-dimensional deformation problem in the $(x_1, x_2)$ plane. Therefore all the variables are constant along the $x_3$ axis. For such two-dimensional deformations where the physical quantities only depend on the coordinates $x_1$ and $x_2$, the general displacement solution to the above equations is

$$
u = \left[ u_1 \quad u_2 \quad u_3 \quad u_4 \right]^T = a f(z), \quad z = x_1 + px_2,
$$

(2–12)
or
\[ u_k = a_k f(z), \quad k = 1, 2, 3, 4, \] (2–13)
where \( u_4 = \phi \) is the electric displacement, \( p \) and \( a \) are constants to be determined, and \( f(z) \) is an arbitrary function of \( z \). Substituting (2–12) into (2–10) and (2–11) yields
\[
\begin{align*}
(c_{1jk1} + p(c_{2jk1} + c_{1jk2}) + p^2 c_{2jk2}) a_k &+ (e_{1j1} + p(e_{1j2} + e_{2j1}) + p^2 e_{2j2}) a_4 = 0, \\
(e_{1k1} + p(e_{1k2} + e_{2k1}) + p^2 e_{2k2}) a_k &- (\varepsilon_{11} + p(\varepsilon_{12} + \varepsilon_{21}) + p^2 \varepsilon_{22}) a_4 = 0,
\end{align*}
\] (2–14)
where \( k = 1, 2, 3 \). In view of (2–7), these equations can be rewritten as
\[
\begin{align*}
(Q + p(R + R^T) + p^2 T) a &= 0, \\
\Phi &= (R^T + pT) a f'(z), \\
\sigma_{1j} &= (c_{1jk1} + p c_{1jk2}) a_k + (e_{1j1} + p e_{1j2}) a_4 f'(z), \\
D_i &= (e_{ik1} + p e_{ik2}) a_k - (\varepsilon_{1i1} + p(\varepsilon_{1i2} + \varepsilon_{2i1}) + p^2 \varepsilon_{2i2}) a_4 f'(z),
\end{align*}
\] (2–18)
(2–19)
or
\[
\{\sigma_2 j, D_2 \}^T = (R^T + pT) a f'(z), \quad \{\sigma_1 j, D_1 \}^T = (Q + pR) a f'(z).
\] (2–20)
Defining
\[ b = (R^T + pT) a, \] (2–21)
and comparing it with (2–16), we get
\[ b = (R^T + pT) a = -\frac{1}{p} (Q + pR) a. \] (2–22)
By introducing the additional solution
\[ \Phi = b f(z), \] (2–23)
then (2–20) can be expressed as
\[
\begin{align*}
\{\sigma_2 j, D_2 \}^T &= \Phi_{1,1}, \\
\{\sigma_1 j, D_1 \}^T &= -\Phi_{1,2}.
\end{align*}
\] (2–24)
The eigenvalue problem (2–16) gives four pairs of complex conjugates and corresponding vectors. \( p_\alpha (\alpha = 1, 2, 3, 4) \) as the eigenvalues with positive imaginary part, and \( a_\alpha \) and \( b_\alpha \) as the associated vectors, we can write
\[
\begin{align*}
p_\alpha + 4 &= \bar{p}_\alpha, \\
a_{\alpha + 4} &= \bar{a}_\alpha, \\
b_{\alpha + 4} &= \bar{b}_\alpha.
\end{align*}
\] (2–25)
where the over-bar denotes the complex conjugate. Assuming that \( p_\alpha \) are distinct, the general solution can be written as

\[
u = \sum_{\alpha=1}^{4} \left( a_\alpha f_\alpha(z_\alpha) + \bar{a}_\alpha f_{\alpha+4}(\bar{z}_\alpha) \right), \quad (2-26)\]

\[
\Phi = \sum_{\alpha=1}^{4} \left( b_\alpha f_\alpha(z_\alpha) + \bar{b}_\alpha f_{\alpha+4}(\bar{z}_\alpha) \right), \quad (2-27)\]

where \( z_\alpha = x_1 + p_\alpha x_2 \) and \( f_l \ (l = 1, 2, 3, 4, 5, 6, 7, 8) \) are arbitrary functions to be determined according to the boundary conditions. In many applications they could be assumed to have the same function form

\[
\begin{align*}
f_\alpha(z_\alpha) &= q_\alpha f(z_\alpha), \\
f_{\alpha+4}(\bar{z}_\alpha) &= \bar{q}_\alpha \bar{f}(\bar{z}_\alpha),
\end{align*} \quad (2-28)\]

where \( q_\alpha \) are constants to be determined, and \( \bar{f}(\bar{z}_\alpha) \) is the conjugate complex of \( f(z_\alpha) \). Defining two \( 4 \times 4 \) complex matrices

\[
A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}, \quad (2-29)\\
B = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}. \quad (2-30)
\]

Equations (2–26) and (2–27) can be written as

\[
u = Af(z) + \bar{A} \bar{f}(\bar{z}), \quad (2-31)\]

\[
\Phi = Bf(z) + \bar{B} \bar{f}(\bar{z}), \quad (2-32)\]

where

\[
f(z) = \langle f(z_\alpha) \rangle q, \quad (2-33)\]

with

\[
\langle f(z_\alpha) \rangle = \text{diag} \left( f(z_1), f(z_2), f(z_3), f(z_4) \right), \quad (2-34)\]

\[
q = \{ q_1, q_2, q_3, q_4 \}^T. \quad (2-35)\]

With the help of (2–22), the eigenvalue problem (2–16) can be expressed in a standard form as

\[
\begin{bmatrix} -T^{-1}R^T & T^{-1} \\ RT^{-1}R^T - Q & -RT^{-1} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = p \begin{bmatrix} a \\ b \end{bmatrix}, \quad (2-36)\]

The \( A \) and \( B \) expressed in (2–29) and (2–30) satisfy the normalized orthogonality relation

\[
\begin{bmatrix} \bar{B}^T & A^T \end{bmatrix} \begin{bmatrix} A & \bar{A} \\ B & \bar{B} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad (2-37)\]

from which three real \( 4 \times 4 \) matrices can be defined

\[
S = i(2AB^T - I), \quad H = 2iAA^T, \quad L = -2iBB^T, \quad (2-38)\]
where, \( I \) is the \( 4 \times 4 \) identity matrix and \( i = \sqrt{-1} \). It is easy to show that

\[
HL = SS = L - S^T S^T = I, \quad LS + S^T T = SH + HS^T = 0.
\]

(2–39)

For a dislocation \( d = [d_1, d_2, d_3, d_4] \) located at \( z_d \) in an infinite homogenous material, the vector \( q \) and the functions \( f(z_o) \) in (2–33) can be written as

\[
q = \frac{1}{2\pi i} B^T d, \quad f(z_o) = \ln(z_o - z_{do}).
\]

(2–40)

Differentiating (2–31) and (2–32) with \( x_1 \), we obtain

\[
\begin{align*}
\Phi_1 &= B F(\bar{z}) + B F(\bar{\alpha}) = i M A F(z) - i M A F(\bar{z}), \\
M &= -i BA^{-1} = H^{-1} + i H^{-1} S.
\end{align*}
\]

(2–43)  (2–44)

3. Statement of the problem

The physical problem to be investigated is shown in Figure 1. A charged dislocation \( d = [d_1, d_2, d_3, d_4] \) is located at the point \( z_d(r_d, \theta_d) \) near some rigid lines \( L_r \) (\( r = 1, 2, \ldots, N \)) embedded in an infinite piezoelectric medium. The rigid lines are assumed to be collinearly located along the \( x_1 \)-axis of a Cartesian coordinate system \( x_1x_2x_3 \). The dislocation is assumed to be straight and infinitely long in the \( x_3 \)-direction, suffering a finite discontinuity in the displacement and electric potential across the slip plane. Assume that the deformations of the solid depend on \( x_1 \) and \( x_2 \) only.

The mechanical boundary conditions at any rigid line surface are

\[
\begin{align*}
\Phi_j(t^+) &= \Phi_j(t^-) = u_j + w_r x_1 \delta_{j2}, \quad j = 1, 2, 3, \quad t \in L_r,
\end{align*}
\]

(3–1)

where the superscript “+” and “−” refer, respectively, to the upper and lower rigid line surfaces, \( u_j0 \) are displacements of the inclusions, \( w_r \) is the counterclockwise rotation with respect to the \( x_3 \) axis, and \( \delta_{j2} \) is the Kronecker coefficient.

Figure 1. A piezoelectric screw dislocation near collinear rigid line inclusions.
The electric boundary conditions at any rigid line surface are

\[ E_1(t)^+ = E_1(t)^-, \quad t \in L_r \]  \hfill (3–2a)
\[ D_2(t)^+ = D_2(t)^-, \quad t \in L_r, \]  \hfill (3–2b)

for the dielectric rigid lines, and

\[ u_4(t)^+ = u_4(t)^- = u_{40}, \quad t \in L_r \]  \hfill (3–3)

for the conducting rigid lines, where \( u_{40} \) is a constant.

By using the perturbation technique, the complex potential vectors for the current problem can be expressed as

\[ F(z) = F_0(z) + F_1(z), \]  \hfill (3–4)

where \( F_0(z) \) is associated with the unperturbed field that is related to the solutions of an infinite homogeneous medium without the inclusions and is holomorphic in the entire domain except at \( z_d \). \( F_0(z) \) can be expressed as

\[ F_0(z) = \frac{1}{2\pi i} \left( \frac{1}{z - \omega} \right) B^T d. \]  \hfill (3–5)

The function \( F_1(z) \) corresponds to the perturbed field due to the introducing of the rigid lines and is holomorphic in the entire domain excluded the rigid lines. It is an unknown function to be determined according to the boundary conditions of the rigid lines.

4. Interaction of a dislocation with rigid dielectric lines

4.1. Determination of the complex potential function. In this case, the boundary conditions (3–1) and (3–2) apply. Conditions (3–1) and (3–2) can be rewritten as

\[ u_j'(t)^+ = u_j'(t)^- = w_r \delta_{j2}, \quad E_1(t)^+ = E_1(t)^-, \quad j = 1, 2, 3, \quad t \in L_r, \]  \hfill (4–1)
\[ D_2(t)^+ = D_2(t)^-, \quad t \in L_r, \]  \hfill (4–2)

where the prime denotes differentiation with respect to \( x_1 \). Using (2–41) and (3–4), condition (4–1) becomes

\[ AF(t)^+ + \overline{AF(t)}^- = h_0, \quad t \in L \]  \hfill (4–3)
\[ AF(t)^- + \overline{AF(t)}^+ = h_0, \quad t \in L \]  \hfill (4–4)

which leads to

\[ \left[ AF(t) - \overline{AF(t)} \right]^+ - \left[ AF(t) - \overline{AF(t)} \right]^- = 0, \quad t \in L, \]  \hfill (4–5)
\[ \left[ AF(t) + \overline{AF(t)} \right]^+ + \left[ AF(t) + \overline{AF(t)} \right]^- = 2h_0, \quad t \in L, \]  \hfill (4–6)

where \( h_0(t) = (0, w_r, 0, -E_1(t))^T \), and \( E_1(t) \) the unknown function that indicates the boundary value of
$E_1(z)$ on the inclusion faces [Gao and Fan 2000]. The substitution of (3–4) into (4–5) and (4–6) yields

$$[AF_1(t) - \overline{AF_1(t)}]^+ - [AF_1(t) - \overline{AF_1(t)}]^- = 0, \quad t \in L$$  \hspace{1cm} (4–7)

$$[AF_1(t) + \overline{AF_1(t)}]^+ + [AF_1(t) + \overline{AF_1(t)}]^- = 2[h_0(t) + h(t)], \quad t \in L,$$  \hspace{1cm} (4–8)

where

$$h(t) = -\frac{A}{2\pi i} \left( \frac{1}{t - z_{d0}} \right) B^T d + \frac{\hat{A}}{2\pi i} \left( \frac{1}{t - \hat{z}_{d0}} \right) \hat{B}^T d.$$  \hspace{1cm} (4–9)

Based on the theory of [Muskhelishvili 1975] and the assumption that $F_1(z)$ vanishes at infinity, the solution of boundary problems (4–7) and (4–8) can be obtained as

$$AF_1(z) - \overline{AF_1(z)} = 0,$$  \hspace{1cm} (4–10)

$$AF_1(z) + \overline{AF_1(z)} = h_0(z) + 2[Z(z) + X_0(z)P(z)],$$  \hspace{1cm} (4–11)

where

$$X_0(z) = \prod_{j=1}^{N} (z - a_j)^{-\frac{1}{2}} (z - b_j)^{-\frac{1}{2}},$$  \hspace{1cm} (4–12)

$$Z(z) = \frac{X_0(z)}{2\pi i} \int_L \frac{h(t) dt}{X_0^*(t)(t - z)},$$  \hspace{1cm} (4–13)

$$P(z) = \epsilon_N z^N + \epsilon_{N-1} z^{N-1} + \ldots + \epsilon_0.$$  \hspace{1cm} (4–14)

Incorporating Equations (4–10) and (4–11) results in

$$AF_1(z) = \frac{h_0(z)}{2} + Z(z) + X_0(z)P(z).$$  \hspace{1cm} (4–15)

Taking the limit $z \to \infty$ in (4–15), and noting that $F_1(\infty) = 0$, and $E_1(\infty) = 0$, the constant $\epsilon_N$ can be obtained as

$$\epsilon_N = (0, -w_r/2, 0, 0)^T.$$  \hspace{1cm} (4–16)

The other constants, that is, the vector $\epsilon_{N-1}, \ldots, \epsilon_0$ and $w_N, \ldots, w_1$ can be determined by single-value displacement, the irrotationality of electric fields and the force equilibrium conditions. With reference to (2–42), these conditions can be written as

$$\oint_{\Lambda} AF_1(z)dz = 0, \quad \hat{H}_2 \oint_{\Lambda} AF_1(z)dz = 0.$$  \hspace{1cm} (4–17)

where $\Lambda$ is the closed path around each inclusion, and $\hat{H}_2$ is the second low of the real $4 \times 4$ matrix $\hat{H} = H^{-1}$. The complex potential is therefore obtained if the function $E_1(z)$ is known.

To obtain $E_1(z)$, we introduce the condition (4–2). Using (2–42), (4–2) can be rewritten as

$$i M_4 AF_1^+(t) - i \tilde{M}_4 \overline{AF_1}(t) = i M_4 AF_1^+(t) - i \tilde{M}_4 \overline{AF_1}(t),$$  \hspace{1cm} (4–18)

where the vector $M_4$ is the fourth low of the matrix $M$ as expressed in (2–44). From [Muskhelishvili 1975] we know that the solution of the Equation (4–18) is

$$\hat{H}_4 AF_1(z) = 0.$$  \hspace{1cm} (4–19)
\[ r_2 \theta_2 (r_2, \theta_2) \]

Figure 2. A piezoelectric screw dislocation near a rigid line inclusion.

where \( \hat{H}_4 = (\hat{H}_{h1}, \hat{H}_{h2}, \hat{H}_{h3}, \hat{H}_{h4})^T \) is the fourth low of the real 4 \( \times \) 4 matrix \( \hat{H} \). Inserting (4–15) into (4–19) yields

\[
E_1(z) = \frac{\hat{H}_{h3}}{\hat{H}_{h4}} w_r + \frac{2\hat{H}_4}{\hat{H}_{h4}} [Z(z) + X_0(z) P(z)]. \tag{4–20}
\]

The complex potentials for the problem are thus determined. After \( F(z) \) has been obtained, we can calculate the stress and the electrical displacement fields. Thus, we can derive the field intensity factors and the force on the dislocation.

As an example, consider a single rigid line as shown in Figure 2. We can then simplify Equations (4–12) to (4–14) as

\[
X_0(z) = (z^2 - a^2)^{-\frac{1}{2}}, \tag{4–21}
\]
\[
P(z) = c_1 z + c_0, \tag{4–22}
\]
\[
Z(z) = \frac{\tilde{A}}{4\pi i} \left( \frac{1}{z_a - z_{da}} - \frac{\sqrt{z_{da}^2 - a^2}}{\sqrt{z_a^2 - a^2} (z_a - z_{da})} - \frac{1}{\sqrt{z_a^2 - a^2}} \right) \vec{B}^T \vec{d}
\]
\[
- \frac{\tilde{A}}{4\pi i} \left( \frac{1}{z_a - z_{da}} - \frac{\sqrt{z_{da}^2 - a^2}}{\sqrt{z_a^2 - a^2} (z_a - z_{da})} - \frac{1}{\sqrt{z_a^2 - a^2}} \right) \vec{B}^T \vec{d}. \tag{4–23}
\]

Substituting (4–15), together with (4–21), (4–22) and (4–23) into (4–17) yields

\[
c_0 = 0, \quad \hat{H}_2 c_1 = 0. \tag{4–24}
\]

Then, substituting (4–16) into (4–24) yields

\[
c_0 = 0, \quad c_1 = 0, \quad w_r = 0. \tag{4–25}
\]

The complex potentials are thus written as

\[
AF_1(z) = (I - Y) Z(z), \tag{4–26}
\]
where $I$ is the $4 \times 4$ identity matrix, and

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{H}_{41}/\hat{H}_{44} & \hat{H}_{42}/\hat{H}_{44} & \hat{H}_{43}/\hat{H}_{44} & 1 \end{bmatrix}. \quad (4–27)$$

When the material is purely elastic, the solution reduces to that of [Fan and Keer 1993].

4.2. **Field intensity factors.** Using (2–24), the field intensity factors at the right rigid line tip can be defined as

$$K = (K_I, K_{II}, K_{III}, K_D)^T = \lim_{x_1 \to a} \sqrt{2\pi(x_1 - a)} \Phi_{11}(x_1), \quad (4–28)$$

where

$$\Phi_{11}(x_1) = 2 \Re i M A F_1(x_1) = -2H^{-1}S(I - Y) \Re Z(x_1). \quad (4–29)$$

Substituting (4–23) into (4–29) yields

$$\Phi_{11}(x_1) = \frac{H^{-1}S(I - Y)}{\pi \sqrt{x_1^2 - a^2}} \left( \Im A \left( \sqrt{x_1^2 - a^2} - \sqrt{z_{da}^2 - a^2} \right) x_1 - z_{da} - 1 \right) B^T d. \quad (4–30)$$

The field intensity factors are thus obtained as

$$K = \frac{H^{-1}S(I - Y)}{\sqrt{\pi a}} \left( \Im A \left( \sqrt{z_{da}^2 + a} \sqrt{z_{da}^2 - a} - 1 \right) B^T d \right). \quad (4–31)$$

When the dislocation lies along the real axis $z_d = (x_d, 0)$, (4–15) reduces to

$$K = -\frac{H^{-1}S(I - Y)Sd}{2\sqrt{\pi a}} \left( \sqrt{x_d^2 + a} \sqrt{x_d^2 - a} - 1 \right). \quad (4–32)$$

4.3. **Force on dislocation.** To analyze the possible balance position of a dislocation, it is of interest to compute the image force acting on the dislocation due to the presence of the rigid lines. The image force per unit length is defined as the negative gradient of the interaction energy with respect to the position of the dislocation. The image force [Pak 1990] can be written as

$$F_{x_1} = d_1\sigma_{21}^1 + d_2\sigma_{22}^1 + d_3\sigma_{23}^1 + d_4D_2^1 = d^T \Phi_{11}, \quad (4–33)$$

$$F_{x_2} = -(d_1\sigma_{11}^1 + d_2\sigma_{12}^1 + d_3\sigma_{13}^1 + d_4D_1^1) = d^T \Phi_{12}, \quad (4–34)$$

where $\Phi^1$ is associated with the perturbed field calculated from $F_1(z)$ with $z_a \to z_{da}$, that is,

$$F_1(z_{da}) = \frac{1}{2\pi i} \left[ A^{-1}(I - Y)\bar{A}(G_1)\bar{B}^T + A^{-1}(I - Y)A(G_2)B^T \right] d. \quad (4–35)$$
with

\[ G_1(z_{da}) = \frac{\sqrt{z_{da}^2 - a^2} - \sqrt{z_{da}^2 - a^2 - (z_{da} - \bar{z}_{da})}}{2(z_{da} - \bar{z}_{da})\sqrt{\bar{z}_{da}^2 - a^2}}, \]

\[ G_2(z_{da}) = -\frac{z_{da} - \sqrt{\bar{z}_{da}^2 - a^2}}{2(\bar{z}_{da}^2 - a^2)}. \]  

(4–36)

As a result, we obtain

\[ \Phi_{11}^1(z_{da}) = \frac{1}{\pi} \text{Im} \left( BA^{-1}(I - Y)\bar{A}(G_1)\bar{B}^T + BA^{-1}(I - Y)A(G_2)B^T \right) d, \]  

(4–37)

\[ \Phi_{12}^1(z_{da}) = \frac{1}{\pi} \text{Im} \left( BA^{-1}(I - Y)\bar{A}(p_0G_1)\bar{B}^T + BA^{-1}(I - Y)A(p_0G_2)B^T \right) d. \]  

(4–38)

When the dislocation lies on the \( x_1 \)-axis, that is, \( z_{ad} = x_{1d} = x_d \), we can simplify the expressions (4–37) and (4–38) as

\[ \Phi_{11}^1(x_d) = -g(x_d)H^{-1}S(I - Y)Sd, \]  

(4–39)

\[ \Phi_{12}^1(x_d) = g(x_d) \text{Im} \left( BA^{-1}(I - Y)(\bar{A}(p_0)\bar{B}^T - A(p_0)B^T) \right) d, \]  

(4–40)

where

\[ g(x_d) = \frac{1}{2\pi} \frac{x_d - \sqrt{x_d^2 - a^2}}{x_d^2 - a^2}. \]  

(4–41)

### 5. Interaction of a dislocation with rigid conducting lines

#### 5.1. Determination of the complex potential function.

In the case of rigid conducting lines, the boundary conditions (3–1) and (3–3) apply. Conditions (3–1) and (3–3) can be rewritten as

\[ u_j'(t)^+ = u_j'(t)^- = w_j\delta_{j2}, \quad j = 1, 2, 3, \quad t \in L_r \]  

(5–1)

\[ u_a^+(t)^+ = u_a^-(t)^- = 0, \quad t \in L_r, \]  

(5–2)

where the prime denotes differentiation with respect to with \( x_1 \). With reference to (2–41) and (3–4), conditions (5–1) and (5–2) arrive at

\[ AF(t)^+ + A\bar{F}(t)^- = h_0, \quad t \in L, \]  

(5–3)

\[ AF(t)^- + A\bar{F}(t)^+ = h_0, \quad t \in L, \]  

(5–4)

which lead to

\[ [AF(t) - A\bar{F}(t)]^+ + [AF(t) - A\bar{F}(t)]^- = 0, \quad t \in L, \]  

(5–5)

\[ [AF(t) + A\bar{F}(t)]^+ + [AF(t) + A\bar{F}(t)]^- = 2h_0, \quad t \in L, \]  

(5–6)
where \( h_0 = (0, w_r, 0, 0)^T \). Substituting (3–4) into (5–5) and (5–6) yields
\[
[AF_1(t) - \overline{AF_1(t)}]^+ - [AF_1(t) - \overline{AF_1(t)}]^- = 0, \quad t \in L, \tag{5–7}
\]
\[
[AF_1(t) + \overline{AF_1(t)}]^- + [AF_1(t) + \overline{AF_1(t)}]^+ = 2[h_0 + h(t)], \quad t \in L, \tag{5–8}
\]
where \( h(t) \) is as defined in (4–9). This problem is a special case of the case solved in the previous section. The solution can be obtained from the previous solution by setting \( Y = 0 \). For a single rigid conducting line as shown in Figure 2, the complex potential corresponding to the perturbed field is
\[
AF_1(z) = Z(z), \tag{5–9}
\]
where \( Z(z) \) is as in (4–23).

5.2. Field intensity factors. The field intensity factors at the right inclusion tip can be defined as
\[
K = \frac{H^{-1}S}{\sqrt{\pi a}} \left( \text{Im} \left( \sqrt{\frac{x_d + a}{x_d - a} - 1} \right) B^T d \right). \tag{5–10}
\]
When the dislocation lies along the real axis \( z_d = (x_d, 0) \), Equation (5–15) reduces to
\[
K = -\frac{H^{-1}S^2 d}{2\sqrt{\pi a}} \left( \sqrt{\frac{x_d + a}{x_d - a} - 1} \right). \tag{5–11}
\]

5.3. Force on dislocation. The image force on dislocation can be written as
\[
F_{s1} = d^T \Phi_{11}(z_{da}), \tag{5–12}
\]
\[
F_{s2} = d^T \Phi_{12}(z_{da}), \tag{5–13}
\]
where
\[
\Phi_{11}(z_{da}) = \frac{1}{\pi} \text{Im} \left[ BA^{-1} \tilde{A} (G_1) \tilde{B}^T + B (G_2) B^T \right] d, \tag{5–14}
\]
\[
\Phi_{12}(z_{da}) = \frac{1}{\pi} \text{Im} \left[ BA^{-1} \tilde{A} (p_a G_1) \tilde{B}^T + B (p_a G_2) B^T \right] d. \tag{5–15}
\]
\( G_1(z_{da}) \) and \( G_2(z_{da}) \) are as defined in (4–36). When the dislocation lies on the \( x_1 \)-axis, that is, \( z_{da} = x_1d = x_d \), we simplify (5–14) and (5–15) as
\[
\Phi_{11}(x_d) = g(x_d) (H^{-1} - L) d, \tag{5–16}
\]
\[
\Phi_{12}(x_d) = g(x_d) \text{Im} \left[ BA^{-1} \left[ \tilde{A} (p_a) \tilde{B}^T - A (p_a) B^T \right] \right] d. \tag{5–17}
\]
where \( g(x_d) \) is as defined in (4–41).

6. Numerical examples

The previous sections derived the explicit expressions for the field intensity factors and the forces on the dislocation. However they are not straightforward since several variables are involved. In this section, we present some numerical illustrations. As an example, we address the case when the dislocation lies
along $\theta_d = \pi/6$. The material is assumed to be PZT-5H, with the $x_1$-axis the polling direction. The material constants [Pak 1992b] are

$$
c_{11} = 117 \text{ GPa}, \quad c_{12} = c_{13} = 53 \text{ GPa}, \\
c_{22} = c_{33} = 126 \text{ GPa}, \quad c_{23} = 55 \text{ GPa}, \\
c_{44} = 35.5 \text{ GPa}, \quad c_{55} = c_{66} = 35.3 \text{ GPa}, \\
e_{11} = 23.3 \text{ C/m}^2, \quad e_{12} = e_{13} = -6.5 \text{ C/m}^2, \\
e_{35} = e_{26} = 17 \text{ C/m}^2, \\
e_{11} = 130 \times 10^{-10} \text{ C/Vm}, \\
e_{22} = e_{33} = 151 \times 10^{-10} \text{ C/Vm}.
$$

(6–1)

For $p_\alpha$ ($\alpha = 1, 2, 3, 4$), the values of $A$ and $B$ are then calculated as follows:

$$
p_1 = -0.17351 + 0.93175i, \\
p_2 = 0.17351 + 0.93175i, \\
p_3 = 0.93367i, \\
p_4 = 0.99718i,
$$

(6–2)

$$
A_{11} = -0.8521 \times 10^{-6} + 0.3117 \times 10^{-5}i, \quad A_{12} = 0.3117 \times 10^{-5} - 0.8521 \times 10^{-6}i, \\
A_{13} = 0.4133 \times 10^{-5} + 0.1433 \times 10^{-5}i, \quad A_{14} = 0, \\
A_{21} = -0.3561 \times 10^{-5} + 0.4268 \times 10^{-6}i, \quad A_{22} = -0.4268 \times 10^{-6} + 0.3561 \times 10^{-5}i, \\
A_{23} = -0.1189 \times 10^{-5} + 0.1189 \times 10^{-5}i, \quad A_{24} = 0, \\
A_{31} = 0, \quad A_{32} = 0, \\
A_{33} = 0, \quad A_{34} = -0.2657 \times 10^{-5} + 0.2657 \times 10^{-5}i, \\
A_{41} = 722.3288 + 2351.6593i, \quad A_{42} = 2351.6593 + 722.3288i, \\
A_{43} = -3006.4445 - 3006.4445i, \quad A_{44} = 0.
$$

(6–3)

$$
B_{11} = -262382.5644 - 27548.9152i, \quad B_{12} = 27548.9157 + 262382.5653i, \\
B_{13} = -41491.1280 + 41491.1353i, \quad B_{14} = 0, \\
B_{21} = -22107.3141 - 277484.1268i, \quad B_{22} = -277484.1276 - 22107.3141i, \\
B_{23} = 44438.6631 - 44438.6550i, \quad B_{24} = 0, \\
B_{31} = 0, \quad B_{32} = 0, \quad B_{33} = 0, \quad B_{34} = -94074.2510 - 94074.2510i, \\
B_{41} = -0.7241 \times 10^{-4} - 0.1944 \times 10^{-4}i, \quad B_{42} = 0.1944 \times 10^{-4} + 0.7241 \times 10^{-4}i, \\
B_{43} = -0.8535 \times 10^{-4} + 0.8535 \times 10^{-4}i, \quad B_{44} = 0.
$$

(6–4)

6.1. **Field intensity factors.** The expression (4–31) gives the field intensity factors at the right rigid line tip arising from the dislocation $d = (d_1, d_2, d_3, d_4)^T$ located at $z_d$ near a rigid dielectric line. Expression (5–10) does the same for a rigid conducting line. When these intensity factors have the same sign as
those arising from the remote applied stress or electric displacement, the total intensity factors increase. The dislocation then anti-shields the rigid line tip; otherwise the dislocation shields it. Shielding effects from \( d_1, d_2, \) and \( d_4 \) on \( K_I, K_{II}, K_{III}, K_D \) for the dislocation located along \( \theta_d = \pi/6 \) near a rigid line are illustrated in Figures 3 to 8, in relation to the normalized dislocation radial location \( r_d/a \). To plot the four field intensity factors in one figure, the values of \( K_I, K_{II}, K_{III}, K_D \) were properly normalized in the figures with positive values. The normalized intensity factors are denoted as \( K^*_I, K^*_{II}, K^*_{III}, K^*_D \) in the figures, where

\[
K^*_j(d_j) = \frac{K_j(d_j)}{K_{j0}(d_j)}, \quad j = I, II, III, D, \tag{6–5}
\]

with

\[
K_{j0}(d_j) = \frac{d_j}{2\sqrt{\pi}a} \times 10^{10} \text{N/m}^2, \quad K_{D0}(d_j) = \frac{d_j}{\sqrt{\pi}a} \times 2 \text{N/Vm}, \quad j = I, II, III
\]

\[
K_{j0}(d_4) = \frac{d_4}{\sqrt{\pi}a} \times 2 \text{N/Vm}, \quad j = I, II, III, \quad K_{D0}(d_4) = \frac{d_4}{\sqrt{\pi}a} \times 10^{-9} \text{N/V}^2. \tag{6–6}
\]

In the above equations, \( d_I = d_1, d_{II} = d_2, d_{III} = d_3 \).

Figure 3 shows that the glide dislocation \( d_1 \) always shields \( K_I \) while anti-shielding \( K_{II} \) and \( K_D \) when it is near a dielectric line tip. The shielding effects from the glide dislocation on \( K_{II} \) and \( K_D \) appear

![Figure 3](image-url)

**Figure 3.** The shielding effect from the glide dislocation \( d_1 \) located along \( \theta_d = \pi/6 \) on the field intensity factors for a rigid dielectric inclusion.
Figure 4. The shielding effect from the glide dislocation $d_1$ located along $\theta_d = \pi/6$ on the field intensity factors for a rigid conducting inclusion.

in a very similar way. But the glide dislocation $d_1$ does not affect $K_{III}$. This occurs because the glide dislocation does not contribute any anti-plane deformations.

Figure 4 also shows that the glide dislocation $d_1$ always shields $K_I$ while anti-shielding $K_{II}$ and $K_D$, but does not affect $K_{III}$ when it is near a conducting line tip. A comparison of Figures 3 and 4 indicates that the conductivity of the inclusion only has apparent effects on $K_I$.

Figures 5 and 6 show the shielding effects from the climb dislocation $d_2$ for a rigid dielectric line and a rigid conducting line, respectively. We find that the two figures are nearly the same, which indicates that the conductivity of the rigid line is not sensitive to the shielding effects from $d_2$.

Figures 7 and 8 show the shielding effects from the electrical dislocation $d_4$ for a rigid dielectric line and a rigid conducting one, respectively. The comparison of these two figures also indicates that the conductivity of the rigid line only has apparent effects on $K_I$. For a rigid dielectric line, it first anti-shields and then shields $K_I$ when increasing $r_d/a$; while for a rigid conducting one, it always shields $K_I$.

6.2. **Image force on dislocation.** Expressions for the image forces on the dislocation due to existence of the inclusion are calculated using (4–33) and (4–34) together with (4–37) and (4–38) for a rigid dielectric line, and by (5–12) to (5–15) for a rigid conducting one. As such, the slip and climb parts of the image forces can be calculated as follows:

$$F_r = F_x \cos \theta_d + F_y \sin \theta_d,$$

$$F_t = -F_x \sin \theta_d + F_y \cos \theta_d.$$  

(6–7)
Figure 5. The shielding effect from the climb dislocation $d_2$ located along $\theta_d = \pi/6$ on the field intensity factors for a rigid dielectric inclusion.

Figure 6. The shielding effect from the climb dislocation $d_2$ located along $\theta_d = \pi/6$ on the field intensity factors for a rigid conducting inclusion.
Figure 7. The shielding effect from the electrical dislocation $d_4$ located along $\theta_d = \pi/6$ on the field intensity factors for a rigid dielectric inclusion.

Figure 8. The shielding effect from the electrical dislocation $d_4$ located along $\theta_d = \pi/6$ on the field intensity factors for a rigid conducting inclusion.
Figures 9 and 10 plot the normalized slip image force $F_r/F_0$ and climb image force $F_t/F_0$ varied with the normalized radial location $r_d/a$, respectively, for a rigid dielectric line.

Figure 9. Variations of the radial normalized image forces on the dislocation located along $\theta_d = \pi/6$ near a rigid dielectric inclusion.

Figure 10. Variations of the tangential normalized image forces on the dislocation located along $\theta_d = \pi/6$ near a rigid dielectric inclusion.
Figures 11 and 12 plot those for a rigid conducting line. The dislocation has four different dislocation strength characteristics \((d_1, d_2, d_3, d_4)\). We allow the dislocation to have only one non-zero strength.

**Figure 11.** Variations of the radial normalized image forces on the dislocation located along \(\theta_d = \pi/6\) near a rigid conducting inclusion.

**Figure 12.** Variations of the tangential normalized image forces on the dislocation located along \(\theta_d = \pi/6\) near a rigid conducting inclusion.
characteristic. The other three are zero in each plotted curve. The normalizing factors in each curve are given by

$$ F_0 = \frac{d^T L d}{4\pi a}. $$

(6–8)

Figures 9 and 10 show that, a rigid dielectric line always repels the mechanical dislocation in the radial direction, while it does little on the electrical dislocation; it always attracts the dislocation to the real axis when it is close to the rigid line tip. On the other hand, Figures 11 and 12 show that a rigid conducting line always repels the dislocation in the radial direction and attracts the dislocation in the tangential direction when the dislocation is close to the inclusion.

7. Conclusions

The interaction problem of a dislocation and collinear rigid lines embedded in a piezoelectric media is addressed. The lines considered are for either conductors or dielectrics. We obtain a closed form solution using the complex potential method, and explicitly derive field intensity factors and the forces on the dislocation for a single inclusion case. We present numerical examples and discuss the results.

References


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