Journal of
Mechanics of
Materials and Structures

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Volume 2, № 8

October 2007

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 2, No. 8, 2007

STARK LADDER RESONANCES IN ACOUSTIC WAVEGUIDES

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We present a theoretical study on how to obtain a Wannier–Stark ladder in the transmission spectra of an acoustic wave traveling through a waveguide of variable cross section. Starting from Webster's equation for the acoustic pressure, we derive the necessary conditions to obtain the Wannier–Stark ladder. Furthermore, we present a numerical calculation for the transmission spectra when a Wannier–Stark ladder is present. This ladder is characterized by a family of well defined peaks, equidistant in frequency.

1. Introduction

Band structures of the energy spectrum of the electrons are the basis of electronic devices. When electrons travel through a periodic structure such as a crystal, the constructive and destructive interference gives rise to bands in the energy spectra of the electrons [Brillouin 1953; Guo 2006]. Similarly, an electromagnetic wave traveling through a structure with a dielectric function that varies periodically will exhibit a band structure in its frequency spectrum. This gives rise to photonic crystals and its applications in light flow control as described by Joannopoulos et al. [1995]. In an elastic structure with a specific impedance that varies periodically, the transmission spectra as a function of frequency, elastic waves will also show a band structure [Esquivel-Sirvent and Cocoletzi 1994].

Physically, the band structure represents regions of allowed and forbidden propagation as shown in Figure 1. λ represents the frequency (or energy) of a wave (or electron) traveling through a system. Figure 1(a) corresponds to a periodic system in which λ can get only certain allowed values indicated by the dark zones. The regions where no values of λ are permissible are known as forbidden bands or gaps. When the periodicity is only slightly modified, for example at only one site of the structure, the band structure shows localized states. This is, there are certain allowed values of λ for which transmission is allowed in an otherwise forbidden region. This is indicated by the dotted lines in Figure 1(b). Finally, Figure 1(c) shows that λ can take any value when there is no periodicity.

An interesting case of broken periodicity give rise to Wannier–Stark ladders (WSL) that will be discussed in the next section. By imposing a particular condition on the configuration of the system, it is possible to destroy the band structure of an otherwise periodic system, and obtain sharply localized states that are equidistant in frequency or energy. This resonances were first predicted in quantum mechanics by Wannier [1962] in connection with the energy spectrum of an electron traveling through a crystal in a dc electric field. As in the case of periodic system, it has been shown that WSL exist in photonic crystals [Monsivais et al. 1990], elastic systems [Mateos and Monsivais 1994] and piezoelectric systems [Monsivais et al. 2003].

Keywords: Stark, acoustic, resonances.
Partial support from CONACyT Grant No. 44306 and DGAPA-UNAM Grant No. IN101605 is acknowledged.

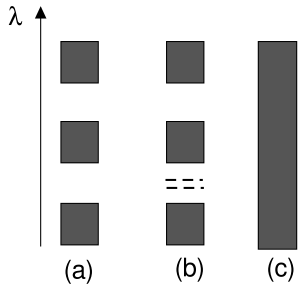

Figure 1. Allowed values of λ in three different structures: (a) periodic; (b) periodic with a defect, where the localized states are shown by dotted lines; (c) disordered.

In this paper we study the formation of WSL in acoustic waveguides. This system is simpler than the elastic case since there is no mode conversion. In addition, for low frequencies, it is easier to realize experimentally.

2. Theory

Up to constants, we can write Schroedinger's equation for stationary solutions as

$$-\frac{d^2\psi}{dx^2} + U(x)\psi = \lambda\psi. \tag{1}$$

If the potential $U(x)$ is periodic, that is, $U(x) = U(x+np)$ where p is the period and n a positive integer, the system will show a band structure [Brillouin 1953]. We can break the periodicity by requiring that

$$U(x+np) = U(x) + npF, \tag{2}$$

where F is a constant. In the case of an electron of charge q traveling through a crystal, $F = qE$, where E is a constant electric field. Making the change of variable $x = x' + npF$, Equation (1) becomes

$$-\frac{d^2\varphi(x')}{dx'^2} + U(x'+ndF)\varphi(x') = \lambda\varphi(x').$$

Finally, using the property of $U(x)$ given by Equation (2) we have

$$-\frac{d^2\varphi(x')}{dx'^2} + U(x')\varphi(x') = (\lambda - npF)\varphi(x'),$$

where $\varphi(x') = \psi(x'+npF)$. Comparing Equations (1) and (2), we see that if λ is an eigenvalue, then so is $\lambda - npF$. The difference between two neighboring eigenvalues is exactly pF; it is this that gives rise to a Wannier–Stark ladder.

We should mention, however, that the mathematics described above presents many subtleties [Zak 1968] and a rigorous description is very difficult. For this reason the existence of WSL in quantum mechanics was a controversial matter for twenty years, from the prediction of Wannier [1962] in the 1960's until the experimental observation of WSL in superlattices [Mendez et al. 1988] and the results of numerical calculations in, both in the 1980's.

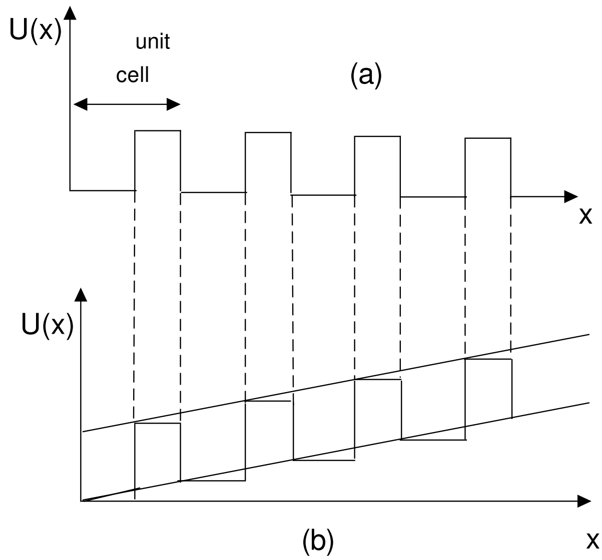

Figure 2. Schematic representation of the function U(x) given by Equation (2). (a) Periodic case obtained by setting $F = 0$. (b) If $F \neq 0$ the periodicity of $U(x)$ is broken and the solution of Equation (3) gives rise to a WSL.

The formulation discussed in this paper associated with the properties of the acoustic waveguide presents several differences compared with the quantum mechanical case. However, it suffers from similar mathematical subtleties, and, therefore, our discussion cannot be a rigorous demonstration of the existence of acoustic WSL. However, our numerical calculation will show that the naive formulation, in fact, predicts the correct result.

Now we will show how the above ideas can be adapted to an acoustic waveguide. Consider the equation for a waveguide with variable cross section $S(x)$ and symmetry axis parallel to the x-axis. Webster's equation for pressure $p(x,t)$ is

$$\frac{\partial^2 p(x,t)}{\partial t^2} = c^2\left(\frac{1}{S(x)}\frac{\partial}{\partial x}\left[S(x)\frac{\partial p(x,t)}{\partial x}\right]\right),$$

where c is the speed of sound in the waveguide. This equation can be transformed to a Schrodinger-like one by introducing a function $f(x,t)$ defined by $p(x,t) = f(x,t)/S(x)^{1/2}$. This particular choice comes from the fact that $f(x,t)$ is proportional to the potential energy per unit area of the acoustic wave [Forbes et al. 2003]. Thus, Webster's equation takes the form

$$\frac{\partial^2 f(x,t)}{\partial t^2} = c^2\left(\frac{\partial^2 f(x,t)}{\partial x^2} - U(x) f(x,t)\right), \qquad U(x) = \frac{1}{S(x)^{1/2}}\frac{d^2 S(x)^{1/2}}{dx^2}. \tag{3}$$

Equation (3) is separable in the variables x, t, and its solutions are of the form $f(x,t) = X(x)T(t) = X(x)\exp(i\omega t)$, where ω is the wave frequency. Substituting this ansatz into Equation (3) yields the

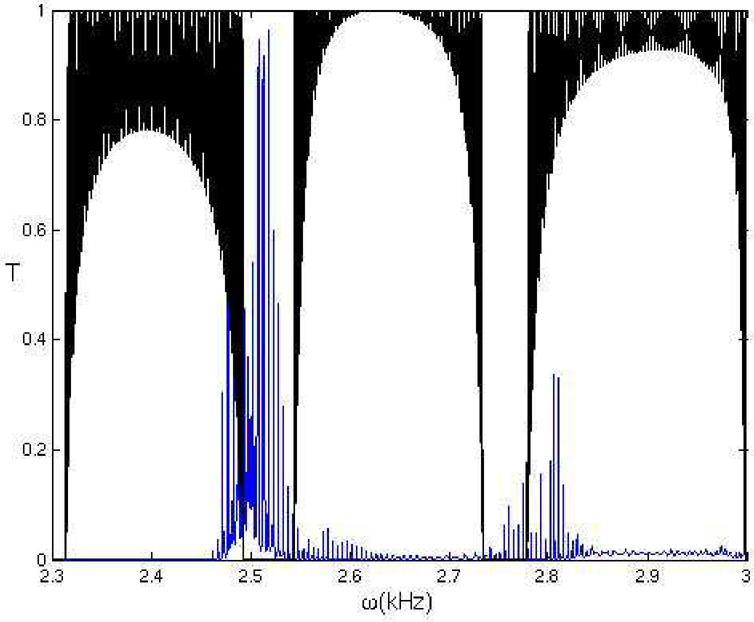

Figure 3. Transmission spectra for an acoustic waveguide. Period of unit cell is 5 cm. Black curve shows result for a periodic system made of 60 unit cells. For $F = 1$ the band structure disappears and WSL emerges, demonstrated by the blue curve.

following equation for $X(x)$:

$$\frac{d^2X(x)}{dx^2} + \left[\frac{\omega^2}{c^2} - U(x)\right]X(x) = 0, \tag{4}$$

which has the form of Schrodinger's equation. By imposing condition (2) for the $U(x)$, the existence of Wannier–Stark ladders can be expected. We can now construct the function $U(x)$ subject to the required condition. The actual variation of the cross section $S(x)$ is obtained by solving Equation (3). However, this is not done in this paper, and will be reported elsewhere. First we notice that when $F = 0$, we have a periodic function constructed by repeating a unit cell as shown in Figure 2(a). When $F \neq 0$ we choose a function profile as that shown in Figure 2(b). This profile will be used in the numerical calculations presented in the next section.

3. Numerical results

In this section we consider a finite system in order to have a model to analyze the existence of WSL numerically. Since for a finite system Equation (2) is only satisfied for a finite region of space, it can be expected that WSL formation will not be perfect.

The transmission spectra are calculated for a particular function $U(x)$ of Figure 2(b). To perform the calculations, we consider that the changes in $U(x)$ imply a change in the cross section of the waveguide. This in turn implies a change in the acoustic impedance that also depends on $S(x)$. At each change

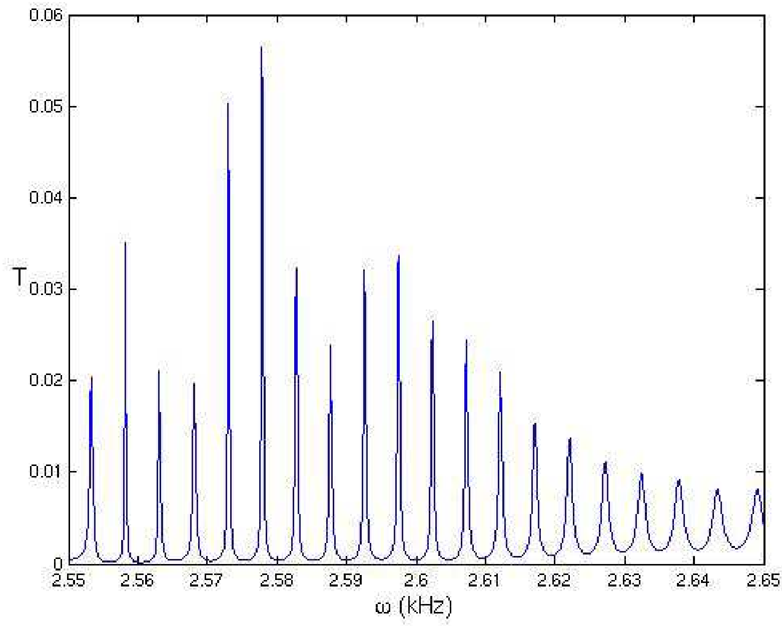

Figure 4. Detail of WSL in Figure 3: sharp, equidistant peaks are evident.

in the cross section, the boundary conditions are given by the continuity of $X(x)$ and of its derivative $dX(x)/dx$. A transfer matrix approach is used to calculate the transmission spectra [Esquivel-Sirvent and Cocoletzi 1994].

In Figure 3 we show two curves. The black line shows the transmission spectra for a periodic system made of 60 unit cells as described in Figure 2(a). The band structure is clearly seen. The number of oscillations in the regions of high transmission is equal to the number of unit cells. In the case of an infinite number of unit cells, the transmission will be equal to one in these regions. If we break the periodicity by setting $F = 1$, we obtain the blue curve. The periodicity is broken and the band structure is replaced by a series of sharp transmission peaks. This is the Wannier–Stark ladder. In Figure 4 we show a detail of the transmission spectra. The spikes are equally spaced and are sharply peaked as expected. For these calculations we took a unit cell of period 5 cm. As mentioned before, the peaks in WSL are equidistant, but their separation is not pF as predicted by the theory since the system we analyzed was finite.

The parameters we choose to construct the function $U(x)$ were adequate to obtain a WSL. However, we have observed that not any choice of parameters gives rise to WSL. It is not possible to know when a Stark ladder will emerge, except by trial and error.

4. Conclusions

In this paper we have demonstrated the basic principles to obtain the Wannier–Stark ladder in the transmission spectra of an acoustic waveguide. Starting with Webster's equation, we find an equivalent

Schrodinger-like equation that exhibits a Stark ladder for a suitable choice of a function $U(x)$. In our case, this function is related to the changes in cross section of the acoustic waveguide.

Our numerical studies show that the Stark ladder in acoustic waveguides in fact exist, even though this cannot be rigorously proven from the theoretical analysis, as was the case in quantum mechanics.

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Received 18 Jul 2006. Accepted 20 Apr 2007.

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