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MICROMECHANICAL APPROACH TO TRANSFORMATION TOUGHENING IN ZIRCONIA-ENRICHED MULTIPHASE COMPOSITES

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A micromechanical model based on the mean-field approach was developed to investigate the effect of the mismatch in material properties of the constituents on the transformation toughening in zirconia enriched multiphase composites. Important results have been obtained for SiC/Al composites, which have a large potential for application in lightweight automotive structures.

1. Introduction

The phenomenon of stress transformation in zirconia-enriched composites was first reported by Garvie et al. [1975]. Some of the early experimental investigations on zirconia composites were performed by Gupta et al. [1978], Evans and Heuer [1980], Lange [1982], and others. In accordance with these investigations, the unconstrained phase transformation of zirconia particles results in approximately 4% dilatation and 16% shear strain. However, the particles embedded in a composite transform with twin bands of an alternative character so that the overall change in shear strain is very small [Simha and Truskinovsky 1994]. Therefore, the strain changes due to stress-induced phase transformation in composites are usually assumed to be dilatant.

When the particles surround a stable or a growing crack, the high stress concentration near the crack tip will trigger the transformation of zirconia particles near the crack tip. The typical transformation zone sizes found experimentally are on the order of 20 μm or less [Casellas et al. 2001]. The dilatant transformation will induce closure tractions on/along the crack faces. The overall stress concentration near the crack tip is, therefore, reduced; hence, the fracture toughness of the composite is enhanced, since a higher remote stress must be applied to reach the critical conditions at the crack tip.

The stress-induced transformation toughening mechanism has been successfully used to increase the fracture toughness of brittle ceramics [Kelly and Rose 2002; Basu et al. 2004]. In recent years, the demand for the development of super high temperature resistant and lightweight materials applied to aerospace structures has stimulated intensive research on the transformation toughening of brittle intermetallics composites, particularly with molybdenum or nickel aluminide matrices [Sbaizero et al. 2003]. The transformation toughening mechanism is now regarded as the most effective way to enhance the fracture toughness of ceramics and ceramic matrix composites [Kelly and Rose 2002; Basu et al. 2004; Cesari et al. 2006].

The continuum models of the phenomenon were developed by Lange [1982], McMeeking and Evans [1982], Budiansky et al. [1983], and Stump and Budiansky [1989]. All of these models are based on the assumption of effectively homogeneous composite material, where only macroscopic aspects of the

Keywords: transformation toughening, micromechanics, multiphase composites, crack, zirconia particles, continuum model, dilatant strain.

material deformation are considered. Strictly speaking, these theories are only applicable to a composite with the same material properties in all constituent phases, which have to be equal to the material properties of zirconia. However, in a real composite, the effect of the mismatch in material properties of the constituent phases on the transformation toughening mechanism can be significant. Consider, for example, a limiting case—a two-phase composite with rigid matrix and soft transformable particles. In such a composite the stress induced transformation in the particles will not affect the dilatation of the representative volume element (RVE) of the composite [Hill 1963] and, in its turn, the high stress concentration in the RVE in the vicinity of the crack tip will generate much less stress in the particles, reducing the size of the transformation zone. Both mechanisms will lead to a greatly reduced toughening effect in comparison with that predicted assuming homogeneous material with effective properties of the composite (continuum models). It can be shown that the opposite situation to that described above takes place for a composite with a soft matrix and rigid particles. When a multiphase composite is enriched with zirconia particles, the final effect of the mismatch in the material properties of the constituent phases is difficult to envisage.

A number of numerical studies have reported a significant influence of local stress concentration, particle size, shape and microstructure on the transformation toughening mechanism. Some of these factors were recently analyzed using a hybrid finite element method [Zeng et al. 2004; Alfano et al. 2006] and a micromechanical approach [Tsukamoto and Kotousov 2006]. The previous studies focused primarily on two-phase composite systems. For multiphase composites, in addition to the factors discussed above, a strong influence of the nontransforming dispersed phases on the transformation toughening mechanism is also expected. This was confirmed by results of some experimental investigations [Chen et al. 2000; Chen and Tuan 2001]. However, it seems that so far there have been no attempts to study theoretically the toughening effect in multiphase composites [Zeng et al. 2004].

In this paper, a micromechanical model is developed and incorporated into the continuum model of transformation toughening of Stump and Budiansky [1989], to investigate the effect of the mismatch of the constituent phases on the transformation toughening mechanism in multiphase composites. This model is based on the mean-field theory of Wakashima and Tsukamoto [1991]. It was reported in the literature that among the averaging methods [Hill 1963; Bui et al. 1972], the Wakashima–Tsukamoto estimate provides the closest prediction to numerical calculations in terms of the effective material properties of the composite [Miyamoto et al. 1999]. Similar methods were discussed in some other works, for example, [Ponte-Castaneda and Willis 1995; Cho and Ha 2001].

2. Continuum theory of transformation toughening

For the sake of completeness of this paper a brief introduction of the continuum model of transformation toughening [Stump and Budiansky 1989] will be presented next. Assume that a transformation zone surrounding the crack tip has undergone an irreversible transformation dilatation of strength $f^{(1)}\theta$ where $f^{(1)}$ is the zirconia particle volume fraction and θ is the unconstrained particle dilatation (Figure 1). Since typical transformation zone sizes are on the order of $20\,\mu$m or less, the small scale zone size approximation can be invoked. Then, the stress field in the vicinity of the crack tip can be described as

$$\sigma_{ik} = \frac{K}{\sqrt{2\pi r}} F_{ik}(\phi), \tag{1}$$

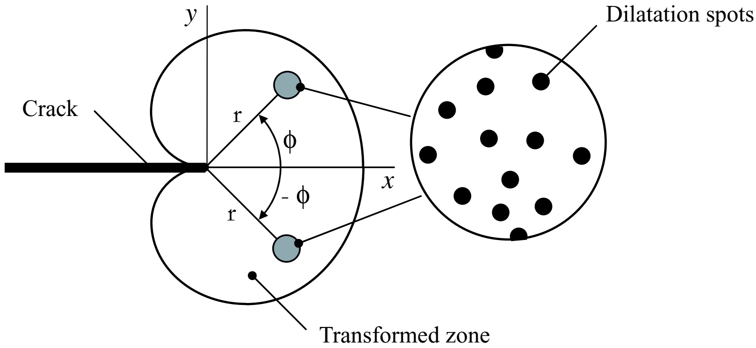

Figure 1. Symmetrically placed dilatant spots at a crack tip.

where K is the stress intensity factor at the crack tip, r is the distance from the crack tip, and $F_{ik}(\phi)$ is the well known trigonometric function.

The mean stress due to dilatation can be calculated using Hutchinson's solution for two small circular spots of dilatation of area dA located at $z_0 = x_0 + iy_0$ and $\bar{z}_0 = x_0 - iy_0$ (Figure 1) [Hutchinson 1974] as

$$\sigma_m = \frac{Ef^{(1)}\theta}{18\pi}\frac{1+\nu}{1-\nu}\mathrm{Re}\left\{\frac{1}{\sqrt{zz_0}(\sqrt{z}+\sqrt{z_0})}+\frac{1}{\sqrt{z\bar{z}_0}(\sqrt{z}+\sqrt{\bar{z}_0})}\right\}dA, \tag{2}$$

where E is Young's modulus and ν is Poisson's ratio.

The equation for the zone boundary, $z = R(\phi)e^{i\phi}$, is then obtained by adding the near field mean stress of Equation (1) to the zone contribution calculated by integrating Equation (2) over the upper half of the transformed zone, and equating the sum to σ_m^c as

$$\sigma_m^c = \frac{K(1+\nu)}{3}\sqrt{\frac{\pi R}{2}}\cos(\phi/2) + \int_A F(z, z_0)dA, \tag{3}$$

where σ_m^c is the critical value for mean stress corresponding to the phase transformation, and $F(z, z_0)$ is given by Equation (2).

The stress intensity factor at the crack tip, K, is equal to the sum of the applied stress intensity factor K_{ap} and that induced by the presence of the transformed zone. Once the transformed zone is found, K as a function of K_{ap} can be obtained by integrating the change of the stress intensity factor ΔK due to the two small circular spots of dilatation, as

$$\Delta K = \frac{Ef^{(1)}\theta}{6\sqrt{2\pi}(1-\nu)}\mathrm{Re}\{z_0^{-3/2} + \bar{z}_0^{-3/2}\}dA \tag{4}$$

over the upper half of the transformed zone

$$K = K_{ap} + \int_A \Delta K(z, z_0)dA. \tag{5}$$

A similar procedure can be adapted to the growing crack case as well. Details can be found in Stump and Budiansky [1989]. The final result of the calculation of the fracture toughening of the composite for a stable and a growing crack is shown in Figure 2.

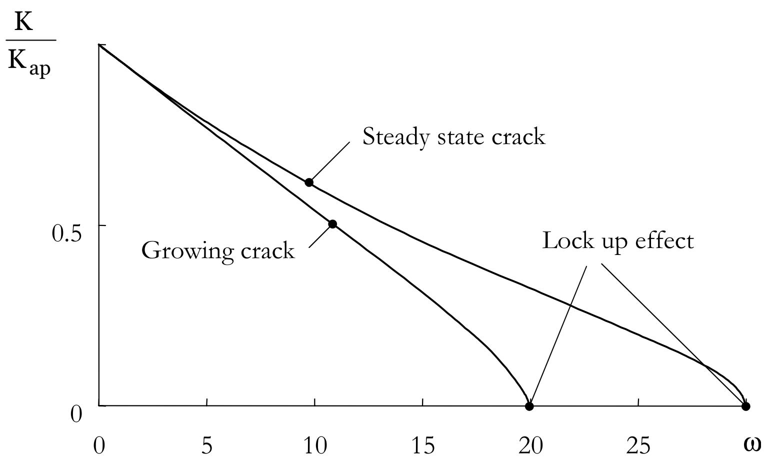

Figure 2. Fracture toughening versus the strength of transformation, ω.

In Figure 2 the nondimensional measure of the strength of the transformation, ω, is defined by

$$\omega = \frac{Ef^{(1)}\theta}{\sigma_m^c}\frac{1+\nu}{1-\nu}. \tag{6}$$

The toughening effect increases with the increase of the strength of the transformation. The most notable feature of Figure 2 is the existence of the *lock-up* effect, that is, infinite toughening of the composite, which was discovered by Rose [1986]. It occurs at $\omega \approx 30$ for the steady state case, and at $\omega \approx 20.2$ for the growing crack.

3. Mean-field micromechanical model

In this section, a composite consisting of N types of inclusions uniformly and randomly distributed in an isotropic continuum matrix is considered. Let us introduce a representative volume element (RVE) of the general heterogeneous media. When the RVE is subjected to macrostress $\bar{\sigma}_{ik}$ (corresponding to the traction vector $t_i = \bar{\sigma}_{ik}n_k$), the volume average of the induced microstress, σ_{ik}^{ap}, over RVE is equal to the macrostress $\bar{\sigma}_{ik}$ such that

$$\frac{1}{V^R}\int_{V^R} \sigma_{ik}^{ap}\, dV = \bar{\sigma}_{ik}, \tag{7}$$

where V^R is the volume of RVE. The volume average of the internal stress, σ_{ik}^{in}, due to the presence of eigenstrain, ε_{ij}^*, which was introduced by Mura [1982], is equal to zero, that is,

$$\frac{1}{V^R}\int_{V^R} \sigma_{ik}^{in}\, dV = 0. \tag{8}$$

For the system under consideration, the potential energy, Φ, is defined as

$$\Phi = \frac{1}{2}\int\limits_{V^R} (\sigma_{ik}^{ap} + \sigma_{ik}^{in}) \cdot (\varepsilon_{ik}^{ap} + \varepsilon_{ik}^{in} - \varepsilon_{ik}^{*}) dv - \int\limits_{S^R} (\sigma_{ik}^{ap} n_k) \cdot (u_i^{ap} + u_i^{in})\, ds, \tag{9}$$

where S^R is the surface of the RVE, and the repeated indexes assume the usual summation convention. Equation (9) can be rewritten in terms of summations of piecewise uniform quantities as

$$\Phi = -\frac{1}{2}\bar{\sigma}_{ik} \sum_{r=0}^{N} f^{(r)} M_{iklm}^{(r)} \bar{\sigma}_{lm}^{ap(r)} - \frac{1}{2}\sum_{r=0}^{N} f^{(r)} \varepsilon_{ik}^{*(r)} \cdot \bar{\sigma}_{ik}^{in(r)} - \sum_{r=0}^{N} f^{(r)} \varepsilon_{ik}^{*(r)} \cdot \bar{\sigma}_{ik}^{ap(r)}, \tag{10}$$

where the superscript (r) is the number identifying the constitutive phases and r runs from 0–N, 0 denotes the matrix, and 1–N denotes the corresponding dispersed particle phase. $M_{iklm}^{(r)}$ is the elastic compliance tensor of r-th phase, and $\varepsilon_{ik}^{*(r)}$ is the eigenstrain in r-th phase. $\bar{\sigma}_{ik}^{ap(r)}$ and $\bar{\sigma}_{ik}^{in(r)}$ are the volumetric average of the microstresses in phase (r) due to external loading and internal factors such as thermal expansion, plastic deformations and the like, respectively. Hill [1963] reported that $\bar{\sigma}_{ik}^{ap(r)}$ is proportional to the applied macrostress, $\bar{\sigma}_{ik}$. This can be written mathematically as

$$\bar{\sigma}_{ik}^{ap(r)} = B_{iklm}^{(r)} \bar{\sigma}_{lm}, \tag{11}$$

where $B_{iklm}^{(r)}$ is the stress concentration factor tensor. Equations (7) and (11) lead to the following relation:

$$\sum_{r=0}^{N} f^{(r)} B_{iklm}^{(r)} = I_{iklm}, \tag{12}$$

where I_{iklm} is the fourth order identity tensor.

According to the mean-field micromechanical theory [Wakashima and Tsukamoto 1991], which is based on Eshelby's equivalent inclusion method [Eshelby 1957; 1959; 1961] and Mori–Tanaka's mean-field approximation [Mori and Tanaka 1973], the following algebraic relations for $B_{iklm}^{(r)}$ can be derived:

$$B_{iklm}^{(r)} = C_{ikop}^{(r)} \left\{ \sum_{s=0}^{N} f^{(s)} C_{oplm}^{(s)} \right\}^{-1}, \tag{13}$$

$$C_{iklm}^{(r)} = \left\{ I_{iklm} + P_{ikop}^{(r)} (M_{oplm}^{(r)} - M_{oplm}^{(0)}) \right\}^{-1}, \tag{14}$$

$$P_{iklm}^{(r)} = L_{ikop}^{(0)} (I_{oplm} - S_{oplm}^{(r)}), \tag{15}$$

where $L_{ikop}^{(0)}$ is the elastic stiffness tensor of the matrix, and $S_{oplm}^{(r)}$ are the Eshelby's tensors for the (r) phase. Details of the derivation of these expressions are given in Appendix A. Further, the internal stress for each phase $\bar{\sigma}_{ik}^{in(r)}$ is also related to the eigenstrain, $\varepsilon_{ik}^{*(r)}$, and the stress concentration tensor, $B_{iklm}^{(r)}$, as

$$\bar{\sigma}_{ik}^{in(r)} = D_{iklm}^{(r)} \Delta\varepsilon_{lm}^{*(r)} - B_{ikop}^{(r)} \sum_{s=0}^{N} f^{(s)} D_{oplm}^{(s)} \Delta\varepsilon_{lm}^{*(s)}, \tag{16}$$

with

$$D_{oplm}^{(r)} = -C_{opik}^{(r)} P_{iklm}^{(r)}, \tag{17}$$

$$\Delta\varepsilon_{lm}^{*(r)} = \varepsilon_{lm}^{*(r)} - \varepsilon_{lm}^{*(0)}. \tag{18}$$

The macro strain $\bar{\varepsilon}_{ik}$ can be related to macro stress $\bar{\sigma}_{ik}$ using the potential energy Φ as

$$\bar{\varepsilon}_{ik} = -\frac{\partial\Phi}{\partial\bar{\sigma}_{ik}}. \tag{19}$$

Thus, from Equations (10), (11), (16), and (19) the constitutive relation of the composites can be written as

$$\bar{\varepsilon}_{ik} = \sum_{r=0}^{r} f^{(r)} M_{ikop}^{(r)} B_{oplm}^{(r)} \bar{\sigma}_{lm} + \sum_{r=0}^{r} f^{(r)} B_{iklm}^{(r)}{}^{T} \varepsilon_{lm}^{*(r)}. \tag{20}$$

The effective elastic properties (bulk modulus and shear modulus) of multiphase composites with randomly distributed spherical particles can be derived by substituting the Eshelby's tensor for the spherical inclusions [Eshelby 1957]:

$$S_{iklm} = \frac{1+\nu^{(0)}}{3(1-\nu^{(0)})}\delta_{ik}\delta_{lm} + \frac{8-10\nu^{(0)}}{15(1-\nu^{(0)})}\left(\delta_{il}\delta_{km} + \delta_{im}\delta_{kl} - \frac{2}{3}\delta_{ik}\delta_{lm}\right) \tag{21}$$

into Equations (15), and from Equations (13), (14), (15), and (20). In Equation (21) $\nu^{(0)}$ is Poisson's ratio of the matrix. Finally, the effective bulk and shear modulus can be written as

$$k^c = \sum_{r=0}^{N} f^{(r)} \xi_{B0}^{(r)} \Bigg/ \sum_{r=0}^{N} \frac{f^{(r)} \xi_{B0}^{(r)}}{k^{(r)}}, \tag{22}$$

$$\mu^c = \sum_{r=0}^{N} f^{(r)} \eta_{B0}^{(r)} \Bigg/ \sum_{r=0}^{N} \frac{f^{(r)} \eta_{B0}^{(r)}}{\mu^{(r)}}, \tag{23}$$

respectively, with

$$\xi_{B0}^{(r)} = \frac{k^{(r)}}{3k^{(0)}} \frac{3k^{(0)} + 4\mu^{(0)}}{3k^{(r)} + 4\mu^{(0)}}, \tag{24}$$

$$\eta_{B0}^{(r)} = \frac{\mu^{(r)}}{\mu^{(0)}} \frac{\mu^{(0)} + \chi^{(0)}}{\mu^{(r)} + \chi^{(0)}}, \tag{25}$$

and

$$\chi^{(0)} = \frac{\mu^{(0)}}{6} \frac{9k^{(0)} + 8\mu^{(0)}}{k^{(0)} + 2\mu^{(0)}}.$$

In the above equations δ_{ij} is the Kronecker delta. $f^{(r)}$ is the volume fraction, $k^{(r)}$ the bulk modulus, and $\mu^{(r)}$ the shear modulus of phase r. k^c and μ^c are overall effective bulk modulus and shear modulus of the multiphase composites, respectively.

Now, consider the unconstrained dilatational strain of the composite, Θ, corresponding to the unconstrained dilatational strain of the particles, $\theta^{(r)}$. The relation between Θ and $\theta^{(r)}$ is given as

$$\Theta\delta_{ik} = \sum_{r=1}^{n} f^{(r)} {B^{(r)}_{iklm}}^{T} \delta_{lm}\theta^{(r)}. \tag{26}$$

In the same way as the effective elastic constants were derived above, the relation between the dilatational strain component Θ of the composite and the dilatational strain of the unconstrained transformable particle $\theta^{(r)}$ can be also written as

$$\Theta = \sum_{r=1}^{n} f^{(r)} \xi_B^{(r)} \theta^{(r)}, \tag{27}$$

where

$$\xi_B^{(r)} = \xi_{B0}^{(r)} \Bigg/ \sum_{r=0}^{N} f^{(r)} \xi_{B0}^{(r)}. \tag{28}$$

Further, in order to take into account the stress concentration effect on the stress transformation criterion, let us consider the mean stress in the particles. The mean stress in each phase, $\sigma_m^{(r)}$, is given as

$$\sigma_m^{(r)} = \frac{\bar{\sigma}_{ii}^{(r)}}{3} = \frac{B^{(r)}_{iikl}\bar{\sigma}_{kl}}{3}. \tag{29}$$

Similarly, the following relations are derived:

$$\sigma_m^{(r)} = \xi_B^{(r)} \bar{\sigma}_m. \tag{30}$$

Here, $\xi_B^{(r)}$ is the stress concentration factor, which is shown in Equation (28). Consequently, using these equations, one can calculate the effective elastic constants, microstress in each phase, and unconstrained dilatation Θ of the composite due to dilatation of each phase $\theta^{(r)}$.

4. Micromechanical model of transformation toughening

Consider the toughening effect in a multiphase composite enriched with partially stabilized spherical zirconia particles. The micromechanical model developed above can be directly incorporated into the continuum model [Stump and Budiansky 1989] by replacing the corresponding material constants, dilatation, and critical stress with those derived from the micromechanical model, Equations (22), (23), (28) and (30). Such substitution will not affect the fracture-toughening curve as shown in Figure 2; however the strength of the transformation will be modified as

$$\omega_m = \xi^2 \frac{E^c f^{(t)}\theta}{\sigma_m^c} \frac{1+\nu^c}{1-\nu^c}, \tag{31}$$

where ξ is the stress concentration factor, which corresponds to the transforming particles and can be calculated using Equations (28) and (24), and $f^{(t)}$ is the volume fraction of the transforming particles. Young's modulus E^c and Poisson's ratio ν^c are related to the bulk modulus k^c and shear modulus μ^c (Equations (22) and (23)), of the composite by well-known relationships.

The strength of the transformation ω_m calculated based on the micromechanical model differs from that obtained from the continuum model by factor ξ^2. This parameter reflects the effect of the mismatch in the constituent material properties on the toughening mechanism. The mismatch in the material properties of the constituent phases results in two effects: the first influences the unconstrained dilatation of the composite, and the second influences the average stress in the particles. Both effects produce the same factor ξ in the strength of the transformation parameter Equation (31) and both act in the same direction, increasing or decreasing the stress transformation parameter depending on the combination of material properties of the constituents.

5. Case study

As an application of the developed theory, a SiC/Al composite will be considered next. Such composites have a significant potential for application in lightweight automotive structures, forgings for suspension, chassis, drive train, and vehicle structures, as well as automotive rolled products and semiproducts for the manufacture of advanced automotive components [Fang et al. 1997; Tung and McMillan 2004]. The greatest interest in aluminum metal-matrix composites is in their ability to provide high specific strength and stiffness. This translates into weight savings by producing lighter components capable of withstanding the required loads, for example, space frames and sheet panels, which is of particular interest in the modern transportation industry. One of the major concerns regarding widespread application in the automotive industry is the relatively high brittleness of this composite in comparison with traditional materials [Ma et al. 2003; Agrawal and Sun 2004].

An estimation of a potential benefit from the enrichment of the SiC/Al composite with zirconia particles, as calculated using the developed model and previously reported results (Figure 2), is shown in Figure 3.

Figure 3 demonstrates a very significant influence of the microstructure on toughening of the composite as the theory under consideration differentiates the matrix phase from the dispersed particle phases. In the case when a crack is located in an SiC matrix, the toughening effect of the stress transformation of partially stabilized zirconia particles is significant and reaches a maximum in the vicinity of 15% of volume fraction of SiC (or 55% of Al). In the case when crack is located in an Al matrix, the toughening effect is not so significant for small volume fractions of SiC. However at large volume fractions of SiC the toughening effect becomes stronger than in the case when SiC is the matrix. It is interesting that the toughening effect can decrease or increase when increasing the volume fraction of SiC, depending on the microstructure of the composite. Consequently, in engineering and toughening such composites special attention should be paid to the microstructure of the composite.

The results obtained for SiC/Al indicate that the stress-induced mechanism of toughening of multiphase composites is pretty effective and visible for these composites. For a multiphase composite the toughening effect can be tailored in accordance with the desired profile based on the developed theory. A greater toughening effect may be expected by increasing the volume fraction of ZrO_2, and theoretically the strength of transformation, ω_m, could reach the lock-up values (Figure 2). It should be mentioned that the present theory is based on a small scale approximation and neglects the actual sizes of the component. At high values of ω_m, the transformation zone could be very large to justify the application of the small scale approximation.

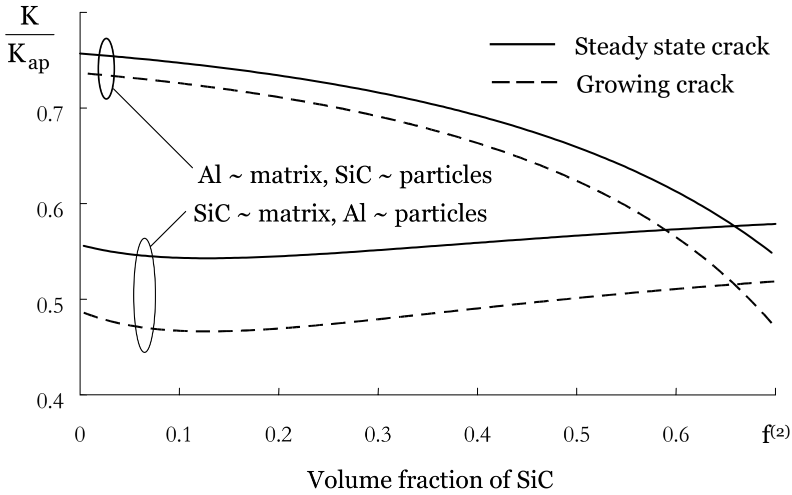

Figure 3. Toughening effect of adding ZrO_2 particles to SiC/Al composite. Material properties: Al $E^{(0)} = 70\,\text{GPa}$, $\nu^{(0)} = 0.3$ [Simmons and Wang 1971], ZrO_2: $E^{(1)} = 200\,\text{GPa}$, $\nu^{(1)} = 0.3$, $\sigma_m^c = 500\,\text{MPa}$ [Pace et al. 1969; Zeng et al. 2004], SiC: $E^{(2)} = 430\,\text{GPa}$, $\nu^{(2)} = 0.17$ [Jackson 2005]. The volume fraction of ZrO_2 particles is set at 30% for each composite.

6. Conclusion

A micromechanical model was developed to investigate the possibility of fracture toughening of multiphase composites using the stress transformation mechanism of partially stabilized zirconia particles. Results obtained within this model demonstrate a very strong influence of the material properties of constituent phases and microstructure on the toughening mechanism. The toughening of SiC/Al composites, which have a significant potential for application in lightweight automotive structures, were studied in detail. A high level of the toughening effect can be reached by adding ZrO_2 particles to the composite. Based on the developed theory the toughening effect for multiphase composites can also be tailored by varying the phase composition in a pre-determined profile. It is recognized that extensive experimental work is needed to validate the developed theory and include other important effects.

Notation

K	Stress intensity factor
r	Distance from the crack tip
E	Young's modulus
ν	Poisson's ratio
σ_m^c	Critical value of mean stress for the stress-induced phase transformation in transformable particles
K_{ap}	Stress intensity factor due to remote applied loading

ΔK	Change of the stress intensity factor due to presence of the transformed zone
ω	Non-dimensional measure of the strength of the transformation, defined by continuum model
V^R	Volume of RVE
S^R	Surface of RVE
n_i	Outward unit vector normal to the surface S of RVE
$\bar{\sigma}_{ik},\ \bar{\varepsilon}_{ik}$	Macrostress and macrostrain
$\sigma_{ik}^{ap},\ \sigma_{ik}^{ap},\ u_i^{ap}$	Microstress, microstrain and microdisplacement due to applied loading, respectively
$\sigma_{ik}^{in},\ \varepsilon_{ik}^{in},\ u_i^{in}$	Microstress, microstrain and displacement due to the presence of inclusions
Φ	Potential energy
$f^{(r)}$	Volume fraction of the rth phase
$\varepsilon_{ik}^{*(r)}$	Eigenstrain in the rth phase
$P_{iklm}^{(r)}$	Fourth-rank tensor defined by Equation (15)
$\bar{C}_{iklm},\ \bar{C}_{iklm}^{(r)}$	Fourth-rank tensor defined by Equations (A.11), (A.12)
I_{iklm}	Fourth rank identity tensor
$L_{iklm}^{(r)},\ M_{iklm}^{(r)}$	Elastic stiffness and compliance, respectively
$\bar{\sigma}_{ik}^{ap(r)},\ \bar{\sigma}_{ik}^{in(r)}$	Average microstress in the rth phase due to applied loading and internal stress, respectively
$k^{(r)}$	Bulk modulus of the rth phase
$\mu^{(r)}$	Shear modulus of the rth phase
$S_{iklm}^{(r)}$	Eshelby's tensor
$B_{iklm}^{(r)}$	Stress concentration factor tensor of the rth phase
$\xi_B^{(r)}$	Stress concentration factor of the rth phase
$\xi_{B0}^{(r)},\ \eta_{B0}^{(r)}$	Parameters for the rth phase defined by Equation (24) and (25)
Θ	Unconstrained dilatation strain of the composite
$\theta^{(r)}$	Unconstrained dilatation strain of the rth phase
ω_m	Modified strength of the stress-induced phase transformation derived from the micromechanical model

Appendix A

The mean-field micromechanical approach reformulated by Wakashima and Tsukamoto [1991], which stems from Eshelby's equivalent inclusion method [Eshelby 1957; 1959; 1961] and Mori–Tanaka's mean-field approximation [Mori and Tanaka 1973] will be mentioned briefly here. Based on Eshelby's equivalent inclusion concept, we can replace the microinhomogeneous material by the homogeneous comparison material (HCM), with the equivalent inclusions and the fictitious eigenstrain distribution to represent the disturbance of stress field in the microinhomogeneous material. The HCM has exactly the same microgeometry as the multiphase composite. For the microinhomogeneous material Hooke's law can be written as

$$\bar{\varepsilon}_{ik}^{ap(r)} = M_{iklm}^{(r)} \bar{\sigma}_{lm}^{ap(r)}, \tag{A.1}$$

while for the homogeneous comparison material (HCM) it can be written as,

$$\bar{\varepsilon}_{ik}^{ap(r)} = M_{iklm}^{(r)} \bar{\sigma}_{lm}^{ap(r)} + \varepsilon_{ik}^{**(r)}, \tag{A.2}$$

where $\varepsilon_{ik}^{**(r)}$ is a uniform fictitious eigenstrain, defined in the equivalent inclusions corresponding to the r-th phase. From Equations (A.1) and (A.2) it follows that

$$\varepsilon_{ik}^{**(r)} = (M_{iklm}^{(r)} - M_{iklm}^{(0)}) \bar{\sigma}_{lm}^{ap(r)}, \tag{A.3}$$

where

$$\varepsilon_{ik}^{**(0)} = 0. \tag{A.4}$$

According to the Eshelby's solution [Eshelby 1957], the following relation is derived:

$$\bar{\sigma}_{ik}^{ap(r)} - \bar{\sigma}_{ik} = -P_{iklm}^{(r)} \varepsilon_{lm}^{**(r)}, \tag{A.5}$$

where

$$P_{iklm}^{(r)} = L_{ikop}^{(0)} (I_{oplm} - S_{oplm}^{(r)}). \tag{A.6}$$

$S_{oplm}^{(r)}$ denotes Eshelby's tensor, whose components are dimensionless and dependent on the axial ratios of the elliptical inclusions and Poisson's ratio of the matrix, which is assumed to be isotropic. This scheme is only applicable to the case when the discrete phases are dilute. To overcome this limitation, the Mori–Tanaka concept will be used. Equation (A.5) is replaced by the following equation:

$$\bar{\sigma}_{ik}^{ap(r)} - \bar{\sigma}_{ik}^{ap(0)} = -P_{iklm}^{(r)} \varepsilon_{lm}^{**(r)}. \tag{A.7}$$

From Equations (A.3) and (A.7), the following relation can be derived:

$$\bar{\sigma}_{ik}^{ap(r)} = C_{iklm}^{(r)} \bar{\sigma}_{lm}^{ap(0)}, \tag{A.8}$$

where

$$C_{iklm}^{(r)} = \left\{ I_{iklm} + P_{ikop}^{(r)} (M_{oplm}^{(r)} - M_{oplm}^{(0)}) \right\}^{-1}. \tag{A.9}$$

By considering Equation (7), the sum of the microstress can be written as

$$\bar{\sigma}_{ik} = \sum_{r=0}^{N} f^{(r)} \bar{\sigma}_{ik}^{ap(r)}. \tag{A.10}$$

Therefore, the stress concentration factor $B_{ijkl}^{(r)}$ is calculated as

$$B_{iklm}^{(r)} = C_{ikop}^{(r)} \bar{C}_{oplm}^{-1}, \tag{A.11}$$

where

$$\bar{C}_{iklm} = \sum_{r=0}^{N} f^{(r)} C^{(r)}_{iklm}. \tag{A.12}$$

Next, the internal stress in the microinhomogeneous material will be considered. For the internal strain in the microinhomogeneous material,

$$\bar{\varepsilon}^{in(r)}_{ik} = M^{(r)}_{iklm} \bar{\sigma}^{in(r)}_{lm} + \varepsilon^{*(r)}_{ik}. \tag{A.13}$$

In accordance with the last Equation (A.13), in the equivalent homogeneous material

$$\bar{\varepsilon}^{in(r)}_{ik} = M^{(0)}_{iklm} \bar{\sigma}^{in(r)}_{lm} + \varepsilon^{*(r)}_{ik} + \varepsilon^{**(r)}_{ik}, \tag{A.14}$$

where $\varepsilon^{**(r)}_{ik}$ is the fictitious eigenstrain, which is found from (A.13) and (A.14) as

$$\varepsilon^{**}{}^{(r)}_{ik} = (M^{(r)}_{iklm} - M^{(0)}_{iklm}) \bar{\sigma}^{in(r)}_{lm}. \tag{A.15}$$

Using the Mori–Tanaka concept, the following equation can be written:

$$\bar{\sigma}^{in(r)}_{ik} - \bar{\sigma}^{in(0)}_{ik} = -P^{(r)}_{iklm} (\Delta\varepsilon^{*(r)}_{lm} + \varepsilon^{**(r)}_{lm}), \tag{A.16}$$

where $\Delta\varepsilon^{*(r)}_{kl}$ is defined by Equation (18). From Equations (A.15) and (A.16), it follows that

$$\bar{\sigma}^{in}{}^{(r)}_{ik} = B^{0(r)}_{iklm} \big(\bar{\sigma}^{in}{}^{(0)}_{lm} - P^{(r)}_{lmop} \Delta\varepsilon^{*}{}^{(r)}_{op} \big). \tag{A.17}$$

The sum of the internal stress must be equal to zero,

$$\sum_{r=0}^{N} f^{(r)} \bar{\sigma}^{in(r)}_{ik} = 0, \tag{A.18}$$

and therefore,

$$\bar{\sigma}^{in(0)}_{ik} = \bar{C}^{-1}_{iklm} \sum_{r=0}^{N} f^{(r)} C^{(r)}_{lmop} P^{(r)}_{opvw} \Delta\varepsilon^{*(r)}_{vw}. \tag{A.19}$$

By comparing Equation (A.19) with Equation (16), $D^{(r)}_{ijkl}$ is obtained as given in Equation (17) as

$$D^{(r)}_{ijkl} = -C^{(r)}_{ikop} P^{(r)}_{oplm}. \tag{A.20}$$

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