Journal of
Mechanics of
Materials and Structures

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Volume 2, № 2 February 2007

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 2, No. 2, 2007

WORST CASE PLASTIC LIMIT ANALYSIS OF TRUSSES UNDER UNCERTAIN LOADS VIA MIXED 0-1 PROGRAMMING

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The paper presents a global optimization method to compute the minimum limit load factor of trusses subjected to unknown but bounded loads. We assume that the external forces consist of a part proportional to a load factor and a part that is uncertain around its nominal value. The worst-case limit load factor is introduced as the smallest limit load factor realized with some uncertain parameters. In order to detect the worst case, we have to find the global optimal solution of a nonconvex optimization problem, which is the major difficulty of the worst-case limit analysis. By reformulating the worst-case determination problem as a mixed 0-1 programming problem, we propose a global optimization algorithm as a combination of a branch-and-bound method based on the linear programming relaxations and a cutting plane method based on the disjunctive or lift-and-project cuts. The worst-case limit loads, as well as the corresponding critical loading patterns, are computed to demonstrate that our method converges to the global optimal solutions successfully.

1. Introduction

In designing civil, mechanical and aerospace structures, plastic limit analysis has been used widely for decades as a means of estimating the ultimate strength of structures. While dead and live loads are uncertain around their nominal values, the disturbance load is applied proportionally with a load factor. This paper discusses a global optimization technique for computing the smallest limit load factor of truss structures, where the applied dead and live loads are imprecisely known.

Limit analysis still receives much attention by numerous researchers with regard to algorithms [Muralidhar and Jagannatha Rao 1997; Andersen et al. 1998; Cocchetti and Maier 2003; Krabbenhoft and Damkilde 2003] and issues relevant to the finite element method [Tin-Loi and Ngo 2003; Lyamin et al. 2005]. Based on the probabilistic uncertainty models of structural systems, various approaches to stochastic limit analysis have also been proposed [Lloyd Smith et al. 1990; Rocho and Sonnenberg 2003; Staat and Heitzer 2003; Marti and Stoeckel 2004]. In the framework of probabilistic uncertainties, reliability-based structural design methods have been investigated extensively [Zang et al. 2005; Kharmanda et al. 2004].

Besides these probabilistic uncertainty models, nonprobabilistic uncertainty models have also been developed, where a mechanical system is assumed to contain uncertain parameters which are unknown but bounded. [Ben-Haim and Elishakoff 1990] developed the well known convex model approach, with which [Ganzerli and Pantelides 1999] proposed a robust truss optimization method. Interval linear algebra has been well developed for uncertain linear equations [Alefeld and Mayer 2000], and has been employed in structural analyses with uncertainties [Chen et al. 2002]. In contrast to probabilistic models, these

Keywords: data uncertainty, limit analysis, integer programming, cutting plane, branch-and-bound, global optimization.

nonprobabilistic uncertainty models require only upper bounds on the magnitude of uncertain parameters, and engineers need not estimate the probabilistic density distributions of uncertain parameters.

[Elishakoff et al. 1994] proposed a structural optimization scheme under unknown-but-bounded uncertainty by using antioptimization. The bilevel optimization problems were formulated and solved numerically for robust structural design against the worst case [Craig et al. 2003; Cheng et al. 2004]. [Gu et al. 2000] proposed an estimation method for the worst case of propagated uncertainty in a multidisciplinary system. A unified methodology which is a robust counterpart of various convex optimization problems was developed by [Ben-Tal and Nemirovski 2002], and was applied to robust compliance minimization of trusses [Ben-Tal and Nemirovski 1997]. The authors proposed methods for robustness analysis and robust optimization of structures [Takewaki and Ben-Haim 2005; Kanno and Takewaki 2006a; Kanno and Takewaki 2006b] based on the info-gap uncertainty model [Ben-Haim 2001].

A serious difficulty in worst-case detection arises when the worst case is defined as an optimal solution of a nonconvex optimization problem in terms of the uncertain parameters. Conventional methods for linear worst-case analysis, for example the convex model of [Ben-Haim and Elishakoff 1990], can be applied only to cases in which sufficiently small variation of the uncertain parameters is allowed, or in which the structural response considered is represented as a linear function of the uncertain parameters. In these cases, the worst case can be detected by solving a convex optimization problem.

Unfortunately, in many practical situations, the variation of uncertain parameters is not small and we are interested in nonlinear responses of structures. Then the worst case is defined through a nonconvex optimization problem. In general, the conventional nonlinear programming approach converges to a local optimal solution of that problem. However, a local minimum solution, which is not globally optimal, does not correspond to the worst case. Obviously, the worst case corresponds to a global optimal solution. Thus, we have to find a global optimal solution of the nonconvex problem and guarantee that the solution obtained is globally optimal, which prevent us from using the conventional nonlinear programming algorithms.

In this paper, we aim at developing a global optimization method for worst-case detection. We consider the limit load factor of a truss structure subjected to uncertain loads. The external forces applied to a truss are supposed to consist of a constant part and a part proportional to a load factor, where the former part cannot be known precisely, but is assumed to be bounded. The worst-case limit load factor is defined as the minimum value among all the possible limit load factors realized by some uncertain parameters belonging to the given closed set.

We define the worst-case limit load factor by using a nonconvex optimization problem, which can be rewritten as a mixed 0-1 programming problem. Based on a linear programming (LP) relaxation, we propose a simple branch-and-bound algorithm to obtain a global optimal solution of the mixed 0-1 programming problem. To strengthen the LP relaxation, we generate some cutting planes at the root node of the branch-and-bound tree. This approach is called the cut-and-branch method [Cordier et al. 1999]. We formulate an LP problem to generate the deepest disjunctive cut. By adding generated cuts to LP relaxation problems, we drastically reduce the number of LP problems to be solved in the branch-and-bound method.

The solution obtained by using the cut-and-branch method is a global optimal solution of the worst-case determination problem, that is, it is assured that there exists no uncertain parameter with which the limit load factor becomes smaller than the obtained optimal value. Through the numerical experiments in

Section 6 we show that the limit load factor can be reduced greatly from its nominal value, by nontrivial combination of uncertain external forces. We also show that the critical load yielding the worst-case limit load factor cannot be detected easily by generating a large sample of loading scenarios. The upper bound of the worst-case limit load factor obtained from such a sample is shown to be too optimistic.

Recently, there has been renewed interest in cutting planes, or cuts, that are valid linear inequalities of a mixed integer programming problem; see for example the review paper [Marchand et al. 2002]. In particular, the branch-and-cut method [Balas et al. 1996; Cordier et al. 1999], that is, a branch-and-bound method with cuts added, is considered as the most successful approach to solving the mixed integer program. Among various types of cuts, a disjunctive cut (or lift-and-project cut) is defined as a linear inequality selected among inequalities valid for a disjunctive programming relaxation of the mixed 0-1 program [Balas et al. 1996; Ceria and Soares 1997; Balas and Perregaard 2002]. We utilize disjunctive cuts to strengthen the LP relaxation problems that are solved at nodes of the branch-and-bound tree.

This paper is organized as follows. In Section 2, we prepare the LP problems for conventional limit analysis and define the notation used in this paper. In Section 3 we introduce the notion of uncertain limit analysis by defining the info-gap model for uncertainty of external load and the worst-case limit load factor. In Section 4, we present the mixed 0-1 programming formulation for the uncertain limit analysis, and propose a branch-and-bound method as the solution. In Section 5 we propose an LP problem that generates the disjunctive cutting plane, to strengthen the LP relaxation problems solved in the branch-and-bound tree. In Section 6 we present numerical experiments for various trusses, made using our cut-and-branch method. Finally, in Section 7 we draw conclusions.

2. Notation and preliminary results

2.1. ***Notation.*** In this paper, we assume all vectors to be column vectors. For an n-tuple $p_{m+1}, \dots, p_{m+n}$, we let $(p_i \mid i = m+1, \dots, m+n)$ and $\{p_i \mid i = m+1, \dots, m+n\}$, respectively, denote the n-dimensional vector $(p_{m+1}, \dots, p_{m+n})^\top$ and the set consisting of $p_{m+1}, \dots, p_{m+n}$. The vector $(p_i \mid i = 1, \dots, n) \in \mathbb{R}^n$ is often simplified as $(p_i) \in \mathbb{R}^n$. The ℓ^1, ℓ^2 (or standard Euclidean), and ℓ^∞ norms of the vector $\boldsymbol{p} = (p_i) \in \mathbb{R}^n$, denoted by $\|\boldsymbol{p}\|_1$, $\|\boldsymbol{p}\|_2$, and $\|\boldsymbol{p}\|_\infty$, respectively, are defined as

$$\|\boldsymbol{p}\|_1 = \sum_{i=1}^{n} |p_i|,$$

$$\|\boldsymbol{p}\|_2 = (\boldsymbol{p}^\top \boldsymbol{p})^{1/2},$$

$$\|\boldsymbol{p}\|_\infty = \max_{i \in \{1,\dots,n\}} |p_i|.$$

For vectors $\boldsymbol{p} = (p_i) \in \mathbb{R}^n$ and $\boldsymbol{q} = (q_i) \in \mathbb{R}^n$, we write $\boldsymbol{p} \geq \boldsymbol{0}$ and $\boldsymbol{p} \geq \boldsymbol{q}$, respectively, if $p_i \geq 0$, $i = 1, \dots, n$ and $\boldsymbol{p} - \boldsymbol{q} \geq \boldsymbol{0}$. The $(m+n)$-dimensional column vector $(\boldsymbol{p}^\top, \boldsymbol{q}^\top)^\top$ is often written simply as $(\boldsymbol{p}, \boldsymbol{q})$. Moreover, $(\boldsymbol{p}, \boldsymbol{q})_i$ denotes the i-th component of the vector $(\boldsymbol{p}^\top, \boldsymbol{q}^\top)^\top$.

We define the vectors $\mathbf{1} \in \mathbb{R}^n$ and $\boldsymbol{e}^j \in \mathbb{R}^n$, $j = 1, \dots, n$, as

$$\mathbf{1} = (1, \dots, 1)^\top,$$
$$\boldsymbol{e}^j = (e_i^j \,|\, i = 1, \dots, n), \quad e_i^j = \begin{cases} e_i^j = 0, & \text{for } i \neq j, \\ e_j^j = 1, \end{cases}$$

that is, $\boldsymbol{e}^j$ is the j-th column vector of the identity matrix. We define $\mathbb{R}^n_+ \subset \mathbb{R}^n$ as

$$\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \boldsymbol{x} \geq \mathbf{0}\}.$$

For two sets $\mathcal{A} \subseteq \mathbb{R}^m$ and $\mathcal{B} \subseteq \mathbb{R}^n$, the Cartesian product is defined as $\mathcal{A} \times \mathcal{B} = \{(\boldsymbol{a}^\top, \boldsymbol{b}^\top)^\top \in \mathbb{R}^{m+n} \,|\, \boldsymbol{a} \in \mathcal{A},\ \boldsymbol{b} \in \mathcal{B}\}$. In particular, we write $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$.

The convex hull of $\mathcal{A}$, that is the smallest convex set that contains $\mathcal{A}$ is denoted 'conv $\mathcal{A}$.' The closure of $\mathcal{A}$, that is smallest closed set that contains $\mathcal{A}$, is denoted 'cl $\mathcal{A}$.' The cardinality of the set $\mathcal{A}$ is denoted $|\mathcal{A}|$. The empty set is denoted as $\emptyset$.

2.2. ***Basic problem for plastic limit analysis.*** Consider an elastic and perfectly-plastic truss in two- or three-dimensional space. Small rotations and small strains are assumed. Let $\boldsymbol{f} \in \mathbb{R}^{n^{\mathrm{d}}}$ denote the vector of the external forces, where n^{d} denotes the number of degrees of freedom of displacements. Where n^{m} is the number of members, the vector of member axial forces is $\boldsymbol{q} = (q_i) \in \mathbb{R}^{n^{\mathrm{m}}}$. The system of equilibrium equations in terms of $\boldsymbol{f}$ and $\boldsymbol{q}$ can be written

$$\boldsymbol{B}\boldsymbol{q} = \boldsymbol{f}, \tag{1}$$

where $\boldsymbol{B} \in \mathbb{R}^{n^{\mathrm{d}} \times n^{\mathrm{m}}}$ is a constant matrix.

Let $\boldsymbol{u} \in \mathbb{R}^{n^{\mathrm{d}}}$ and c_i denote the vector of nodal displacements and the corresponding elongation of the i-th member, respectively. We often write $\boldsymbol{c} = (c_i) \in \mathbb{R}^{n^{\mathrm{m}}}$. The i-th column vector of $\boldsymbol{B}$ is denoted by $\boldsymbol{b}_i \in \mathbb{R}^{n^{\mathrm{d}}}$, $i = 1, \dots, n^{\mathrm{m}}$. The compatibility relation between $\boldsymbol{u}$ and c_i can be written as

$$c_i = \boldsymbol{b}_i^\top \boldsymbol{u}, \quad i = 1, \dots, n^{\mathrm{m}}. \tag{2}$$

The external load $\boldsymbol{f}$ consists of a constant part $\boldsymbol{f}_{\mathrm{D}}$ and a proportionally increasing part $\lambda \boldsymbol{f}_{\mathrm{R}}$, that is,

$$\boldsymbol{f} = \boldsymbol{f}_{\mathrm{D}} + \lambda \boldsymbol{f}_{\mathrm{R}}. \tag{3}$$

Here, $\lambda \boldsymbol{f}_{\mathrm{R}}$ is defined by the monotonically increasing load parameter $\lambda \in \mathbb{R}$ and the constant reference load $\mathbb{R}^{n^{\mathrm{d}}} \ni \boldsymbol{f}_{\mathrm{R}} \neq \mathbf{0}$. In civil engineering, $\boldsymbol{f}_{\mathrm{D}}$ consists of the dead load, live load, etc., while $\lambda \boldsymbol{f}_{\mathrm{R}}$ consists of live or disturbance load caused by earthquakes, winds, and the like. For the sake of simplicity, in this paper $\boldsymbol{f}_{\mathrm{D}}$ is simply called *dead load* and $\boldsymbol{f}_{\mathrm{R}}$ is called *reference disturbance load.*

Let $\sigma_i^{\mathrm{y}} > 0$ and $-\sigma_i^{\mathrm{y}}$ denote the yield stresses of the i-th member in tension and in compression, respectively. We assume for simplicity that the yield stresses in tension and compression share the common absolute value. The member cross-sectional area is denoted by $a_i > 0$. The absolute value of the admissible axial force can be expressed as

$$q_i^{\mathrm{y}} = a_i \sigma_i^{\mathrm{y}}, \quad i = 1, \dots, n^{\mathrm{m}}.$$

Then, the yield functions can be written as

$$|q_i| - q_i^{\mathrm{y}} \le 0, \quad i = 1, \dots, n^{\mathrm{m}}. \tag{4}$$

From the static or lower-bound principle [Hodge 1959], and by using (1), (3), and (4), the limit load factor is obtained by solving the following LP problem

$$\max \left\{ \lambda : \boldsymbol{B}\boldsymbol{q} = \boldsymbol{f}_{\mathrm{D}} + \lambda \boldsymbol{f}_{\mathrm{R}}, \ \ |q_i| - q_i^{\mathrm{y}} \le 0, \ \ i = 1, \dots, n^{\mathrm{m}} \right\}, \tag{5}$$

where the variables are λ and $\boldsymbol{q}$.

3. Uncertain limit analysis

In this section, we introduce the uncertainty model of the external load, and rigorously define the worst-case limit load factor.

3.1. ***Uncertainty model.*** In this paper, we suppose that only $\boldsymbol{f}_{\mathrm{D}}$ in Equation (3) possesses uncertainty, that is, it cannot be known precisely. The model of the uncertainty of $\boldsymbol{f}_{\mathrm{D}}$ is motivated by a nonprobabilistic information-gap model [Ben-Haim 2001].

Let $\tilde{\boldsymbol{f}}_{\mathrm{D}} \in \mathbb{R}^{n^{\mathrm{d}}}$ denote the nominal value (or the best estimate) of $\boldsymbol{f}_{\mathrm{D}}$. We describe the uncertainty of $\boldsymbol{f}_{\mathrm{D}}$ in terms of the m-dimensional vector $\boldsymbol{\zeta} \in \mathbb{R}^m$, which is considered to be unknown but bounded. Suppose that $\boldsymbol{f}_{\mathrm{D}}$ depends on $\boldsymbol{\zeta}$ affinely as

$$\boldsymbol{f}_{\mathrm{D}} = \tilde{\boldsymbol{f}}_{\mathrm{D}} + \boldsymbol{T}\boldsymbol{\zeta}, \tag{6}$$

where $\boldsymbol{T} \in \mathbb{R}^{n^{\mathrm{d}} \times m}$ is a constant matrix satisfying the following assumption:

Assumption 3.1. The matrix $\boldsymbol{T}$ in Equation (6) satisfies the following conditions:

(i) $\{\boldsymbol{T}^{\top}\boldsymbol{u} \,|\, \boldsymbol{u} \in \mathbb{R}^{n^{\mathrm{d}}}\} = \mathbb{R}^m$;

(ii) $\boldsymbol{f}_{\mathrm{R}}^{\top}\boldsymbol{T}\boldsymbol{\zeta} = 0$ for any $\boldsymbol{\zeta} \in \mathbb{R}^m$.

Assumption 3.1 (ii) implies that the reference disturbance load $\boldsymbol{f}_{\mathrm{R}}$ does not have uncertainty.

For a given parameter $\alpha \in \mathbb{R}_+$, the uncertain set $\mathcal{Z}(\alpha) \subset \mathbb{R}^m$ is defined as

$$\mathcal{Z}(\alpha) = \left\{ \boldsymbol{\zeta} \in \mathbb{R}^m \,\middle|\, \alpha \ge \|\boldsymbol{\zeta}\|_\infty \right\}, \tag{7}$$

where the uncertain parameters vector $\boldsymbol{\zeta}$ is assumed to be running through $\mathcal{Z}(\alpha)$, that is,

$$\boldsymbol{\zeta} \in \mathcal{Z}(\alpha). \tag{8}$$

From (6), (7), and (8) it follows that the uncertain $\boldsymbol{f}_{\mathrm{D}}$ satisfies

$$\boldsymbol{f}_{\mathrm{D}} \in \mathcal{F}_{\mathrm{D}}(\alpha) := \left\{ \boldsymbol{f} \in \mathbb{R}^{n^{\mathrm{d}}} \,\middle|\, \boldsymbol{f} = \tilde{\boldsymbol{f}}_{\mathrm{D}} + \boldsymbol{T}\boldsymbol{\zeta}, \ \alpha \ge \|\boldsymbol{\zeta}\|_\infty \right\}. \tag{9}$$

Roughly speaking, $\boldsymbol{f}_{\mathrm{D}}$ moves around the center-point $\tilde{\boldsymbol{f}}_{\mathrm{D}}$. The greater the value of α, the greater the range of possible variation of $\boldsymbol{f}_{\mathrm{D}}$. In the context of the info-gap uncertainty model [Ben-Haim 2001], α is called the uncertainty parameter. Throughout, we suppose that the bound α on the uncertain variation given in Equation (9) is a constant. Note that the uncertain set $\mathcal{F}_{\mathrm{D}}(\alpha)$ is bounded for any $\alpha \in \mathbb{R}_+$.

Moreover, $\mathscr{F}_{\mathrm{D}}(\alpha)$ satisfies the two basic axioms for the info-gap model: nesting, in which $0 \leq \alpha_1 < \alpha_2$ implies $\mathscr{F}_{\mathrm{D}}(\alpha_1) \subset \mathscr{F}_{\mathrm{D}}(\alpha_2)$, and contraction, in which $\mathscr{F}_{\mathrm{D}}(0)$ is the singleton set $\{\tilde{\boldsymbol{f}}_{\mathrm{D}}\}$.

3.2. ***Worst-case limit load factor.*** For a given (but uncertain) dead load $\boldsymbol{f}_{\mathrm{D}} \in \mathbb{R}^{n^{\mathrm{d}}}$, we define a set $\mathcal{Q}(\boldsymbol{f}_{\mathrm{D}}) \subseteq \mathbb{R}^{n^{\mathrm{m}}+1}$ as

$$\mathcal{Q}(\boldsymbol{f}_{\mathrm{D}}) := \left\{ (\lambda, \boldsymbol{q}) \in \mathbb{R} \times \mathbb{R}^{n^{\mathrm{m}}} \,\middle|\, \boldsymbol{B}\boldsymbol{q} = \boldsymbol{f}_{\mathrm{D}} + \lambda \boldsymbol{f}_{\mathrm{R}},\ |q_i| - q_i^{\mathrm{y}} \leq 0,\ i = 1, \dots, n^{\mathrm{m}} \right\}, \tag{10}$$

which is the set of all statically admissible vectors $(\lambda, \boldsymbol{q}) \in \mathbb{R} \times \mathbb{R}^{n^{\mathrm{m}}}$ associated with the fixed $\boldsymbol{f}_{\mathrm{D}}$. We also define $\lambda^* : \mathbb{R}^{n^{\mathrm{d}}} \to \mathbb{R}$ as

$$\lambda^*(\boldsymbol{f}_{\mathrm{D}}) = \max_{\lambda, \boldsymbol{q}} \left\{ \lambda : (\lambda, \boldsymbol{q}) \in \mathcal{Q}(\boldsymbol{f}_{\mathrm{D}}) \right\}. \tag{11}$$

According to the static principle (5), $\lambda^*(\boldsymbol{f}_{\mathrm{D}})$ corresponds to the limit load factor under the dead load $\boldsymbol{f}_{\mathrm{D}}$. the external load in the decomposition (3) of the total external load $\boldsymbol{f}$, while $\boldsymbol{f}_{\mathrm{D}}$ is called the dead load, for the sake of simplicity. In civil engineering, it is usually held that $\boldsymbol{f}_{\mathrm{D}}$ consists of the conventional dead load caused by the weight of the truss itself and the live load caused by the nonstructural masses. In contrast, $\boldsymbol{f}_{\mathrm{R}}$ is the reference disturbance load, and $\lambda \boldsymbol{f}_{\mathrm{R}}$ is regarded as the load caused by earthquakes, winds, and the like. Hence, the uncertainty of live load, as well as that of dead load, can be represented by the uncertainty of $\boldsymbol{f}_{\mathrm{D}}$. For consistency in the concept of the limit load factor, we make the following assumption throughout our study:

Assumption 3.2. The uncertainty set $\mathscr{F}_{\mathrm{D}}(\alpha)$ is chosen so that

$$\lambda^*(\boldsymbol{f}_{\mathrm{D}}) > 0, \qquad \text{for all } \boldsymbol{f}_{\mathrm{D}} \in \mathscr{F}_{\mathrm{D}}(\alpha).$$

This assumption guarantees that the structure does not collapse without applying the load $\lambda \boldsymbol{f}_{\mathrm{R}}$ $(\lambda > 0)$. In other words, if it is not satisfied, there exists $\boldsymbol{f}'_{\mathrm{D}} \in \mathscr{F}_{\mathrm{D}}(\alpha)$ such that the structure collapses only with the dead load $\boldsymbol{f}'_{\mathrm{D}}$.

We can now introduce a concept of the worst-case limit load factor, by considering that the limit load factor can be regarded as a function of $\boldsymbol{f}_{\mathrm{D}}$ as seen in Equation (11), while $\boldsymbol{f}_{\mathrm{D}}$ is uncertain and running through $\mathscr{F}_{\mathrm{D}}(\alpha)$. Certainly, to evaluate robustness of trusses quantitatively, we are interested in the most severe situation, if any, in which the limit load factor happens to decrease unexpectedly from the nominal limit load factor corresponding to $\tilde{\boldsymbol{f}}_{\mathrm{D}}$ because of the uncertainty of $\boldsymbol{f}_{\mathrm{D}}$. To this end, we attempt to compute the minimum value of the limit load factor that can be attained at some load satisfying $\boldsymbol{f}_{\mathrm{D}} \in \mathscr{F}_{\mathrm{D}}(\alpha)$. This is naturally realized by introducing $\lambda_{\min} : \mathbb{R}_+ \to \mathbb{R}$ as

$$\lambda_{\min}(\alpha) = \min_{\boldsymbol{f}_{\mathrm{D}}} \left\{ \lambda^*(\boldsymbol{f}_{\mathrm{D}}) : \boldsymbol{f}_{\mathrm{D}} \in \mathscr{F}_{\mathrm{D}}(\alpha) \right\}. \tag{12}$$

Substitution of (9) into (12) yields

$$\lambda_{\min}(\alpha) = \min_{\boldsymbol{\zeta}} \left\{ \lambda^*(\boldsymbol{f}_{\mathrm{D}}(\boldsymbol{\zeta})) : \alpha \geq \|\boldsymbol{\zeta}\|_\infty \right\}. \tag{13}$$

Let $\boldsymbol{\zeta}^{\mathrm{cr}}$, which we call the *critical uncertain parameters vector*, denote an optimal solution of (13). Given the uncertainty parameter α, we refer to $\lambda_{\min}(\alpha)$ defined by (13) as the worst-case limit load factor, that is the minimum value among limit load factors $\lambda^*(\boldsymbol{f}_{\mathrm{D}})$ corresponding to $\boldsymbol{f}_{\mathrm{D}} \in \mathscr{F}_{\mathrm{D}}(\alpha)$. The corresponding dead load $\boldsymbol{f}_{\mathrm{D}}(\boldsymbol{\zeta}^{\mathrm{cr}})$ is called the critical load. In the case without uncertainty, we easily see that the following relationship holds:

$$\lambda_{\min}(0) = \lambda^*(\widetilde{\boldsymbol{f}}_{\mathrm{D}}) = \lambda^*\big(\boldsymbol{f}_{\mathrm{D}}(\mathbf{0})\big),$$

where $\lambda^*(\widetilde{\boldsymbol{f}}_{\mathrm{D}})$ is the nominal limit load factor, that is, the limit load factor corresponding to the nominal dead load $\widetilde{\boldsymbol{f}}_{\mathrm{D}}$.

The objective of this paper, then, is to propose a solution technique for computing both $\lambda_{\min}(\alpha)$ and $\boldsymbol{\zeta}^{\mathrm{cr}}$.

Remark 3.3. In this paper we suppose that only the external load has uncertainty, and that the strength (or the admissible axial force) q_i^{y} of each member is certain. This is because the worst case associated with the uncertainty of member strength can be found easily for the limit analysis. Indeed, the limit load factor monotonically decreases if member strength decreases. Hence, the set of critical member strength corresponds to the trivial case in which the strength of each member coincides with its lower bound. On the contrary, the loading pattern that gives the worst case is not trivial, which motivates us to confine attention to the uncertainty of $\boldsymbol{f}_{\mathrm{D}}$. We also assume throughout that $\boldsymbol{f}_{\mathrm{R}}$ is certain. Suppose that $\boldsymbol{f}_{\mathrm{R}}$ varies proportionally with the fixed direction. Then the worst-case limit load factor is obtained simply by scaling the limit load factor corresponding to the nominal value of $\boldsymbol{f}_{\mathrm{R}}$. In future work, it may be interesting to consider the case where the distribution and/or the direction of $\boldsymbol{f}_{\mathrm{R}}$ include uncertainty.

Remark 3.4. The limit load factor can be computed easily if the loading pattern of the additional dead load is fixed. Suppose that $\boldsymbol{\zeta}$ in Equation (9) is defined as $\boldsymbol{\zeta} = \beta\boldsymbol{\zeta}^0$ with a given constant $\boldsymbol{\zeta}^0$ and a parameter β. After finding the nominal limit load factor $\lambda^*(\boldsymbol{f}_{\mathrm{D}}(\mathbf{0}))$ by employing conventional limit analysis, the variation of $\lambda^*(\boldsymbol{f}_{\mathrm{D}}(\beta\boldsymbol{\zeta}^0))$ with respect to β can be computed by simply using a parametric linear programming [Chvátal 1983] approach. In our problem, the direction of $\boldsymbol{\zeta}$ is unknown and should be determined so as to minimize $\lambda^*(\boldsymbol{f}_{\mathrm{D}}(\boldsymbol{\zeta}))$. Again, note that here we are trying to find a global optimal solution of (13) which is essentially nonconvex.

3.3. ***Some relevant problems.*** In the remainder of this section, we prepare a reformulation of (13) into the mixed 0-1 programming problem we will present in Section 4. Defining

$$\mathcal{U} = \left\{(\boldsymbol{u}, \boldsymbol{z}) \in \mathbb{R}^{n^{\mathrm{m}}} \times \mathbb{R}^{n^{\mathrm{d}}} \;\middle|\; \boldsymbol{f}_{\mathrm{R}}^{\top}\boldsymbol{u} = 1,\ z_i \geq |\boldsymbol{b}_i^{\top}\boldsymbol{u}|,\ i = 1, \dots, n^{\mathrm{m}}\right\}, \tag{14}$$

consider the following problem in the variables $(\boldsymbol{u}, \boldsymbol{z}) \in \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^{n^{\mathrm{m}}}$:

$$v^*(\boldsymbol{f}_{\mathrm{D}}) := \min_{\boldsymbol{u},\boldsymbol{z}} \left\{-\boldsymbol{f}_{\mathrm{D}}^{\top}\boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top}\boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U}\right\}. \tag{15}$$

Proposition 3.5 (Relation between problems (5) and (15)). Let $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{z}})$ denote an optimal solution of Problem (15). Then,

(i) $v^*(\boldsymbol{f}_{\mathrm{D}}) = \lambda^*(\boldsymbol{f}_{\mathrm{D}})$;

(ii) $\bar{\boldsymbol{u}}$ corresponds to a collapse mode associated with $\boldsymbol{f}_{\mathrm{D}}$;

(iii) $\bar{z}_i$ corresponds to the member elongation compatible to $\bar{\boldsymbol{u}}$.

Proof. We prove this proposition by showing that Equation (15) is dual to the static principle problem (5). Regarding the constraint

$$q_i^{\mathrm{y}} \geq |q_i| \tag{16}$$

in (5), observe that q_i satisfies (16) if and only if

$$(q_i^{\mathrm{y}}, q_i) \cdot (z_i, w_i) \geq 0$$

holds for any (z_i, w_i) where

$$z_i \geq |w_i|. \tag{17}$$

As a result, the function

$$L(\lambda, \boldsymbol{q}, \boldsymbol{u}, \boldsymbol{z}, \boldsymbol{w}) = \begin{cases} \lambda + \boldsymbol{u}^{\top}\left(\boldsymbol{B}\boldsymbol{q} - \boldsymbol{f}_{\mathrm{D}} - \lambda \boldsymbol{f}_{\mathrm{R}}\right) + (\boldsymbol{z}^{\top}\boldsymbol{q}^{\mathrm{y}} + \boldsymbol{w}^{\top}\boldsymbol{q}), & \text{if } z_i \geq |w_i|,\ i = 1, \dots, n^{\mathrm{m}}, \\ -\infty, & \text{otherwise,} \end{cases}$$

corresponds to the Lagrangian of (5), where $\boldsymbol{u} \in \mathbb{R}^{n^{\mathrm{d}}}$, $\boldsymbol{z} \in \mathbb{R}^{n^{\mathrm{m}}}$, and $\boldsymbol{w} \in \mathbb{R}^{n^{\mathrm{m}}}$ are the Lagrangian multipliers. Then the Lagrangian dual of (5) is

$$\min_{\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{w}} \sup \left\{ L(\lambda, \boldsymbol{q}, \boldsymbol{u}, \boldsymbol{z}, \boldsymbol{w}) : (\lambda, \boldsymbol{q}) \in \mathbb{R} \times \mathbb{R}^{n^{\mathrm{m}}} \right\},$$

the explicit form of which is easily obtained as

$$\min \left\{ -\boldsymbol{f}_{\mathrm{D}}^{\top}\boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top}\boldsymbol{z} : w_i = -\boldsymbol{b}_i^{\top}\boldsymbol{u}, \quad \boldsymbol{f}_{\mathrm{R}}^{\top}\boldsymbol{u} = 1, \quad z_i \geq |w_i|, \quad i = 1, \dots, n^{\mathrm{m}} \right\}. \tag{18}$$

Eliminating $\boldsymbol{w}$ from (18) yields (15). Hence, the LP problem (15) is dual to the LP problem (5). From this, and the strong duality of LP [Chvátal 1983] , we obtain the assertions (i) and (ii). Optimal solutions of (5) and (18) satisfy the complementarity condition

$$z_i q_i^{\mathrm{y}} + w_i q_i = 0 \tag{19}$$

over the constraints (16) and (17). Since (16), (17), and (19) imply $z_i = -w_i$, we see that $\bar{z}_i = \boldsymbol{b}_i^{\top}\bar{\boldsymbol{u}}$ is satisfied at an optimal solution of (15), and we obtain assertion (iii). □

Remark 3.6. Note that the upper bound principle (15) is different from the well-known formulation for trusses (see, for example, [Muralidhar and Jagannatha Rao 1997]). Observe that the yield condition (4) in (5) can be rewritten as

$$q_i^{\mathrm{y}} - q_i \geq 0, \qquad q_i^{\mathrm{y}} + q_i \geq 0, \qquad i = 1, \dots, n^{\mathrm{m}}. \tag{20}$$

The elongation c_i defined in (2) is divided into the two parts as

$$c_i = c_i^{+} - c_i^{-}, \qquad c_i^{+} \geq 0, \qquad c_i^{-} \geq 0, \qquad i = 1, \dots, n^{\mathrm{m}}. \tag{21}$$

Using (20) and (21), the set of relations governing the elastic and plastic behavior can be written as

$$\begin{aligned}
&\boldsymbol{B}\boldsymbol{q} = \boldsymbol{f}_{\mathrm{D}} + \lambda \boldsymbol{f}_{\mathrm{R}}, && \text{(equilibrium)}\\
&\boldsymbol{q}^{\mathrm{y}} - \boldsymbol{q} \geq \boldsymbol{0}, \quad \boldsymbol{q}^{\mathrm{y}} + \boldsymbol{q} \geq \boldsymbol{0}, && \text{(yield conditions)}\\
&\boldsymbol{f}_{\mathrm{R}}^{\top}\boldsymbol{u} = 1, && \text{(normalization)}\\
&\boldsymbol{c}^{+} - \boldsymbol{c}^{-} = \boldsymbol{B}^{\top}\boldsymbol{u}, && \text{(compatibility)}\\
&\boldsymbol{c}^{+} \geq \boldsymbol{0}, \quad \boldsymbol{c}^{-} \geq \boldsymbol{0}, && \text{(plastic elongation)}\\
&(\boldsymbol{q}^{\mathrm{y}} - \boldsymbol{q})^{\top}\boldsymbol{c}^{+} = 0, \quad (\boldsymbol{q}^{\mathrm{y}} + \boldsymbol{q})^{\top}\boldsymbol{c}^{-} = 0. && \text{(complementarity)}
\end{aligned} \tag{22}$$

From (21) and (22) it also follows that the dual to (5) can be formulated in the variables $\boldsymbol{u} \in \mathbb{R}^{n^{\mathrm{d}}}$, $\boldsymbol{c}^{+} \in \mathbb{R}^{n^{\mathrm{m}}}$, and $\boldsymbol{c}^{-} \in \mathbb{R}^{n^{\mathrm{m}}}$ as

$$\min\left\{-\boldsymbol{f}_{\mathrm{D}}^{\top}\boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top}(\boldsymbol{c}^{+} + \boldsymbol{c}^{-}) : \boldsymbol{f}_{\mathrm{R}}^{\top}\boldsymbol{u} = 1, \ \ \boldsymbol{c}^{+} - \boldsymbol{c}^{-} = \boldsymbol{B}^{\top}\boldsymbol{u}, \ \ \boldsymbol{c}^{+} \geq \boldsymbol{0}, \ \ \boldsymbol{c}^{-} \geq \boldsymbol{0}\right\}, \tag{23}$$

which coincides with the conventional formulation of upper-bound principle [Muralidhar and Jagannatha Rao 1997]. However, the number of variables in (23) is larger than that of (15), which may imply an advantage of (15) over (23).

For $\alpha \in \mathbb{R}_{+}$, consider the following nonconvex problem in the variables $(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\zeta}) \in \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^{n^{\mathrm{m}}} \times \mathbb{R}^{m}$:

$$v_{\min}(\alpha) := \min_{\boldsymbol{u},\boldsymbol{z},\boldsymbol{\zeta}} \left\{-(\widetilde{\boldsymbol{f}}_{\mathrm{D}} + \boldsymbol{T}\boldsymbol{\zeta})^{\top}\boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top}\boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U}, \ \alpha \geq \|\boldsymbol{\zeta}\|_{\infty}\right\}. \tag{24}$$

The following proposition shows that (24) corresponds to the kinematic version of the worst-case limit analysis (13):

Proposition 3.7 (Relation between problems (13) and (24)). Let $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}, \overline{\boldsymbol{\zeta}})$ denote an optimal solution of Problem (24). Then,

(i) $v_{\min}(\alpha) = \lambda_{\min}(\alpha)$;

(ii) $\overline{\boldsymbol{\zeta}}$ is an optimal solution of (13);

(iii) $\overline{\boldsymbol{u}}$ corresponds to a collapse mode associated with the external dead load $\boldsymbol{f}_{\mathrm{D}}(\overline{\boldsymbol{\zeta}})$;

(iv) $\overline{z}_i$ corresponds to the member elongation compatible with $\overline{\boldsymbol{u}}$.

Proof. By using the definition (11) of λ^{*}, (13) can be rewritten equivalently as

$$\min_{\boldsymbol{\zeta}}\left\{\max_{\lambda,\boldsymbol{q}}\left\{\lambda : (\lambda, \boldsymbol{q}) \in \mathcal{Q}(\boldsymbol{f}_{\mathrm{D}}(\boldsymbol{\zeta}))\right\} : \alpha \geq \|\boldsymbol{\zeta}\|_{\infty}\right\} \tag{25}$$

without changing the optimal value. Let $(\widehat{\boldsymbol{\zeta}}, \widehat{\lambda}, \widehat{\boldsymbol{q}})$ denote an optimal solution of (25). It is obvious that $\widehat{\boldsymbol{\zeta}}$ is an optimal solution of (13), and that $\widehat{\lambda} = \lambda_{\min}(\alpha)$. Since the inner problem of (25) coincides with the static principle (13), $\widehat{\boldsymbol{q}}$ corresponds to the vector of axial forces at the collapse mode. By using Proposition 3.5, we can rewrite the inner problem of (25) as

$$\min_{\boldsymbol{\zeta}}\left\{\min_{\boldsymbol{u},\boldsymbol{z}}\left\{-\boldsymbol{f}_{\mathrm{D}}(\boldsymbol{\zeta})^{\top}\boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top}\boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U}\right\} : \alpha \geq \|\boldsymbol{\zeta}\|_{\infty}\right\} \tag{26}$$

without changing the optimal value. Obviously, $\widehat{\boldsymbol{\zeta}}$ is an optimal solution of (25) if and only if it is an optimal solution of (26). Moreover, Proposition 3.5 guarantees that, at an optimal solution of (26), $\boldsymbol{u}$ and z_i, respectively, coincide with the collapse mode and the member elongation corresponding to $\widehat{\boldsymbol{q}}$. From (6), we see that Problems (24) and (26) share the same optimal value and same optimal solutions, which concludes the proof. □

Proposition 3.7 justifies solving (24) instead of the bilevel optimization problem (13), that is, the worst-case limit load factor is obtained as the optimal value of (24). The critical load and the corresponding collapse mode can be obtained simultaneously as the optimal variables of (24). Note that (24) is a nonconvex (but single-level) problem, since the objective function includes the nonconvex quadratic term $\boldsymbol{\zeta}^{\top} \boldsymbol{T}^{\top} \boldsymbol{u}$. Hence, the conventional nonlinear programming approach converges to a local optimal or stationary solution in general. It should be emphasized that, for the purpose of the robustness analysis, the proof of global optimum of (24) is strongly desired, since it guarantees that the limit load factor cannot be smaller than the obtained optimal objective value. This is the major difficulty of the worst-case limit analysis. To overcome this difficulty, in the following section we propose an algorithm that converges to a global optimal solution of (24).

4. Global optimization for uncertain limit analysis

In this section, we propose an algorithm to find a global optimal solution of (13) based on enumeration.

4.1. ***Mixed 0-1 programming formulation.*** We start by reformulating (13) as a mixed 0-1 programming problem. Letting

$$\mathscr{C}^0 := \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^{n^{\mathrm{m}}} \times \mathbb{R}^m \times \mathbb{R}^m,$$

we define a set $\mathscr{K} \subseteq \mathscr{C}^0$ as

$$\mathscr{K} = \left\{ (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathscr{C}^0 \;\middle|\; \begin{array}{l} \boldsymbol{f}_{\mathrm{R}}^{\top} \boldsymbol{u} = 1, \\ \boldsymbol{z} - \boldsymbol{B}^{\top} \boldsymbol{u} \geq \boldsymbol{0}, \quad \boldsymbol{z} + \boldsymbol{B}^{\top} \boldsymbol{u} \geq \boldsymbol{0}, \\ \boldsymbol{\gamma} - \boldsymbol{T}^{\top} \boldsymbol{u} \leq M(\boldsymbol{1} - \boldsymbol{\tau}), \quad \boldsymbol{\gamma} + \boldsymbol{T}^{\top} \boldsymbol{u} \leq M \boldsymbol{\tau}, \\ \boldsymbol{0} \leq \boldsymbol{\tau} \leq \boldsymbol{1} \end{array} \right\}, \tag{27}$$

where $M \in \mathbb{R}_+$ is a sufficiently large constant. Let

$$\mathscr{K}^{\mathbb{Z}} = \left\{ (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathscr{K} \;\middle|\; \boldsymbol{\tau} \in \{0, 1\}^m \right\}. \tag{28}$$

Consider the following optimization problem in the variables $(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathscr{C}^0$:

$$\min \left\{ -\alpha \boldsymbol{1}^{\top} \boldsymbol{\gamma} - \widetilde{\boldsymbol{f}}_{\mathrm{D}}^{\top} \boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top} \boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathscr{K}^{\mathbb{Z}} \right\}. \tag{29}$$

We refer to (29) as the mixed 0-1 programming problem. It has binary constraints on $\boldsymbol{\tau}$, linear inequality constraints, and a linear objective function.

Proposition 4.1 (Relation between problems (24) and (29)). A feasible solution $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}, \overline{\boldsymbol{\zeta}})$ of (24) satisfying

$$\overline{\boldsymbol{\zeta}}^{\top} \boldsymbol{T}^{\top} \overline{\boldsymbol{u}} = \alpha \| \boldsymbol{T}^{\top} \overline{\boldsymbol{u}} \|_1 \tag{30}$$

is optimal if and only if a feasible solution $(\overline{\boldsymbol{u}}, \overline{z}, \overline{\boldsymbol{\gamma}}, \overline{\boldsymbol{\tau}})$ of Problem (29) satisfying

$$\overline{\gamma}_j = |\boldsymbol{t}_j^\top \overline{\boldsymbol{u}}|, \quad j = 1, \dots, m, \tag{31}$$

$$\begin{cases} \overline{\tau}_j = 1, & \text{if } \boldsymbol{t}_j^\top \overline{\boldsymbol{u}} > 0, \\ \overline{\tau}_j = 0, & \text{if } \boldsymbol{t}_j^\top \overline{\boldsymbol{u}} < 0, \\ \overline{\tau}_j \in \{0, 1\}, & \text{if } \boldsymbol{t}_j^\top \overline{\boldsymbol{u}} = 0, \end{cases} \tag{32}$$

is optimal. Moreover, Problems (24) and (29) share the same optimal value, that is, equal to $\lambda_{\min}(\alpha)$.

Proof. Observe that, in (24), only $\alpha \geq \|\boldsymbol{\zeta}\|_\infty$ is the constraint on $\boldsymbol{\zeta}$, which is independent of the remaining variables $\boldsymbol{z}$ and $\boldsymbol{u}$. Hence, (24) is equivalently rewritten as

$$\min_{\boldsymbol{u}, \boldsymbol{z}} \left\{ \min_{\boldsymbol{\zeta}} \left\{ -(\widetilde{\boldsymbol{f}}_{\mathrm{D}} + \boldsymbol{T}\boldsymbol{\zeta})^\top \boldsymbol{u} : \alpha \geq \|\boldsymbol{\zeta}\|_\infty \right\} + \boldsymbol{q}^{\mathrm{y}\top} \boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U} \right\} \tag{33}$$

without changing the optimal value and optimal solution. From the Hölder inequality [Marti and Stoeckel 2004, Chap. 9], we see that

$$(\boldsymbol{T}\boldsymbol{\zeta})^\top \boldsymbol{u} \geq \|\boldsymbol{\zeta}\|_\infty \|\boldsymbol{T}^\top \boldsymbol{u}\|_1 \tag{34}$$

holds for any fixed $\boldsymbol{u}$. Moreover, Assumption 3.1 (i) guarantees that there exists a $\boldsymbol{\zeta}$ satisfying

$$(\boldsymbol{T}\boldsymbol{\zeta})^\top \boldsymbol{u} = \|\boldsymbol{\zeta}\|_\infty \|\boldsymbol{T}^\top \boldsymbol{u}\|_1. \tag{35}$$

From (34) and (35), we obtain

$$\min_{\boldsymbol{\zeta}} \left\{ -(\boldsymbol{T}\boldsymbol{\zeta})^\top \boldsymbol{u} : \alpha \geq \|\boldsymbol{\zeta}\|_\infty \right\} = \min_{\boldsymbol{\zeta}} \left\{ -\|\boldsymbol{\zeta}\|_\infty \|\boldsymbol{T}^\top \boldsymbol{u}\|_1 : \alpha \geq \|\boldsymbol{\zeta}\|_\infty \right\} = -\alpha \|\boldsymbol{T}^\top \boldsymbol{u}\|_1,$$

where an optimal $\boldsymbol{\zeta}$ satisfies (35). Consequently, the variable $\boldsymbol{\zeta}$ can be eliminated from (33) as

$$\min_{\boldsymbol{u}, \boldsymbol{z}} \left\{ -\alpha \|\boldsymbol{T}^\top \boldsymbol{u}\|_1 - \widetilde{\boldsymbol{f}}_{\mathrm{D}}^\top \boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top} \boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U} \right\}. \tag{36}$$

Note that an optimal solution $(\overline{\boldsymbol{u}}, \overline{z})$ of (36) can be converted to an optimal solution $(\overline{\boldsymbol{u}}, \overline{z}, \overline{\boldsymbol{\zeta}})$ of (33) by defining $\overline{\boldsymbol{\zeta}}$ as in (30), and these two problems share the same objective value. By introducing new variables $\boldsymbol{\gamma} \in \mathbb{R}^m$, (36) is equivalently rewritten as

$$\min \left\{ -\alpha \mathbf{1}^\top \boldsymbol{\gamma} - \widetilde{\boldsymbol{f}}_{\mathrm{D}}^\top \boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top} \boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U}, \quad (\gamma_j = \boldsymbol{t}_j^\top \boldsymbol{u}) \vee (\gamma_j = -\boldsymbol{t}_j^\top \boldsymbol{u}), \quad j = 1, \dots, m \right\}. \tag{37}$$

where $\vee$ denotes logical ‘or’. Note that (31) holds at an optimal solution of (37). By using a sufficiently large constant M, the disjunction

$$\left(\gamma_j \leq \boldsymbol{t}_j^\top \boldsymbol{u}\right) \vee \left(\gamma_j \leq -\boldsymbol{t}_j^\top \boldsymbol{u}\right)$$

is equivalently rewritten as

$$\gamma_j \leq \boldsymbol{t}_j^\top \boldsymbol{u} + M(1 - \tau_j), \quad \gamma_j \leq -\boldsymbol{t}_j^\top \boldsymbol{u} + M\tau_j, \quad \tau_j \in \{0, 1\},$$

with the relation (32), which completes the proof. □

4.2. ***Branch-and-bound method for problem* (29).** The LP relaxation of the mixed 0-1 programming problem (29) is obtained by ignoring the binary constraints on $\boldsymbol{\tau}$ as

$$\min \left\{ -\alpha \mathbf{1}^\top \boldsymbol{\gamma} - \tilde{\boldsymbol{f}}_{\mathrm{D}}^\top \boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top} \boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathcal{K} \right\}. \tag{38}$$

Define a set $\mathcal{C}$ as

$$\mathcal{C} = \left\{ (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathcal{C}^0 \;\middle|\; \boldsymbol{A}_u^\top \boldsymbol{u} + \boldsymbol{A}_z^\top \boldsymbol{z} + \boldsymbol{A}_\gamma^\top \boldsymbol{\gamma} + \boldsymbol{A}_\tau^\top \boldsymbol{\tau} \geq \boldsymbol{b} \right\}, \tag{39}$$

where $\boldsymbol{A}_u \in \mathbb{R}^{n^{\mathrm{d}} \times n^{\mathrm{c}}}$, $\boldsymbol{A}_z \in \mathbb{R}^{n^{\mathrm{m}} \times n^{\mathrm{c}}}$, $\boldsymbol{A}_\gamma \in \mathbb{R}^{m \times n^{\mathrm{c}}}$, $\boldsymbol{A}_\tau \in \mathbb{R}^{m \times n^{\mathrm{c}}}$, and $\boldsymbol{b} \in \mathbb{R}^{n^{\mathrm{c}}}$ are constant matrices and a constant vector. Assume that $\mathcal{C}$ satisfies

$$\mathrm{cl\ conv}\, \mathcal{K} \subseteq \mathcal{C} \subseteq \mathcal{C}^0. \tag{40}$$

Note that in this section we set $\mathcal{C} := \mathcal{C}^0$, while in Section 5 we discuss how to generate a proper subset $\mathcal{C}$ of $\mathcal{C}^0$.

Let $\mathcal{J}_0^k$ and $\mathcal{J}_1^k$ denote the subsets of indices satisfying

$$\mathcal{J}_0^k \subseteq \{1, \dots, m\}, \qquad \mathcal{J}_1^k \subseteq \{1, \dots, m\}, \qquad \mathcal{J}_0^k \cap \mathcal{J}_1^k = \varnothing.$$

Let

$$\mathcal{K}(\mathcal{C}, \mathcal{J}_0^k, \mathcal{J}_1^k) = \left\{ (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathcal{K} \cap \mathcal{C} \;\middle|\; \tau_j = 0 \text{ for } j \in \mathcal{J}_0^k,\ \tau_j = 1 \text{ for } j \in \mathcal{J}_1^k \right\},$$

where $\mathcal{K}$ and $\mathcal{C}$ have been defined in (27) and (39). Consider the following LP problem in the variables $(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathcal{C}^0$:

$$\mathrm{LP}(\mathcal{C}, \mathcal{J}_0^k, \mathcal{J}_1^k): \quad v^k := \min \left\{ -\alpha \mathbf{1}^\top \boldsymbol{\gamma} - \tilde{\boldsymbol{f}}_{\mathrm{D}}^\top \boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top} \boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathcal{K}(\mathcal{C}, \mathcal{J}_0^k, \mathcal{J}_1^k) \right\}. \tag{41}$$

Explicitly, (41) is written as

$$\begin{aligned}
\min \quad & \{ -\alpha \mathbf{1}^\top \boldsymbol{\gamma} - \tilde{\boldsymbol{f}}_{\mathrm{D}}^\top \boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top} \boldsymbol{z} : && \text{(42a)} \\
& \boldsymbol{f}_{\mathrm{R}}^\top \boldsymbol{u} = 1, && \text{(42b)} \\
& \boldsymbol{z} - \boldsymbol{B}^\top \boldsymbol{u} \geq \mathbf{0}, && \text{(42c)} \\
& \boldsymbol{z} + \boldsymbol{B}^\top \boldsymbol{u} \geq \mathbf{0}, && \text{(42d)} \\
& \boldsymbol{\gamma} - \boldsymbol{T}^\top \boldsymbol{u} \leq M(\mathbf{1} - \boldsymbol{\tau}), && \text{(42e)} \\
& \boldsymbol{\gamma} + \boldsymbol{T}^\top \boldsymbol{u} \leq M \boldsymbol{\tau}, && \text{(42f)} \\
& \mathbf{0} \leq \boldsymbol{\tau} \leq \mathbf{1}, && \text{(42g)} \\
& \boldsymbol{A}_u^\top \boldsymbol{u} + \boldsymbol{A}_z^\top \boldsymbol{z} + \boldsymbol{A}_\gamma^\top \boldsymbol{\gamma} + \boldsymbol{A}_\tau^\top \boldsymbol{\tau} \geq \boldsymbol{b}, && \text{(42h)} \\
& \tau_j = 0 \text{ for } j \in \mathcal{J}_0^k, \quad \tau_j = 1 \text{ for } j \in \mathcal{J}_1^k \}. && \text{(42i)}
\end{aligned}$$

We solve $\mathrm{LP}(\mathcal{C}, \mathcal{J}_0^k, \mathcal{J}_1^k)$ at the nodes of enumeration tree. Note that $\mathrm{LP}(\mathcal{C}^0, \varnothing, \varnothing)$ coincides with the LP relaxation (38).

The following is a branch-and-bound method for solving the mixed 0-1 programming problem (29) based on the LP relaxation.

Algorithm 4.2. Branch-and-bound algorithm for (29).

Step 0: Initialization. Set $k=0$, $\mathcal{J}_1^0=\varnothing$, $\mathcal{J}_0^0=\varnothing$, and $v^{\mathrm{U}}=\infty$. Choose the small tolerance $\epsilon>0$ and $\mathscr{C}$ satisfying (40) (set $\mathscr{C}:=\mathscr{C}^0$ in this section).

Step 1: Solving the subproblem. Solve the linear program LP($\mathscr{C}$, $\mathcal{J}_0^k$, $\mathcal{J}_1^k$) defined in (41). If the problem is infeasible, go to Step 5; otherwise, let $(\boldsymbol{u}^k, \boldsymbol{z}^k, \boldsymbol{\gamma}^k, \boldsymbol{\tau}^k)$ and v^k denote its optimal solution and optimal objective value, respectively.

Step 2: Fathoming. If $v^k \geq v^{\mathrm{U}}$, go to Step 5.

Step 3: Branching. If $(\boldsymbol{\tau}^k)^\top(1-\boldsymbol{\tau}^k)\leq\epsilon$, go to Step 4; otherwise, select an index j_1 such that

$$j_1=\arg\max_{j\in\{1,\dots,m\}}\left\{\tau_j^k(1-\tau_j^k)\right\}.$$

Set $\mathcal{J}_1^{k+1}:=\mathcal{J}_1^k\cup\{j_1\}$ and $\boldsymbol{p}^{k+1}:=(\boldsymbol{p}^k, j_1)^\top$. Update $k\leftarrow k+1$, and go to Step 1.

Step 4: Updating. Put $v^{\mathrm{U}}:=v^k$ and $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}, \overline{\boldsymbol{\gamma}}, \overline{\boldsymbol{\tau}}):=(\boldsymbol{u}^k, \boldsymbol{z}^k, \boldsymbol{\gamma}^k, \boldsymbol{\tau}^k)$. Go to Step 5.

Step 5: Backtracking. If $\boldsymbol{p}^k<\boldsymbol{0}$, go to Step 6; otherwise branch to a new live node as follows:

Define $l_1=\max\left\{l\in\{1,\dots,L\}\,\middle|\,p_l^k>0\right\}$, where L denotes the size of the vector $\boldsymbol{p}^k$. Divide $\boldsymbol{p}^k$ into the three parts

$$\boldsymbol{p}_1=\left(p_l^k\,\middle|\,l=1,\dots,l_1-1\right),\quad p_2=p_{l_1}^k,\quad \mathcal{P}_3=\left\{-p_l^k\,\middle|\,l=l_1+1,\dots,L\right\}.$$

Set

$$\begin{aligned}
\boldsymbol{p}^{k+1}&:=\left(\boldsymbol{p}_1^\top, -p_2\right)^\top,\\
\mathcal{J}_0^{k+1}&:=\left\{\mathcal{J}_0^k\cup\{p_2\}\right\}\setminus\mathcal{P}_3,\\
\mathcal{J}_1^{k+1}&:=\mathcal{J}_1^k\setminus\{p_2\}.
\end{aligned}$$

Update $k\leftarrow k+1$, and go to Step 1.

Step 6: Termination. Declare $(\overline{\boldsymbol{z}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{\gamma}}, \overline{\boldsymbol{\tau}})$ as the optimal solution, and stop.

Remark 4.3. Essentially, Algorithm 4.2 is designed by using depth-first search (see, for example [Clausen and Perregaard 1999]) as a strategy for selecting the next live subproblem at Step 5. The condition $\boldsymbol{p}^k<\boldsymbol{0}$ implies that there exists no live node. Among the live subproblems, we always select the subproblem with the largest level in the branch-and-bound tree. The vector $\boldsymbol{p}^k$ plays the role of bookkeeping the path from the root node to the current node in the branch-and-bound tree. The size L of $\boldsymbol{p}^k=(p_j^k)$ coincides with the current depth of the tree, and we see that the following relations hold:

$$\mathcal{J}_0^k=\left\{p_j^k\,\middle|\,p_j^k\leq 0,\ \ j=1,\dots,L\right\},$$

$$\mathcal{J}_1^k=\left\{-p_j^k\,\middle|\,p_j^k\geq 0,\ \ j=1,\dots,L\right\},$$

that is, the components of $\boldsymbol{p}^k$ correspond to the indices of τ_j, possibly with opposite signs, which are fixed in the current subproblem LP($\mathscr{C}$, $\mathcal{J}_0^k$, $\mathcal{J}_1^k$). The remaining τ_j are not fixed in LP($\mathscr{C}$, $\mathcal{J}_0^k$, $\mathcal{J}_1^k$). The order of an element p_j^k of $\boldsymbol{p}^k$ is determined by its level in the tree.

Remark 4.4. Observe that the binary constraints $\boldsymbol{\tau} \in \{0, 1\}^m$ are equivalent to the following complementarity conditions:

$$\boldsymbol{\tau} \geq \mathbf{0}, \quad \mathbf{1} - \boldsymbol{\tau} \geq \mathbf{0}, \tag{43}$$

$$\tau_j (1 - \tau_j) = 0, \quad j = 1, \dots, m. \tag{44}$$

Notice here that any feasible solution of LP($\mathscr{C}, \mathscr{J}_0^k, \mathscr{J}_1^k$) satisfies (43). At Step 3, we make a check if the current solution $(\boldsymbol{u}^k, \boldsymbol{z}^k, \boldsymbol{\gamma}^k, \boldsymbol{\tau}^k)$ satisfies the complementarity conditions (44) or not. Satisfaction (possibly with small tolerance in practice) implies that $(\boldsymbol{u}^k, \boldsymbol{z}^k, \boldsymbol{\gamma}^k, \boldsymbol{\tau}^k)$ is a feasible solution of (29). Alternatively, if (44) is not satisfied, then the variable τ_j with the largest residual of the complementarity (44) is used as the branching variable in Step 3.

Remark 4.5. Note that it is not difficult to randomly generate $\boldsymbol{f}'_{\mathrm{D}}$ satisfying $\boldsymbol{f}'_{\mathrm{D}} \in \mathscr{F}_{\mathrm{D}}(\alpha)$. Then the corresponding limit load factor $\lambda^*(\boldsymbol{f}'_{\mathrm{D}})$ provides an upper bound of the mixed 0-1 (29). At Step 0, we can obtain an upper bound v^{U} by solving (11) several times for randomly sampled $\boldsymbol{f}'_{\mathrm{D}}$. We simply set $v^{\mathrm{U}} = \infty$ if this process is skipped.

Remark 4.6. In this paper, we focus on finding the global optimal solution of (13). Alternatively, for very large structures, it is also important to develop an efficient algorithm for computing a lower bound of $\lambda_{\min}(\alpha)$. For this purpose, it may be interesting to investigate a relaxation of (13) based on the concept of uncertain LP, as presented in [Ben-Tal et al. 2004].

4.3. ***Duality and simplification.*** The remainder of this section is devoted to some practical issues regarding implementation of Algorithm 4.2. In fact, to obtain $(\boldsymbol{u}^k, \boldsymbol{z}^k, \boldsymbol{\gamma}^k, \boldsymbol{\tau}^k)$ at Step 1, we do not solve (41) directly but use the simplex method to solve its Lagrangian dual problem, denoted by LP*($\mathscr{C}, \mathscr{J}_0^k, \mathscr{J}_1^k$). Then we obtain the solution of (41) as the optimal Lagrange multipliers. From preliminary numerical experiments, we observed that the CPU time required to solve the dual problem is much smaller than that required to solve the original (41). Indeed, after the branching process of Step 3, at the new node it is easy to obtain a feasible solution of the dual problem from an optimal solution of the dual problem solved at the previous node. Let

$$\mathscr{C}^{0*} = \mathbb{R} \times \mathbb{R}^{n^{\mathrm{m}}} \times \mathbb{R}^{n^{\mathrm{m}}} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m.$$

From the LP duality [Chvátal 1983] it follows that the dual problem of the LP relaxation LP($\mathscr{C}, \varnothing, \varnothing$) is formulated in the variables $(\rho^\lambda, \boldsymbol{q}^+, \boldsymbol{q}^-, \boldsymbol{\zeta}^+, \boldsymbol{\zeta}^-, \boldsymbol{\rho}^+, \boldsymbol{\rho}^-, \boldsymbol{\mu}) \in \mathscr{C}^{0*} \times \mathbb{R}^{n^{\mathrm{c}}}$ as

$$\begin{aligned}
\mathrm{LP}^*(\mathscr{C}, \varnothing, \varnothing): \quad \max \quad & \big\{\rho^\lambda - \mathbf{1}^\top (M\boldsymbol{\rho}^+ + \boldsymbol{\rho}^-) + \boldsymbol{b}^\top \boldsymbol{\mu} : && (45a)\\
& \boldsymbol{B}(\boldsymbol{q}^+ - \boldsymbol{q}^-) = \tilde{\boldsymbol{f}}_{\mathrm{D}} + \boldsymbol{T}(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) + \boldsymbol{f}_{\mathrm{R}}\rho^\lambda + \boldsymbol{A}_u \boldsymbol{\mu}, && (45b)\\
& \boldsymbol{q}^+ + \boldsymbol{q}^- + \boldsymbol{A}_z \boldsymbol{\mu} = \boldsymbol{q}^{\mathrm{y}}, && (45c)\\
& \boldsymbol{\zeta}^+ + \boldsymbol{\zeta}^- = \alpha \mathbf{1} + \boldsymbol{A}_\gamma \boldsymbol{\mu}, && (45d)\\
& M(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) = \boldsymbol{\rho}^+ - \boldsymbol{\rho}^- + \boldsymbol{A}_\tau \boldsymbol{\mu}, && (45e)\\
& \boldsymbol{q}^+, \boldsymbol{q}^-, \boldsymbol{\zeta}^+, \boldsymbol{\zeta}^-, \boldsymbol{\rho}^+, \boldsymbol{\rho}^-, \boldsymbol{\mu} \geq \mathbf{0}\big\}. && (45f)
\end{aligned}$$

Let LP*($\mathscr{C}, \mathscr{J}_0^k, \mathscr{J}_1^k$) denote the dual of LP($\mathscr{C}, \mathscr{J}_0^k, \mathscr{J}_1^k$) in which some variables are fixed. To obtain LP*($\mathscr{C}, \mathscr{J}_0^k, \mathscr{J}_1^k$), we modify and simplify (45) as follows:

(a) For $j \in \mathcal{J}_0^k$, the variable τ_j in the primal problem (42) is set as $\tau_j = 0$. This is realized in the dual problem as follows:

(i) The j-th row of (42f) should be rewritten as

$$-\boldsymbol{t}_j^\top \boldsymbol{u} - \gamma_j \geq 0.$$

Consequently, the variable ζ_j^- should be eliminated from the j-th row of (45e).

(ii) The j-th row of (42e) becomes redundant. Hence, the variable ζ_j^+ can be eliminated from (45).

(iii) The j-th row of (42g) becomes redundant. Hence, the variables ρ_j^+ and ρ_j^- can be eliminated from (45).

(iv) If the j-th row vector of $\boldsymbol{A}_\tau$ is a zero vector, then the constraint (45e) itself can be eliminated.

(b) For $j \in \mathcal{J}_1^k$, the variable τ_j in the primal problem (42) is set as $\tau_j = 1$. This is realized in the dual problem as follows:

(i) The j-th row of (42e) should be rewritten as

$$\boldsymbol{t}_j^\top \boldsymbol{u} - \gamma_j \geq 0.$$

Consequently, the variable ζ_j^+ should be eliminated from the j-th row of (45e).

(ii) The j-th row of (42f) becomes redundant. Hence, the variable ζ_j^- can be eliminated from (45).

(iii) The j-th row of (42g) becomes redundant. Hence, the variables ρ_j^+ and ρ_j^- can be eliminated from (45).

(iv) If the j-th row vector of $\boldsymbol{A}_\tau$ is a zero vector, then the constraint (45e) itself can be eliminated.

Note that the LP (45) originally has $(2n^{\mathrm{m}} + 4m + n^{\mathrm{c}} + 1)$ variables and $(n^{\mathrm{d}} + n^{\mathrm{m}} + 2m)$ linear equality constraints besides side constraints. With the simplification proposed in this section, we can reduce the number of variables and constraints when some binary variables τ_j are fixed in (41).

5. Cutting plane algorithm

It is guaranteed that Algorithm 4.2 converges to a global optimal solution of the mixed 0-1 programming problem (29). However, it is possible that the algorithm is no better than the enumeration of all binary variables $\boldsymbol{\tau}$. The efficiency of the algorithm depends partially on the tightness of the LP relaxation problem solved at each node of the branch-and-bound tree.

5.1. ***Disjunctive cut generation.*** To strengthen the LP relaxation problems, we now propose an algorithm that generates the disjunctive cutting planes. Recall that $\mathcal{K}^{\mathbb{Z}}$ and $\mathcal{K}$, defined in (28) and (27), correspond to the feasible sets of the mixed 0-1 programming problem (29) and its LP relaxation (38), respectively. Let $(\widehat{\boldsymbol{u}}, \widehat{z}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})$ denote the optimal solution of the LP relaxation (38). Suppose that $\widehat{\boldsymbol{\tau}}$ does not satisfy the binary constraints in $\mathcal{K}^{\mathbb{Z}}$, that is,

$$(\widehat{\boldsymbol{u}}, \widehat{z}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}}) \notin \mathcal{K}^{\mathbb{Z}}.$$

The cutting plane, then, is an additional linear inequality that the point $(\widehat{\boldsymbol{u}}, \widehat{z}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})$ does not satisfy, but is valid for $\mathcal{K}^{\mathbb{Z}}$. If a cutting plane is generated successfully, we can add it to the LP relaxation as the constraint without cutting off any feasible solution in $\mathcal{K}^{\mathbb{Z}}$. If the new optimal solution of the obtained LP problem is feasible for $\mathcal{K}^{\mathbb{Z}}$, it is a global optimal solution of the original mixed 0-1 program problem (29); otherwise, we may continue to generate cutting planes.

In the following, the cutting plane generation is performed over the so-called *disjunctive programming* relaxation of $\mathcal{K}^{\mathbb{Z}}$ instead of $\mathcal{K}^{\mathbb{Z}}$ itself. A valid inequality obtained in this way is called the disjunctive cut [Ceria and Soares 1997]. We define the sets

$$P_j(\mathcal{K}) = \mathrm{cl}\ \mathrm{conv}\left\{(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathcal{K} \,\middle|\, \tau_j \in \{0, 1\}\right\}, \ j = 1, \dots, m \ ,$$

each of which is a disjunctive programming relaxation of the closure of conv $\mathcal{K}^{\mathbb{Z}}$. We attempt to find a linear inequality that cuts off $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})$ but is valid for $P_j(\mathcal{K})$. Although the characterization of $P_j(\mathcal{K})$ is essentially nonlinear, a polyhedral representation can be obtained easily [Balas et al. 1996; Balas and Perregaard 2002], that is, the condition

$$(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in P_j(\mathcal{K})$$

is satisfied if and only if there exist

$$(\boldsymbol{w}^u, \boldsymbol{w}^z, \boldsymbol{w}^\gamma, \boldsymbol{w}^\tau) \in \mathscr{C}^0, \quad w_0 \in \mathbb{R},$$
$$(\boldsymbol{y}^u, \boldsymbol{y}^z, \boldsymbol{y}^\gamma, \boldsymbol{y}^\tau) \in \mathscr{C}^0, \quad y_0 \in \mathbb{R}$$

satisfying

$$(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) = (\boldsymbol{w}^u, \boldsymbol{w}^z, \boldsymbol{w}^\gamma, \boldsymbol{w}^\tau) + (\boldsymbol{y}^u, \boldsymbol{y}^z, \boldsymbol{y}^\gamma, \boldsymbol{y}^\tau), \tag{46a}$$
$$\boldsymbol{f}_{\mathrm{R}}^\top \boldsymbol{w}^u = w_0, \tag{46b}$$
$$\boldsymbol{w}^z - \boldsymbol{B}^\top \boldsymbol{w}^u \geq \boldsymbol{0}, \quad \boldsymbol{w}^z + \boldsymbol{B}^\top \boldsymbol{w}^u \geq \boldsymbol{0}, \tag{46c}$$
$$\boldsymbol{T}^\top \boldsymbol{w}^u - \boldsymbol{w}^\gamma - M\boldsymbol{w}^\tau \geq -M w_0 \boldsymbol{1}, \quad -\boldsymbol{T}^\top \boldsymbol{w}^u - \boldsymbol{w}^\gamma + M\boldsymbol{w}^\tau \geq \boldsymbol{0}, \tag{46d}$$
$$\boldsymbol{0} \leq \boldsymbol{w}^\tau \leq w_0 \boldsymbol{1}, \tag{46e}$$
$$w_j^\tau \leq 0, \tag{46f}$$
$$\boldsymbol{f}_{\mathrm{R}}^\top \boldsymbol{y}^u = y_0, \tag{46g}$$
$$\boldsymbol{y}^z - \boldsymbol{B}^\top \boldsymbol{y}^u \geq \boldsymbol{0}, \quad \boldsymbol{y}^z + \boldsymbol{B}^\top \boldsymbol{y}^u \geq \boldsymbol{0}, \tag{46h}$$
$$\boldsymbol{T}^\top \boldsymbol{y}^u - \boldsymbol{y}^\gamma - M\boldsymbol{y}^\tau \geq -M y_0 \boldsymbol{1}, \quad -\boldsymbol{T}^\top \boldsymbol{y}^u - \boldsymbol{y}^\gamma + M\boldsymbol{y}^\tau \geq \boldsymbol{0}, \tag{46i}$$
$$\boldsymbol{0} \leq \boldsymbol{y}^\tau \leq y_0 \boldsymbol{1}, \tag{46j}$$
$$y_j^\tau \leq y_0, \tag{46k}$$
$$w_0 + y_0 = 1. \tag{46l}$$

We define a set $P_j^*(\mathcal{K}) \subseteq \mathscr{C}^0 \times \mathbb{R}$ so that

$$(\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau, \beta) \in P_j^*(\mathcal{K}) \tag{47}$$

holds if and only if there exist

$$(\xi^{\lambda}, \boldsymbol{\xi}^{q+}, \boldsymbol{\xi}^{q-}, \boldsymbol{\xi}^{\zeta+}, \boldsymbol{\xi}^{\zeta-}, \boldsymbol{\xi}^{\rho+}, \boldsymbol{\xi}^{\rho-}) \in \mathscr{C}^{0*}, \quad \xi^{0} \in \mathbb{R},$$
$$(\eta^{\lambda}, \boldsymbol{\eta}^{q+}, \boldsymbol{\eta}^{q-}, \boldsymbol{\eta}^{\zeta+}, \boldsymbol{\eta}^{\zeta-}, \boldsymbol{\eta}^{\rho+}, \boldsymbol{\eta}^{\rho-}) \in \mathscr{C}^{0*}, \quad \eta^{0} \in \mathbb{R}$$

satisfying

$$\boldsymbol{\alpha}_u = \boldsymbol{f}_{\mathrm{R}}\xi^{\lambda} - \boldsymbol{B}(\boldsymbol{\xi}^{q+} - \boldsymbol{\xi}^{q-}) + \boldsymbol{T}(\boldsymbol{\xi}^{\zeta+} - \boldsymbol{\xi}^{\zeta-}), \tag{48a}$$
$$\boldsymbol{\alpha}_z = \boldsymbol{\xi}^{q+} + \boldsymbol{\xi}^{q-}, \tag{48b}$$
$$\boldsymbol{\alpha}_\gamma = -\boldsymbol{\xi}^{\zeta+} - \boldsymbol{\xi}^{\zeta-}, \tag{48c}$$
$$\boldsymbol{\alpha}_\tau = -M(\boldsymbol{\xi}^{\zeta+} - \boldsymbol{\xi}^{\zeta-}) + \boldsymbol{\xi}^{\rho+} - \boldsymbol{\xi}^{\rho-} - \xi^{0}\boldsymbol{e}^{j}, \tag{48d}$$
$$\beta = \xi^{\lambda} - \mathbf{1}^{\top}(M\boldsymbol{\xi}^{\zeta+} + \boldsymbol{\xi}^{\rho-}), \tag{48e}$$
$$\boldsymbol{\xi}^{q+}, \boldsymbol{\xi}^{q-}, \boldsymbol{\xi}^{\zeta+}, \boldsymbol{\xi}^{\zeta-}, \boldsymbol{\xi}^{\rho+}, \boldsymbol{\xi}^{\rho-}, \xi^{0} \geq \mathbf{0}, \tag{48f}$$
$$\boldsymbol{\alpha}_u = \boldsymbol{f}_{\mathrm{R}}\eta^{\lambda} - \boldsymbol{B}(\boldsymbol{\eta}^{q+} - \boldsymbol{\eta}^{q-}) + \boldsymbol{T}(\boldsymbol{\eta}^{\zeta+} - \boldsymbol{\eta}^{\zeta-}), \tag{48g}$$
$$\boldsymbol{\alpha}_z = \boldsymbol{\eta}^{q+} + \boldsymbol{\eta}^{q-}, \tag{48h}$$
$$\boldsymbol{\alpha}_\gamma = -\boldsymbol{\eta}^{\zeta+} - \boldsymbol{\eta}^{\zeta-}, \tag{48i}$$
$$\boldsymbol{\alpha}_\tau = -M(\boldsymbol{\eta}^{\zeta+} - \boldsymbol{\eta}^{\zeta-}) + \boldsymbol{\eta}^{\rho+} - \boldsymbol{\eta}^{\rho-} + \eta^{0}\boldsymbol{e}^{j}, \tag{48j}$$
$$\beta = \eta^{\lambda} - \mathbf{1}^{\top}(M\boldsymbol{\eta}^{\zeta+} + \boldsymbol{\eta}^{\rho-}) + \eta^{0}, \tag{48k}$$
$$\boldsymbol{\eta}^{q+}, \boldsymbol{\eta}^{q-}, \boldsymbol{\eta}^{\zeta+}, \boldsymbol{\eta}^{\zeta-}, \boldsymbol{\eta}^{\rho+}, \boldsymbol{\eta}^{\rho-}, \eta^{0} \geq \mathbf{0}. \tag{48l}$$

Then, the following inequality is valid for $P_j(\mathscr{K})$ if it satisfies (47):

$$(\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau) \cdot (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \geq \beta. \tag{49}$$

Thus, for a point $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}}) \notin P_j(\mathscr{K})$, we are interested in the following problem in the variables $(\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau, \beta) \in \mathscr{C} \times \mathbb{R}$:

$$\max\bigl\{\beta - (\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau) \cdot (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}}) : (\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau, \beta) \in P_j^{*}(\mathscr{K})\bigr\}, \tag{50}$$

because a feasible solution of (50) defines a valid inequality (in the form of (49)) for $P_j(\mathscr{K})$, that is violated at $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})$.

However, since (50) itself is unbounded, some normalization constraints should be appended to it. We add constraints restricting the magnitude of the vector $(\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau)$ [Ceria and Soares 1997]. The index sets $\mathscr{I}$ and $\overline{\mathscr{I}}$ are defined as subsets of $\{1, \dots, n^{\mathrm{d}} + n^{\mathrm{m}} + 2m\}$ by

$$\overline{\mathscr{I}} = \left\{ i \in \{1, \dots, n^{\mathrm{d}} + n^{\mathrm{m}} + 2m\} \,\middle|\, (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})_i = 0 \right\},$$
$$\mathscr{I} = \{1, \dots, n^{\mathrm{d}} + n^{\mathrm{m}} + 2m\} \setminus \overline{\mathscr{I}}.$$

We assume $\mathcal{J} \neq \varnothing$. We then consider the cut generation problem

$$(\mathrm{CGLP})_j: \ \max \ \begin{aligned}[t] &\{\beta - (\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau) \cdot (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}}) : \\ &(\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau, \beta) \in P_j^*(\mathcal{K}), \\ &\|(\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau)_{\mathcal{J}}\|_\infty \leq 1\}, \end{aligned} \tag{51}$$

where $P_j^*(\mathcal{K})$ has been defined in (48). In (51), which is an LP problem, we attempt to find the deepest cut in the sense that a distance from $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})$ to a separating hyperplane is maximized. The dual to (51) is formulated as in [Ceria and Soares 1997] as

$$\min \ \begin{aligned}[t] &\{\|(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) - (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})\|_1 : \\ &(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in P_j(\mathcal{K}), \\ &(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau})_{\overline{\mathcal{J}}} = (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})_{\overline{\mathcal{J}}}\}, \end{aligned} \tag{52}$$

where $P_j(\mathcal{K})$ has been defined in (46). At the root node of the enumeration tree of Algorithm 4.2, we employ the following procedure for generating some disjunctive cuts.

Algorithm 5.1. Cut generation for problem (29).

Step 0: Set $\mathcal{C}^0 = \mathbb{R}^{n^{\mathrm{m}}} \times \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m \times \mathbb{R}^m$, $\mathcal{J}_{\mathrm{res}}^0 = \{1, \dots, m\}$, $\mathcal{J} = \{1, \dots, m\}$, and $k = 1$. Let $(\boldsymbol{u}^0, \boldsymbol{z}^0, \boldsymbol{\gamma}^0, \boldsymbol{\tau}^0)$ denote an optimal solution of $\mathrm{LP}(\mathcal{C}^0, \varnothing, \varnothing)$.

Step 1: Select $j_2 \in \mathcal{J}$ by

$$j_2 = \arg\max_{j \in \mathcal{J}} \left\{ \tau_j^{k-1}(1 - \tau_j^{k-1}) \right\}.$$

Step 2: Solve $(\mathrm{CGLP})_{j_2}$, with the definition (48) of $P_j^*(\mathcal{K})$, at

$$(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}}) = (\boldsymbol{u}^{k-1}, \boldsymbol{z}^{k-1}, \boldsymbol{\gamma}^{k-1}, \boldsymbol{\tau}^{k-1})$$

to find an optimal solution $(\boldsymbol{\alpha}_u^k, \boldsymbol{\alpha}_z^k, \boldsymbol{\alpha}_\gamma^k, \boldsymbol{\alpha}_\tau^k, \beta^k)$.

Step 3: Letting

$$\mathcal{C}_{\mathrm{cur}} := \left\{ (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathcal{C}^{k-1} \;\middle|\; (\boldsymbol{\alpha}_u^k, \boldsymbol{\alpha}_z^k, \boldsymbol{\alpha}_\gamma^k, \boldsymbol{\alpha}_\tau^k) \cdot (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \geq \beta^k \right\},$$

solve $\mathrm{LP}(\mathcal{C}_{\mathrm{cur}}, \varnothing, \varnothing)$ to find an optimal solution $(\boldsymbol{u}^k, \boldsymbol{z}^k, \boldsymbol{\gamma}^k, \boldsymbol{\tau}^k)$.

Step 4: Let

$$\mathcal{J}_{\mathrm{res}}^k = \left\{ j \in \{1, \dots, m\} \;\middle|\; \tau_j^k(1 - \tau_j^k) > \epsilon \right\}.$$

If $|\mathcal{J}_{\mathrm{res}}^k| \leq |\mathcal{J}_{\mathrm{res}}^{k-1}|$, let $\mathcal{C}^k := \mathcal{C}_{\mathrm{cur}}$ and $\mathcal{J} := \{1, \dots, m\}$; otherwise,

$$\mathcal{C}^k := \mathcal{C}^{k-1} \quad \text{and} \quad \mathcal{J} := \{1, \dots, m\} \setminus j_2.$$

Step 5: If the termination condition is satisfied, stop; otherwise, update $k \leftarrow k + 1$, and go to Step 1.

Remark 5.2. If $\mathcal{J}_{\mathrm{res}}^k = \varnothing$ at Step 4, then stop, because the current solution $(\boldsymbol{u}^k, \boldsymbol{z}^k, \boldsymbol{\gamma}^k, \boldsymbol{\tau}^k)$ is a global optimal solution of the original problem (29). However, it often requires large computational time to solve (29) only by Algorithm 5.1. In practice, we restrict the maximum number of iterations as $k \leq 1.8m$

and then employ Algorithm 4.2. The set of disjunctive cuts $\mathscr{C}^k$ generated by Algorithm 5.1 plays a role in strengthening the LP relaxation problems solved in Algorithm 4.2.

Remark 5.3. At Step 1, as is done in Step 3 of Algorithm 4.2, we select the variable τ_j with the largest residual of the complementarity condition (44). Then the variable τ_{j_2} is used at Step 2 to define the disjunctive constraint.

5.2. ***Simplification of cut generating LP.*** In this section, we further analyze simplifications to the cut generating LP problem (51), motivated by the fact that in the dual problem (52), we can eliminate some of variables corresponding to the index set $\overline{\mathcal{I}}$. Recall that $P_j(\mathcal{H})$ in the constraints of (52) has been defined in (46). Then we can consider the following points:

(a) For i such that $\widehat{z}_i = 0$:
Observe that $(\boldsymbol{\alpha}_z)_i$ does not contribute to the objective function of (51). By replacing the i-th rows of (48b) and (48h) with

$$\xi_i^{q+} + \xi_i^{q-} = \eta_i^{q+} + \eta_i^{q-}, \tag{53}$$

we can remove the variable $(\boldsymbol{\alpha}_z)_i$ from (48) without changing the optimal solution. At an optimal solution, we can complete $(\overline{\boldsymbol{\alpha}}_z)_i$ by letting

$$(\overline{\boldsymbol{\alpha}}_z)_i := \overline{\xi}_i^{q+} + \overline{\xi}_i^{q-}, \tag{54}$$

where $\overline{\xi}_i^{q+}$ and $\overline{\xi}_i^{q-}$ are components of the optimal solution obtained. However, instead of (53), we append more restrictive constraints

$$\xi_i^{q+} = \eta_i^{q-}, \quad \xi_i^{q-} = \eta_i^{q+}$$

to (48), which enables us to remove the variables η_i^{q+} and η_i^{q-}. Then an optimal solution of the simplified problem can be completed to an optimal solution of (51) by using (54).

(b) For i such that $\widehat{\tau}_i = 0$:
In the system of (46), observe that (46a), (46e), (46j), and $\widehat{\tau}_i = 0$ imply

$$w_i^{\tau} = y_i^{\tau} = 0. \tag{55}$$

  (i) Assume that there exists an $l \in \{1, \dots, m\}$ such that $\widehat{\tau}_l \neq 0$. Then, in the system (46), (55) and the l-th row of (46e) make the constraint

$$w_i^{\tau} \leq w_0 \tag{56}$$

  redundant. Similarly, it follows from the l-th row of (46j) and (55) that the constraint

$$y_i^{\tau} \leq y_0 \tag{57}$$

  is redundant. Then, we see that eliminating (56) and (57) from (46) is equivalent to eliminating the variables $\xi_i^{\rho-}$ and $\eta_i^{\rho-}$ from (48).

  (ii) In (46), it follows from (55) that the j-th rows of (46d) can be replaced with

$$\boldsymbol{t}_i^{\top} \boldsymbol{w}^u - w_i^{\gamma} \geq -M w_0, \quad -\boldsymbol{t}_i^{\top} \boldsymbol{w}^u - w_i^{\gamma} \geq 0$$

without changing $P_j(\mathcal{K})$. Then, in (48), the i-th row of (48d) is replaced with

$$(\boldsymbol{\alpha}^{\tau})_i = \xi_i^{\rho+} - \xi_0 e_i^j. \tag{58}$$

Similarly, the i-th row of (48j) is replaced with

$$(\boldsymbol{\alpha}^{\tau})_i = \eta_i^{\rho+} + \eta_0 e_i^j. \tag{59}$$

Note that, in (51), the variables $\xi_i^{\rho+}$ and $\eta_i^{\rho+}$ appear only in the constraints (58) and (59), respectively. Moreover, $(\boldsymbol{\alpha}_{\tau})_i$ does not contribute to the objective function. Consequently, from (51), we can eliminate the variables $(\boldsymbol{\alpha}_{\tau})_i$, $\xi_i^{\rho+}$, and $\eta_i^{\rho+}$ and the constraints (58) and (59). From the nonnegativity of $\xi_i^{\rho+}$ and $\eta_i^{\rho+}$ it follows that an optimal solution of the simplified problem can be completed to an optimal solution of (51) by defining the eliminated variables as

$$(\overline{\boldsymbol{\alpha}}_{\tau})_i := \max\left\{-\overline{\xi}_0 e_i^j,\ \overline{\eta}_0 e_i^j\right\}.$$

The LP problem (51) originally has $(n^{\mathrm{d}} + 5n^{\mathrm{m}} + 10m + 3)$ variables and $(2n^{\mathrm{d}} + 2n^{\mathrm{m}} + 4m + 2)$ linear equality constraints besides side constraints. However, we shall show in the numerical examples in Section 6.2 that the simplification proposed in this section greatly reduces the numbers both of variables and constraints. The size of the simplified problem depends on $(\widehat{\boldsymbol{u}}, \widehat{z}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})$, and hence differs at each iteration of Algorithm 5.1.

6. Numerical experiments

The worst-case limit load factors are computed for trusses by using Algorithms 4.2 and 5.1. Computation was carried out on a Pentium M (1.5 GHz with 1 GB memory) with MATLAB Version 6.5.1. The LP problems are solved by using the simplex method at Step 1 of Algorithm 4.2 and at Steps 2 and 3 of Algorithm 5.1. As an implementation of the simplex method, we use MATLAB built-in function `linprog` of Optimization Toolbox, Version 2.1 [MATLAB 2000], with the options '`LargeScale`' set to '`off`', and '`Simplex`' set to '`on`'.

In the following examples, the yield stress is $\sigma_i^{\mathrm{y}} = 400$ MPa and cross-sectional area is $a_i = 20.0\ \mathrm{cm}^2$ for each member. We set $M = 5.0$ in Algorithm 4.2 and Algorithm 5.1.

6.1. 3 × 3 *truss*. Consider a plane truss illustrated in Figure 1, where $W = 70.0$ cm, $H = 50.0$ cm, $n^{\mathrm{d}} = 28$, and $n^{\mathrm{m}} = 42$. The nodes (a) and (b) are pin-supported.

As the nominal dead load $\widetilde{\boldsymbol{f}}_{\mathrm{D}}$, we apply the external forces $(0, -120.0)$ kN at the nodes (e) and (f) as shown in Figure 1. Note again that $\boldsymbol{f}_{\mathrm{D}}$ represents the sum of conventional live load and dead load. It is possible that the conventional live load has moderately large magnitude of uncertain variation. In this case, the level of uncertainty of $\boldsymbol{f}_{\mathrm{D}}$, that is, α in (9), is supposed to be a moderately large value. The reference disturbance load $\boldsymbol{f}_{\mathrm{R}}$ is defined such that $(40.0, 0)$ kN and $(20.0, 0)$ kN, respectively, are applied at the nodes (c) and (d). The limit load factor under the nominal dead loads is computed as $\lambda^*(\widetilde{\boldsymbol{f}}_{\mathrm{D}}) = 48.4$ by employing the usual limit analysis, that is, by solving the LP problem (5). The collapse mode corresponds to the sway-type with horizontal displacements of the joints shown in Figure 2, where the vanishing members experience plastic deformations.

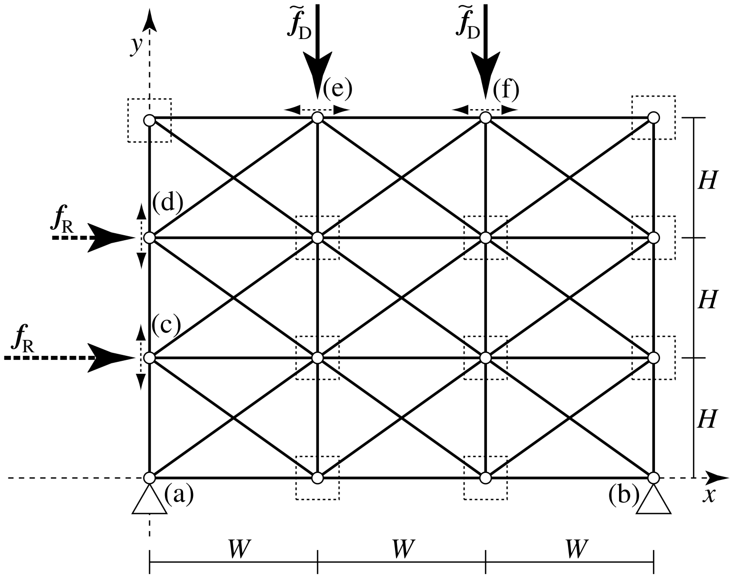

Figure 1. 3×3 plane grid truss.

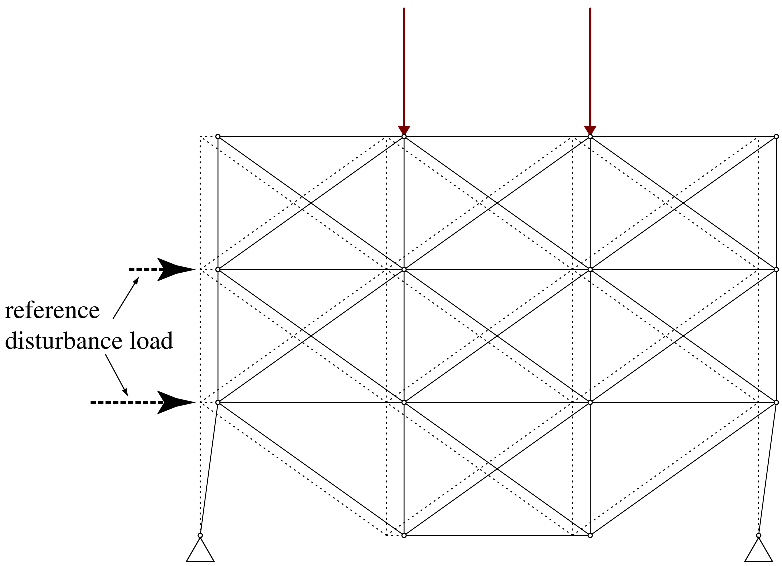

Figure 2. Collapse mode and the dead load of the 3×3 truss without the uncertainty in dead load ($\lambda^*(\widetilde{f}_D) = 48.4$).

We assume that the uncertain load $\boldsymbol{T\zeta}$ can possibly exist at free nodes, so that Assumption 3.1 is satisfied and the condition

$$(\boldsymbol{T\zeta})^{\top}\widetilde{\boldsymbol{f}}_{\mathrm{D}} = 0, \quad \boldsymbol{\zeta} \in \mathbb{R}^{m} \tag{60}$$

holds with $m = 24$. Accordingly, the uncertain load $\boldsymbol{T\zeta}$ is running through the squares and arrows depicted with the dotted lines in Figure 1. For $\alpha_1 = 40.0$ kN, the worst-case limit load factor is computed as $\lambda_{\min}(\alpha_1) = 37.0$ by using Algorithms 4.2 and 5.1. Let $\boldsymbol{\zeta}_1^{\mathrm{cr}}$ denote the optimal solution of (13). The corresponding critical load $\boldsymbol{f}_{\mathrm{D}}(\boldsymbol{\zeta}_1^{\mathrm{cr}})$ and collapse mode are shown in Figure 3.

Figure 3 shows that the collapse mode in the worst case is different from the sway-type mode observed in the nominal case of Figure 2. On the contrary, for $\alpha_2 = 20.0$ kN, the collapse mode in the worst case coincides with the sway-type as illustrated in Figure 4. The corresponding worst-case limit load factor is $\lambda_{\min}(\alpha_2) = 44.4$. The distribution of critical load $\boldsymbol{f}_{\mathrm{D}}(\boldsymbol{\zeta}_2^{\mathrm{cr}})$ is shown in Figure 4, which is different from the critical load in the case of Figure 3.

We next investigate the variation of the limit load factor by proportionally increasing the uncertain dead load, that is, we employ the usual limit analyses repeatedly by putting $\boldsymbol{\zeta} = \beta\boldsymbol{\zeta}_1^{\mathrm{cr}}$ and gradually increasing β. In Figure 5, the solid curve (A)→(B)→(C) shows the variation of $\lambda^*(\boldsymbol{f}_{\mathrm{D}}(\beta\boldsymbol{\zeta}_1))$ with respect to β. The collapse mode coincides with the sway-type shown in Figure 2 between the points (A) and (B), while the mode of Figure 3 is observed between (B) and (C). The variation of $\lambda^*(\boldsymbol{f}_{\mathrm{D}}(2\beta\boldsymbol{\zeta}_2^{\mathrm{cr}}))$ with respect to β is indicated by the dashed line (A)→(D) in Figure 5. Note that $\alpha_1 = 2\alpha_2$ implies $\|\boldsymbol{\zeta}_1^{\mathrm{cr}}\| = 2\|\boldsymbol{\zeta}_2^{\mathrm{cr}}\|$. The collapse mode coincides with the sway type shown in Figure 4 between (A) and (D). The curve (A)→(E)→(C) corresponds to the variation of the worst-case limit load factor $\lambda_{\min}(\beta\alpha_1)$ with respect to β. This illustrates that the critical loads as well as the corresponding collapse modes depend on the level of uncertainty α.

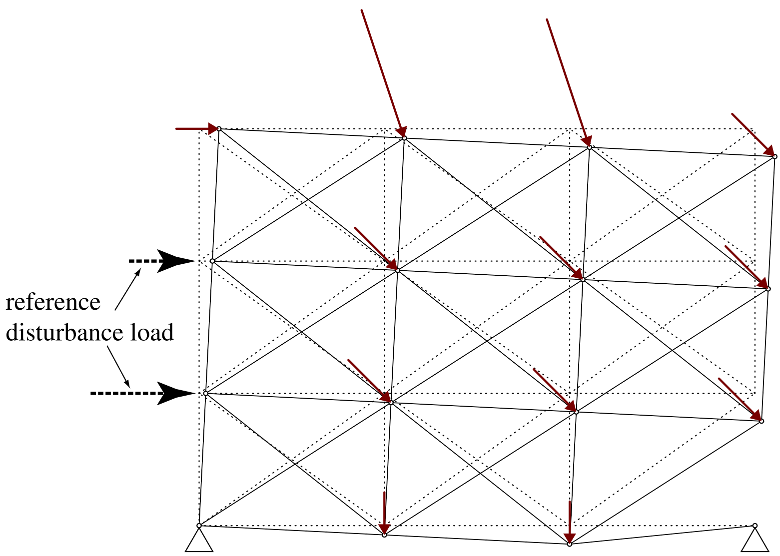

Figure 3. Collapse mode and the critical load of the 3×3 truss in the worst case for $\alpha_1 = 40.0$ kN ($\lambda_{\min}(\alpha_1) = 37.0$).

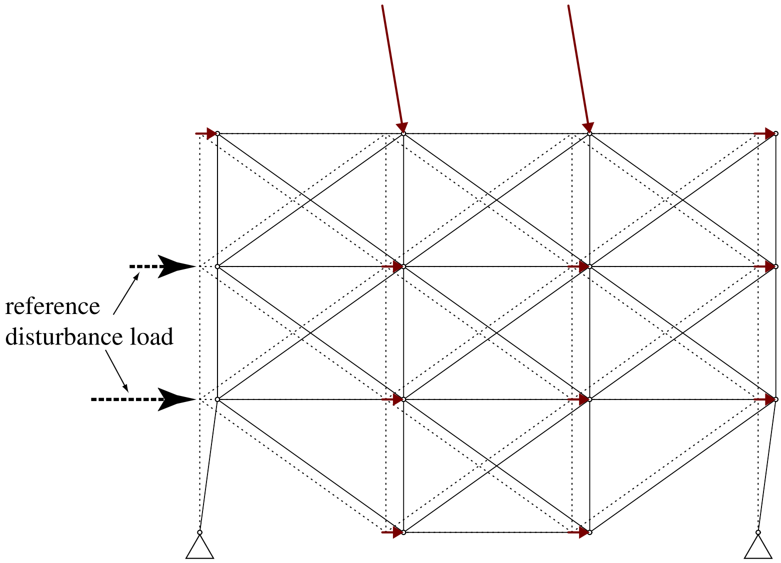

Figure 4. Collapse mode and the critical load of the 3×3 truss in the worst case for $\alpha_2 = 20.0$ kN ($\lambda_{\min}(\alpha_2) = 44.4$).

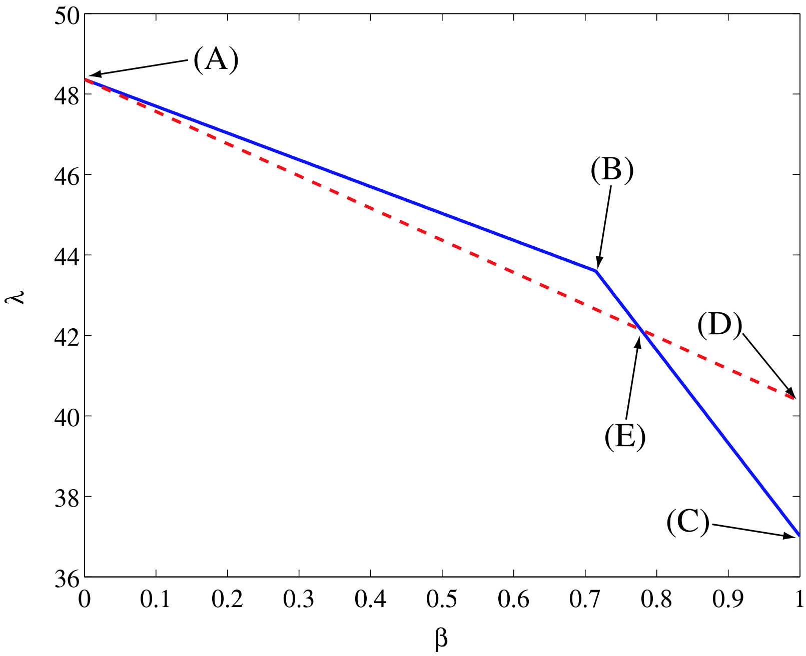

Figure 5. Limit load factor of the 3×3 truss; $\lambda^*(\boldsymbol{f}_{\mathrm{D}}(\beta\boldsymbol{\zeta}_1))$: solid line; $\lambda^*(\boldsymbol{f}_{\mathrm{D}}(2\beta\boldsymbol{\zeta}_2))$: dashed line.

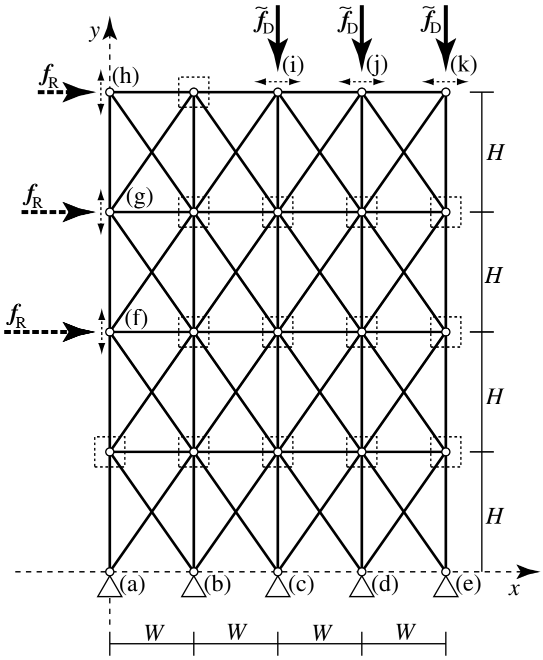

Figure 6. 4×4 plane truss.

6.2. ***4×4 truss.*** Consider the 68-bar plane truss illustrated in Figure 6, where $n^{\mathrm{m}} = 68$, $n^{\mathrm{d}} = 40$, $W = 35.0$ cm, and $H = 50.0$ cm. The nodes (a)–(e) are pin-supported. As the nominal dead load $\tilde{\boldsymbol{f}}_{\mathrm{D}}$, we apply the external forces $(0, -800.0)$ kN at the nodes (i)–(k). The reference disturbance load $\boldsymbol{f}_{\mathrm{R}}$ is defined such that $(52.0, 0)$ kN, $(40.0, 0)$ kN, and $(28.0, 0)$ kN are applied at the nodes (f), (g), and (h), respectively. The nominal limit load factor is computed as $\lambda^*(\tilde{\boldsymbol{f}}_{\mathrm{D}}) = 14.3$ by employing the usual limit analysis. The corresponding collapse mode is shown in Figure 7, where the vanishing members experience plastic deformations.

In a manner similar to Section 6.1, we assume that the uncertain load $\boldsymbol{T\zeta}$ can possibly exist at all free nodes so that Assumption 3.1 and (60) are satisfied with $m = 34$. Accordingly, the uncertain load $\boldsymbol{T\zeta}$ is running through the squares and arrows depicted with the dotted lines in Figure 6. We set $\alpha = 40.0$ kN. By using Algorithms 4.2 and 5.1, the worst-case limit load factor is computed as $\lambda_{\min}(\alpha) = 7.73$. The CPU time required by Algorithm 5.1 is 229.5 sec, and 42 cutting planes are generated within 61 iterations. Afterward, Algorithm 4.2 terminates by solving only 9 LP problems, where the CPU time required is 28.5 sec. This result demonstrates that the generated cutting planes at the root node of the branch-and-bound tree can reduce the number of nodes drastically that have to be visited in Algorithm 4.2.

In Step 2 of Algorithm 5.1, the LP problem (51) originally has 723 variables and 354 linear equality constraints besides side constraints in this example. By using the simplification proposed in Section 5.2, the problems to be solved have 534 variables and 240 linear equality constraints in average. Thus, the simplification scheme can reduce the numbers both of variables and constraints drastically. In Step 1 of Algorithm 4.2, the LP problem (45) originally has 315 variables and 176 linear equality constraints

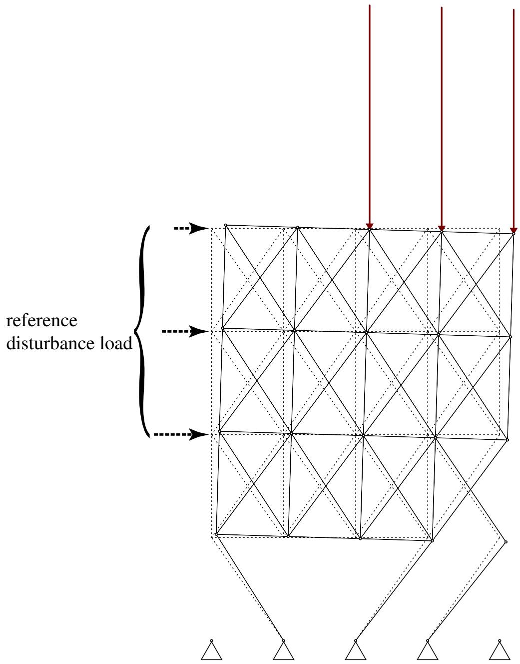

Figure 7. Collapse mode and the dead load of the 4×4 truss without the uncertainty in dead load ($\lambda^*(\widetilde{\boldsymbol{f}}_{\mathrm{D}}) = 14.3$).

besides side constraints, because $n^{\mathrm{c}} = 42$. After applying the simplification proposed in Section 4.3, the problems to be solved have 309 variables and 174 linear equality constraints in average.

Note that the worst-case limit load factor is almost half of the nominal one, in spite of the fact that the level of uncertainty α is relatively small compared with the norm of the nominal dead loads vector $\widetilde{\boldsymbol{f}}_{\mathrm{D}}$. The critical load $\boldsymbol{f}_{\mathrm{D}}(\boldsymbol{\zeta}^{\mathrm{cr}})$ and the corresponding collapse mode are shown in Figure 8 (left). It is observed from Figure 8 (left) that the collapse mode in the worst case is the same as that in the nominal case illustrated in Figure 7.

For comparison, we select a sample of the uncertain parameters vector $\boldsymbol{\zeta}'$ satisfying $\|\boldsymbol{\zeta}'\|_\infty = \alpha$ as the nodal forces shown in Figure 8 (right). The corresponding limit load factor is $\lambda^*(\boldsymbol{f}_{\mathrm{D}}(\boldsymbol{\zeta}')) = 8.43$, which is larger than the worst case. The corresponding collapse mode is shown in Figure 8 (right), which is different from the mode shown in Figure 8 (left). Thus, it is not easy to find the critical loads vector in a heuristic way.

We randomly generate a number of $\boldsymbol{\zeta}$ satisfying $\|\boldsymbol{\zeta}\|_\infty = \alpha$, and perform conventional limit analyses. The limit load factors $\lambda^*(\boldsymbol{f}_{\mathrm{D}}(\boldsymbol{\zeta}))$ obtained are plotted in Figure 9. We observe that all generated $\lambda^*(\boldsymbol{\zeta})$ are larger than the worst-case limit load factor $\lambda^*(\boldsymbol{\zeta}^{\mathrm{cr}}) = \lambda_{\min}(\alpha)$. This supports the assertion that using our proposed algorithms yields a global optimal solution of the nonconvex problem (13).

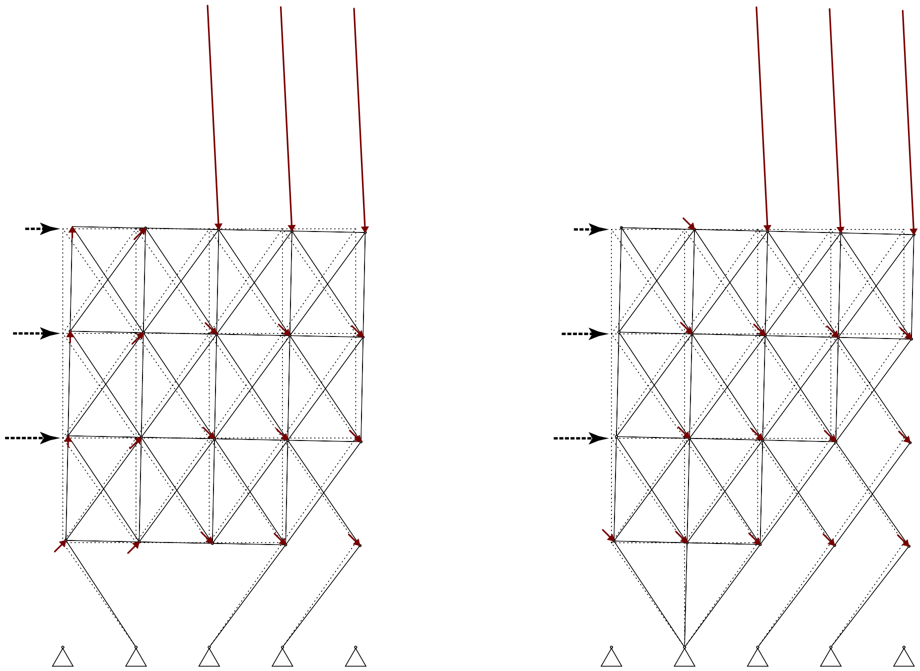

Figure 8. Left: collapse mode and the critical load of the 4×4 truss in the worst case for $\alpha = 40.0$ kN ($\lambda_{\min}(\alpha) = 7.73$). Right: definition of the dead load with $\boldsymbol{\zeta}'$ and the corresponding collapse mode of the 4×4 truss ($\lambda^*(\boldsymbol{f}_{\rm D}(\boldsymbol{\zeta}')) = 8.43$).

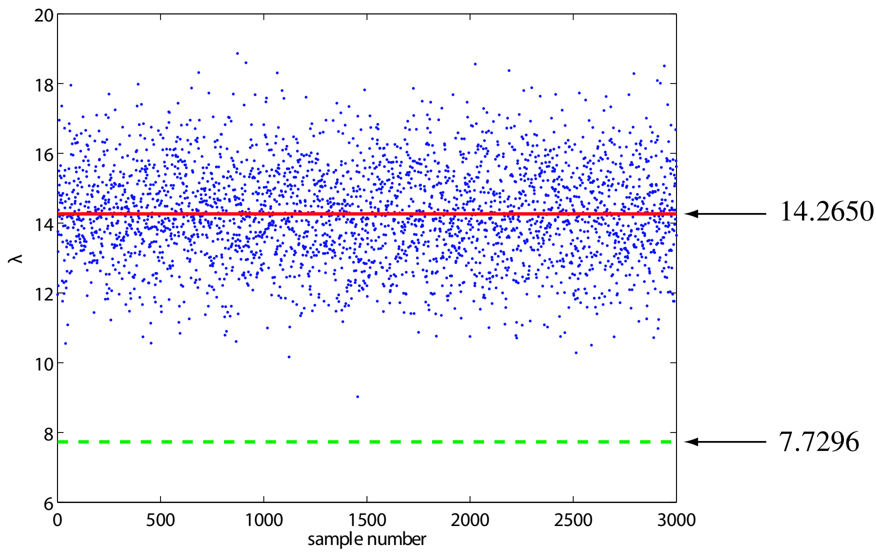

Figure 9. Limit load factor for randomly generated $\boldsymbol{\zeta}$: (solid line) $\lambda^*(\widetilde{\boldsymbol{f}}_{\rm D})$; (dashed line) $\lambda_{\min}(\alpha)$.

7. Conclusions

In this paper, we have investigated the worst-case detection in the plastic limit analysis of trusses affected by unknown-but-bounded dead and live loads. While the imprecisely known dead and live loads are constrained into a given bounded set, the live or disturbance loads are amplified with the load factor. A global optimization technique has been presented to compute the worst-case limit load factor as well as the critical load.

We supposed that the dead and live loads applied to a truss contain bounded errors around the nominal values. The level, or 'width', of uncertain variation is assumed to be known. We defined the worst case limit load factor as the minimum value among limit load factors attained with some loading patterns belonging to a given closed set. Then the worst-case detection problem has been formulated as a mixed 0-1 programming problem. To obtain a global optimal solution of the present problem, we have proposed a cut-and-branch method based on the LP relaxation and the disjunctive cut, where a cutting plane is generated by solving another LP problem. Since the proposed method converges to a global optimal solution, it is theoretically guaranteed that there exits no uncertain parameter such that the limit load factor becomes smaller than the obtained optimal value.

We showed in the numerical examples that the proposed cut-and-branch method can find the worst-case limit load factors. The comparison with the limit load factors for randomly generated dead and live loads demonstrates that the limit load factors we obtained correspond to the global optimal solutions of the mixed 0-1 programming problem presented. We have also illustrated through numerical examples that the process of cutting plane generation at the root node of the enumeration tree can reduce the number of LP relaxation problems that should be solved in the successive branch-and-bound procedure, although no theoretical result is to date available that suggests how many cutting planes should be generated.

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Received 30 Jun 2006. Accepted 23 Sep 2006.

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