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# GEOMETRIC ANALYSIS OF THE DYNAMICS OF A DOUBLE PENDULUM

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In this paper we make use of Riemannian geometry to analyze the dynamics of a simple low dimensional system with constraints, namely a double physical pendulum. The dynamics are analyzed by means of the Jacobi–Levi–Civita equation and its solutions. We show that this geometrical approach is in qualitative agreement with the classical techniques devoted to the study of dynamical systems.

## 1. Introduction

The classical approach to analysis of Hamiltonian systems has been widely applied, providing a classical explanation of the onset of chaos in these systems. In addition to the classical techniques for analyzing Hamiltonian systems, the geometric approach plays an important role. The geometric approach is based on the relation between Riemannian geometry and Hamiltonian dynamics, but is distinct from the geometric formulation of Hamiltonian mechanics in terms of symplectic geometry. This technique has been successfully applied [Cerruti-Sola and Pettini 1995; 1996, Casetti et al. 1996; Di Bari and Cipriani 1998; Casetti et al. 2000], especially to systems with many degrees of freedom. It has also been widely applied in general relativity [Szydłowski 2000] and to low dynamical systems with a nondiagonal metric tensor [Awrejcewicz et al. 2006]. It is believed that the geometric approach can provide an alternative to the classical explanation for the onset of chaos in Hamiltonian systems, which involves the homoclinic intersections [Lichtenberg and Lieberman 1992]. In the geometric approach to Hamiltonian dynamics, the analysis of dynamical trajectories and behavior of a system is cast into the analysis of a geodesic flow in a corresponding Riemannian space. The main tool of this approach is the so-called Jacobi–Levi–Civita (JLC) equation [do Carmo 1992; Di Bari and Cipriani 1998]. In general, the JLC equation is a system of second-order differential equations with respect to a geodesic length, and it describes the evolution of a tangent vector (so-called Jacobi vector) along the geodesic. Although there are many dynamical systems that can be described in this manner, there are some that can not, namely systems with velocity-dependent potentials. However, this kind of dynamical system can be analyzed by means of the Finslerian geometry [Di Bari and Cipriani 1998]. In this paper, we confine ourselves to conservative Hamiltonian systems, which can be geometrized within the Riemannian geometry approach. The main idea of this approach is to make use of the fact that Hamilton’s least action principle,

$$\delta \int_{t_1}^{t_2} L(q^i, \dot{q}^i, t) dt = 0,$$

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can be connected with the condition of minimizing the arc-length functional in the Riemannian space between two points  $A, B$ . The condition has the form

$$\delta \int_A^B ds = 0.$$

The point is that motion of a Hamiltonian system can be viewed as the motion of a single virtual particle along a geodesic in a suitable Riemannian space  $\mathcal{Q}$ . From the above condition one can obtain the geodesics equation, which has the following form in local coordinates [do Carmo 1992]

$$\frac{d^2 q^i}{ds^2} + \Gamma^i_{jk} \frac{dq^j}{ds} \frac{dq^k}{ds} = 0. \tag{1}$$

The Riemannian space is endowed with a metric tensor, which is obtained from the dynamics of the analyzed system. In order to make use of the geometric approach we must choose a Riemannian manifold and the metric tensor. There are several choices for a Riemannian manifold and metric tensor: a space-time configuration manifold and the Eisenhart metric [Szydłowski 1998], a configuration manifold and the Jacobi metric [Casetti et al. 1996; Cerruti-Sola and Pettini 1996], etc. In this paper we choose a configuration space of an analyzed system for a Riemannian manifold. Hence, the metric tensor is the Jacobi metric  $g$ , which is connected to the dynamics by the following relationship [Casetti et al. 2000]

$$g_{ij} = 2W a_{ij}(q), \quad W \equiv E - V,$$

where  $E$  is a total energy and  $V$  is potential energy. The matrix  $a$  is a kinetic energy matrix (we use the Einstein summation convention):

$$L = \frac{1}{2} \dot{q}^i a_{ij} \dot{q}^j - V. \tag{2}$$

This relationship follows from the Maupertuis principle, which gives

$$ds = 2W dt.$$

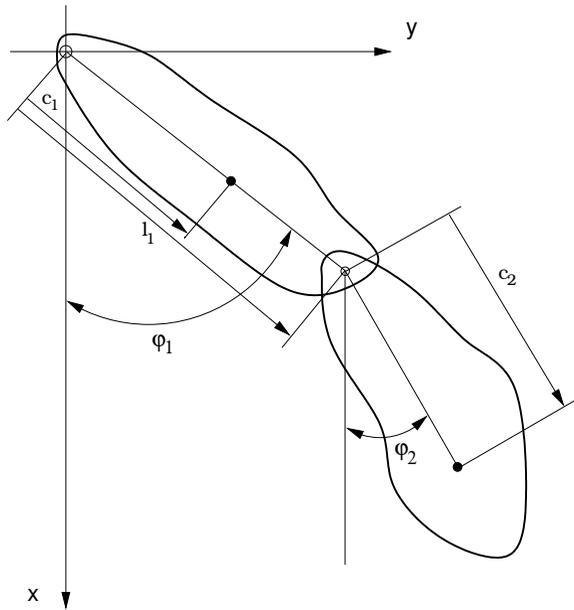
The main tool of the geometric approach, namely the JLC equation in a local coordinate system, has the following form [do Carmo 1992; Casetti et al. 2000]

$$\frac{\delta^2 J^n}{\delta s^2} + J^i \frac{dq^j}{ds} \frac{dq^k}{ds} R^n_{kij} = 0, \quad n = 1, 2, \dots, \dim \mathcal{Q}, \tag{3}$$

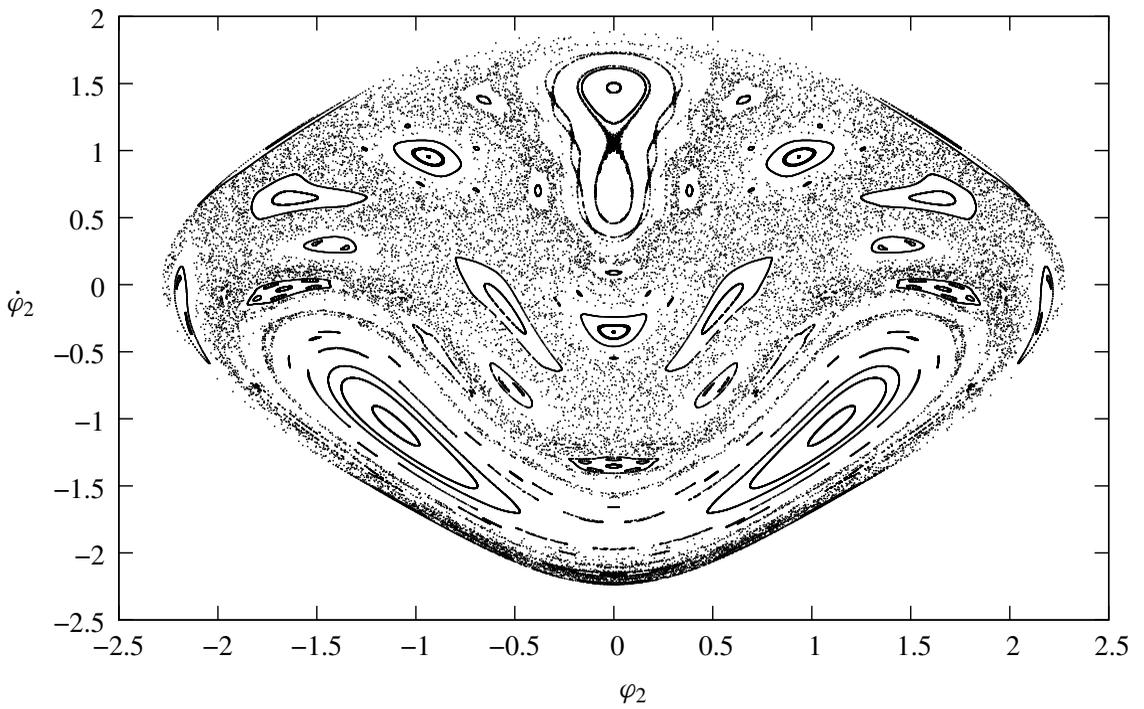
where  $q^j$  satisfy the geodesics Equation (1),  $J^n$  are components of the Jacobi vector,  $R^n_{kij}$  are components of the Riemann curvature tensor, and

$$\frac{\delta J^n}{\delta s} = \frac{dJ^n}{ds} + \Gamma^n_{kl} J^k \frac{dq^l}{ds}$$

are so-called absolute derivatives. The above equation has a similar form to the tangent dynamics equation, which is used to evaluate Lapunov’s exponents. In fact, Equation (3) takes exactly the same form in the case of the Eisenhart metric [Casetti et al. 1996]. This means that there is a connection between the JLC equation and the tangent dynamics equation. Moreover, it is possible to find Lapunov’s exponents using the Riemannian geometry approach. This has been done only for systems with many degrees of freedom and diagonal metric tensor [Casetti et al. 2000].



**Figure 1.** Double physical pendulum.



**Figure 2.** Poincaré section for  $\mathcal{E}=1.1$ .

Because we are interested in systems of only two degrees of freedom, the Riemannian space  $\mathcal{Q}$  is two-dimensional. This implies that we have only one nonzero component  $R_{2121}$  of the Riemann curvature tensor [Nakahara 1990; do Carmo 1992]. In this case, the JLC equation (3) takes the form

$$\frac{d^2\Psi}{ds^2} + \frac{R_{2121}}{\det \mathbf{g}} \Psi = 0,$$

where  $\Psi$  is a normal component of the Jacobi vector relative to the geodesic. The tangent component of the Jacobi vector evolves only linearly in a geodesic length, so it does not contribute to the character of the solution [Di Bari and Cipriani 1998]. Next, making use of the fact that

$$\mathcal{R} = \frac{2R_{2121}}{\det \mathbf{g}}, \tag{4}$$

we obtain a single differential equation which carries information about the system behavior

$$\frac{d^2\Psi}{ds^2} + \frac{1}{2}\mathcal{R}\Psi = 0, \tag{5}$$

where  $\mathcal{R}$  is the scalar curvature which, in general, is not periodic in  $\tau$ . At this point, we can see where a possible explanation of the onset of chaos in Hamiltonian system lies. The component  $\Psi$  of the Jacobi vector represents a distance between two nearby geodesics, which in turn represent trajectories of the analyzed system. The solutions of Equation (5) can exhibit exponential growth due to parametric excitations in the scalar curvature. Hence, this formulation and description of Hamiltonian dynamics gives us a qualitatively different explanation of the onset of chaos as a parametric instability of geodesics [Cerruti-Sola and Pettini 1996].

In order to solve Equation (5), we need to transform it into a differential equation with respect to the real time,  $t$ . Taking into account Equation (2) we find

$$\ddot{\Psi} - \frac{\dot{W}}{W} \dot{\Psi} + 2\mathcal{R}W^2\Psi = 0.$$

The above equation can be easily transformed into another form by means of the following substitution [Cerruti-Sola and Pettini 1996]

$$\Psi = J\sqrt{W},$$

which gives

$$\ddot{J} + \Omega(\tau)J = 0, \tag{6}$$

where

$$\Omega(\tau) \equiv \frac{1}{2} \left( \frac{\ddot{W}}{W} - \frac{1}{2} \left( \frac{\dot{W}}{W} \right)^2 + 4\mathcal{R}W^2 \right).$$

It should be emphasized here that  $\Omega$  is not, in general, periodic in  $\tau$ -time. Although  $\Omega$  is written as a function of  $\tau$ , it does not depend on  $\tau$  explicitly. In fact, it depends on a particular trajectory of the system.

### 2. The pendulum

In this paper we analyze a mechanical system with constraints, namely a double physical pendulum. The dynamics of the pendulum are described by the following lagrangian  $L$

$$L = \frac{1}{2} (m_1 c_1^2 + J_1 + m_2 l_1^2) \dot{\varphi}_1^2 + \frac{1}{2} (m_2 c_2^2 + J_2) \dot{\varphi}_2^2 + m_2 c_2 l_1 \dot{\varphi}_1 \dot{\varphi}_2 \cos (\varphi_1 - \varphi_2) - V (\varphi_1, \varphi_2),$$

where

$$V (\varphi_1, \varphi_2) = g (m_2 l_1 + m_2 c_2 + m_1 c_1) - g (m_1 c_1 + m_2 l_1) \cos \varphi_1 - m_2 g c_2 \cos \varphi_2,$$

$m_1$  and  $m_2$  are masses,  $J_1$  and  $J_2$  are moments of inertia, and  $c_1$  and  $c_2$  are the positions of centers of masses of the first and second link, respectively (see Figure 1). In order to cast the above lagrangian into a nondimensional form, we introduce the following scaling

$$\begin{aligned} \tau &\equiv t \sqrt{\frac{m_1 g c_1 + m_2 g l_1}{J_1 + m_1 c_1^2 + m_2 l_1^2}}, & \beta &\equiv \frac{J_2 + m_2 c_2^2}{J_1 + m_1 c_1^2 + m_2 l_1^2} \\ \kappa &\equiv \frac{m_2 c_2 l_1}{J_1 + m_1 c_1^2 + m_2 l_1^2}, & \mu &\equiv \frac{m_2 c_2}{m_1 c_1 + m_2 l_1}. \end{aligned}$$

Hence, the lagrangian takes the nondimensional form

$$\begin{aligned} L &= \frac{1}{2} \dot{\varphi}_1^2 + \frac{\beta}{2} \dot{\varphi}_2^2 + \kappa \dot{\varphi}_1 \dot{\varphi}_2 \cos \phi - 1 - \mu + \cos \varphi_1 + \mu \cos \varphi_2 \\ &= \frac{1}{2} (\dot{\varphi}_1 \dot{\varphi}_2) \mathbf{a} \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} - 1 - \mu + \cos \varphi_1 + \mu \cos \varphi_2, \quad \phi \equiv \varphi_1 - \varphi_2, \end{aligned}$$

where

$$\mathbf{a} = \begin{pmatrix} 1 & \kappa \cos \phi \\ \kappa \cos \phi & \beta \end{pmatrix},$$

The dot over  $\varphi$  denotes  $\tau$  derivative. Using the Euler–Lagrange equations we obtain the equations of motion

$$\begin{cases} \ddot{\varphi}_1 = \frac{-\kappa \sin \phi (\kappa \cos \phi \dot{\varphi}_1^2 + \beta \dot{\varphi}_2^2) - \beta \sin \varphi_1 + \kappa \mu \sin \varphi_2 \cos \phi}{\beta - \kappa^2 \cos^2 \phi}, \\ \ddot{\varphi}_2 = \frac{\kappa \sin \phi (\dot{\varphi}_1^2 + \kappa \cos \phi \dot{\varphi}_2^2) - \mu \sin \varphi_2 + \kappa \sin \varphi_1 \cos \phi}{\beta - \kappa^2 \cos^2 \phi}. \end{cases}$$

### 3. Geometrization

Let us consider the Jacobi metric  $\mathfrak{g}$  of the physical pendulum

$$\mathfrak{g} = 2^{\mathcal{W}} \mathbf{a} = 2^{\mathcal{W}} \begin{pmatrix} 1 & \kappa \cos \phi \\ \kappa \cos \phi & \beta \end{pmatrix}, \quad \mathcal{W} \equiv \mathcal{E} - 1 - \mu + \cos \varphi_1 + \mu \cos \varphi_2.$$

Next, we find the connection coefficients  $\Gamma^i_{jk}$ :

$$\Gamma^1_{11} = \frac{1}{2^{\circ}W \det \mathbf{a}} (2\kappa^2 \sin \varphi_1 \cos^2 \phi + {}^{\circ}W\kappa^2 \sin (2\phi) - \mu\kappa \sin \varphi_2 \cos \phi - \beta \sin \varphi_1),$$

$$\Gamma^2_{22} = \frac{1}{2^{\circ}W \det \mathbf{a}} (2\mu\kappa^2 \sin \varphi_2 \cos^2 \phi - {}^{\circ}W\kappa^2 \sin (2\phi) - \beta\kappa \sin \varphi_1 \cos \phi - \beta\mu \sin \varphi_2),$$

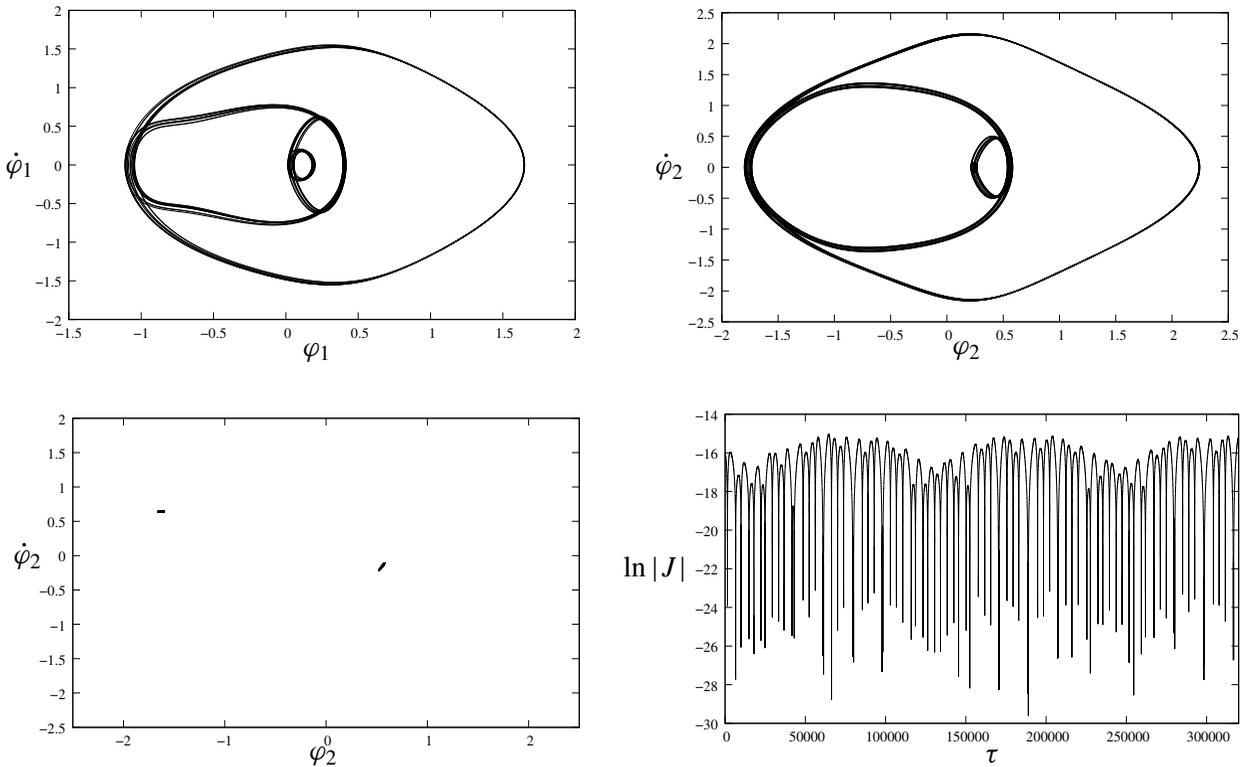
$$\Gamma^2_{11} = \frac{1}{2^{\circ}W \det \mathbf{a}} (\mu \sin \varphi_2 - \kappa \sin \varphi_1 \cos \phi - 2^{\circ}W\kappa \sin \phi),$$

$$\Gamma^1_{22} = \frac{\beta}{2^{\circ}W \det \mathbf{a}} (\beta \sin \varphi_1 - \mu\kappa \sin \varphi_2 \cos \phi + 2^{\circ}W\kappa \sin \phi),$$

$$\Gamma^2_{12} = \frac{1}{2^{\circ}W \det \mathbf{a}} (\mu\kappa \sin \varphi_2 \cos \phi - \beta \sin \varphi_1),$$

$$\Gamma^1_{12} = \frac{\beta}{2^{\circ}W \det \mathbf{a}} (\kappa \sin \varphi_1 \cos \phi - \mu \sin \varphi_2).$$

In a two-dimensional space there is only one nonzero component of the Riemann curvature tensor, namely,



**Figure 3.** Initial conditions:  $\varphi_2 = -1.63, \dot{\varphi}_2 = 0.63$ .

$$R_{2121} = \mu \cos \varphi_2 + 2^{\circ}W\kappa \cos \phi + \beta \cos \varphi_1 + \frac{1}{^{\circ}W} (\kappa \sin \varphi_1 \cos \phi - \mu \sin \varphi_2)^2 + \frac{\sin^2 \varphi_1 \det \mathbf{a}}{^{\circ}W} - \frac{\kappa \sin \phi}{\det \mathbf{a}} (\beta \kappa \sin \varphi_1 \cos \phi - \mu \kappa \sin \varphi_2 \cos \phi - \beta \mu \sin \varphi_2 + \beta \sin \varphi_1) - \frac{2^{\circ}W\kappa^3 \sin^2 \phi \cos \phi}{\det \mathbf{a}}.$$

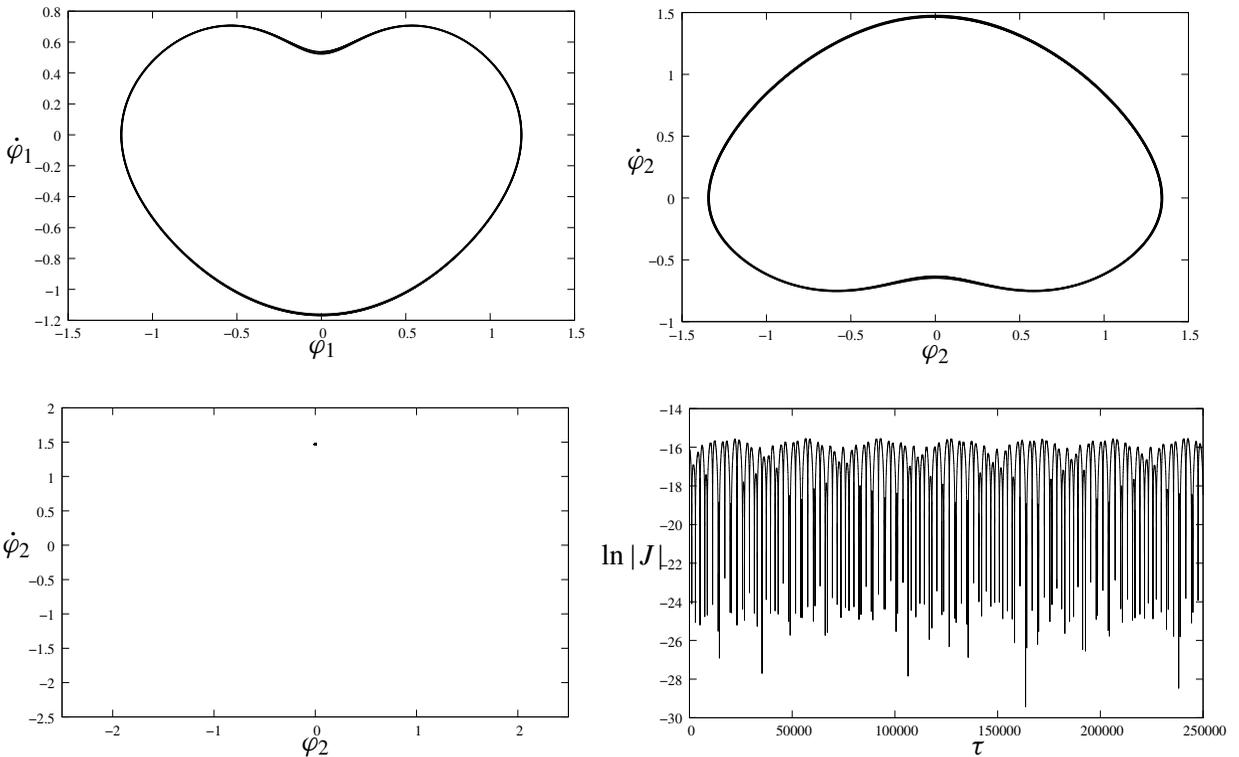
Making use of Equation (4) we find the scalar curvature:

$$\mathcal{R} = \frac{\kappa \cos(\phi)}{^{\circ}W \det \mathbf{a}} - \frac{\kappa^3 \sin^2 \phi \cos \phi}{^{\circ}W \det^2 \mathbf{a}} + \frac{\mu \cos \varphi_2 + \beta \cos \varphi_1}{2^{\circ}W^2 \det \mathbf{a}} - \frac{\kappa \sin \phi (\beta \kappa \sin \varphi_1 \cos \phi - \mu \kappa \sin \varphi_2 \cos \phi - \beta \mu \sin \varphi_2 + \beta \sin \varphi_1)}{2^{\circ}W^2 \det^2 \mathbf{a}} + \frac{\sin^2 \varphi_1}{2^{\circ}W^3} + \frac{(\kappa \sin \varphi_1 \cos \phi - \mu \sin \varphi_2)^2}{2^{\circ}W^3 \det \mathbf{a}}.$$

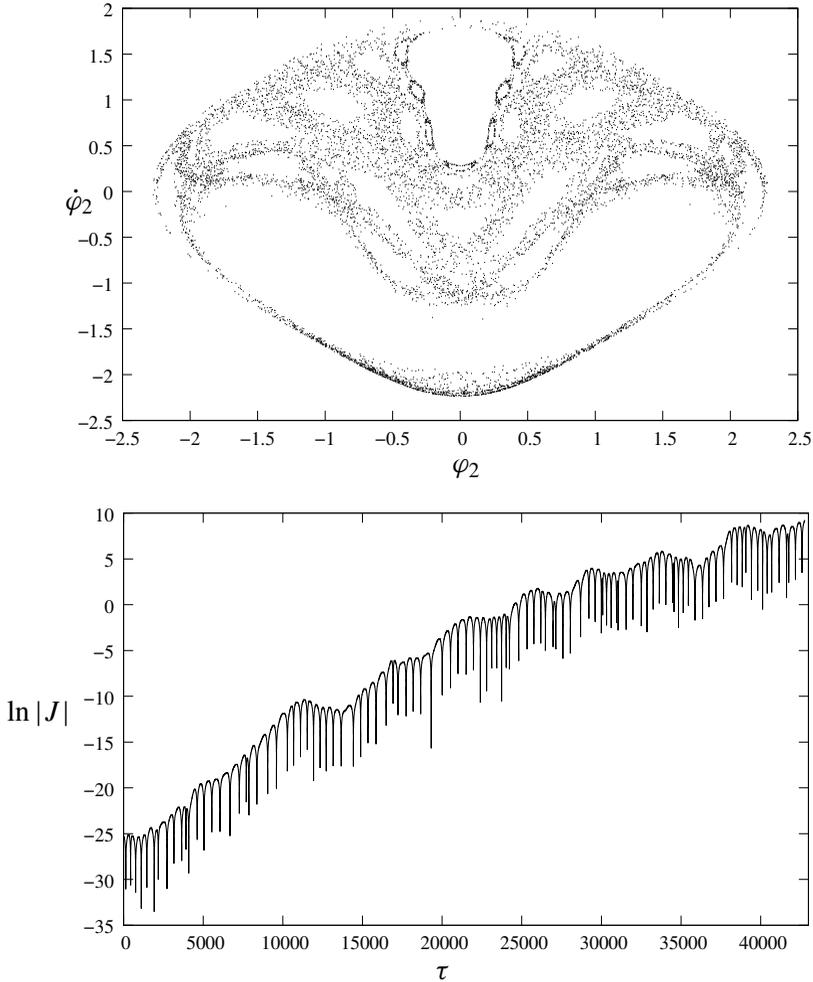
Finally, inserting the obtained scalar curvature into Equation (6) we get the JLC equation for the physical pendulum.

### 4. Numerical simulations

The equations of motion have been numerically solved by means of the symplectic algorithm of Strömer–Verlet [Hairer et al. 2006], whilst the JLC equation (6) has been solved by the Dormand-Prince 5(4)



**Figure 4.** Initial conditions:  $\varphi_2 = 0, \dot{\varphi}_2 = 1.46$ .



**Figure 5.** Initial conditions:  $\varphi_2 = -1.93, \dot{\varphi}_2 = -0.23$ .

algorithm with variable time-step size and the energy correction. Numerical simulation parameters were given the following values:  $\beta = 0.6, \kappa = 0.4, \mu = 0.66667$ . The simulation was performed for the total energy  $\mathcal{E} = 1.1$ . Below, we present the Poincaré section, in which one can observe chaotic regions as well as islands of regular behavior. Thus, we can analyze the system's behavior on the same energy level. The numerical results include three cases, namely two of them (Figures Figure 3, and Figure 4) from regions of regular behavior and the last (Figure 5) one from the chaotic region. The initial conditions of the regular behavior cases have been taken from the interior of the regular islands, so that trajectories stay in regular regions regardless of numerical errors. The presented figures include two projections of the phase trajectories (only in the case of regular behavior), the corresponding Poincaré section of a particular trajectory, and the graph, which presents the evolution rate of a solution of the JLC equation. One can easily observe that in the case of regular trajectories (Figures Figure 3, and Figure 4) the evolution of the Jacobi vector along the geodesic is bounded. However, in Figure 5 we can observe the unbounded evolution of the Jacobi vector, which means that two nearby geodesics originating from

the neighborhood of the initial condition move away from each other and hence the distance between them grows exponentially. This is caused by the parametric resonance occurring in the JLC Equation (5).

## 5. Concluding remarks

We have applied the Riemannian approach to a low dimensional system with constraints, and have shown that the geometric approach gives results that are in qualitative agreement with those obtained from the classical approach. The existence of constraints is manifest in the metric tensor, which has a nondiagonal form in this case. Although the obtained results show that there is an agreement between classical and geometric approaches, a more thorough analysis is needed. The aim of this approach is to make use of the Riemannian geometry tools to gain information about a system's behavior without referring to the geodesic evolution. The geometric approach has already been applied to systems that have no constraints and many degrees of freedom [Di Bari and Cipriani 1998; Casetti et al. 2000]. However, systems with few degrees of freedom and constraints are more difficult to analyze in this manner. Nevertheless, the obtained results are very promising and enable us to work out a more analytical way to analyze such systems within the geometric approach.

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