Journal of
Mechanics of
Materials and Structures

ELASTIC WANNIER–STARK LADDERS IN TORSIONAL WAVES

Guillermo Monsivais, Rafael A. Méndez-Sánchez,
Alfredo Díaz de Anda, Jorge Flores, Luis Gutiérrez
and Alejandro Morales

Volume 2, Nº 8

October 2007

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 2, No. 8, 2007

ELASTIC WANNIER–STARK LADDERS IN TORSIONAL WAVES

GUILLERMO MONSIVAIS, RAFAEL A. MÉNDEZ-SÁNCHEZ, ALFREDO DÍAZ DE ANDA, JORGE FLORES, LUIS GUTIÉRREZ AND ALEJANDRO MORALES

We study the normal modes of torsional waves in an elastic beam consisting of a set of N cuboids of varying heights. We present experimental, theoretical, and numerical results. We show that some analogies to the Wannier–Stark ladders resonances, originally introduced by Wannier in 1962, are exhibited by this classical system. The original ladders studied by Wannier consist of a series of equidistant energy levels for the electrons in a crystal in the presence of a static external electric field with the nearest-neighbor level spacing proportional to the intensity of the external field. For the case of torsional waves in the beam we have observed a similar behavior, namely, the vibrations of the beam show resonances of equidistant frequencies with the nearest-neighbor spacing proportional to parameter γ associated with the geometry of the beam analogously to the electric field. However, this analogy is not perfect; we address the origin of the differences.

1. Introduction

Since the discovery of the wavelike behavior of particles whose size is on the order of atomic dimensions or smaller, several analogies between quantum systems, that is, systems whose dynamics is governed by quantum mechanics, and classical systems have been observed. This is particularly true when the undulatory properties of the particles are important and interference phenomena play the relevant role. Thus, one frequently finds analogies in optics, electromagnetism, acoustics, elasticity and the like. It is in the case of optics where more analogies have been studied; see [Monsivais et al. 1990; Sheng 1995; Joannopoulos et al. 1995; Soukoulis 1996; de Sterke et al. 1998; Sapienza et al. 2003; Agarwal et al. 2004] and references therein. In some cases the analogies are not exact, and new and interesting characteristic effects appear for each field. In other cases the classical systems that potentially can present analogies, do not exist in a natural way, but can be built from an appropriate combination of other systems.

In this paper we study the analogy of the quantum mechanical phenomenon known as Wannier–Stark Ladder Resonances (WSLR) in a special type of classical elastic system. The study includes experimental, theoretical and numerical results. The existence of WSLR and their associated Bloch Oscillations (BO) in quantum mechanics have been controversial for many years, but by now some of their properties seem to be theoretically on firm ground. The BO were predicted by Bloch [1928]; see also [Zener 1934; James 1949; Wannier 1955]. They consist of a counterintuitive behavior of electrons in a crystal, which is under the action of a static external electric field. According to Bloch, this static field produces an oscillatory movement of the electrons inside the crystal. This strange prediction was a controversial matter for more than 60 years [Hart and Emin 1988]. The controversy waxed due to Wannier's 1962 discovery [Wannier

Keywords: Wannier–Stark ladders, elastic waves, EMAT.
This work has been supported by UNAM DGAPA-PAPIIT Projects IN111307 and IN104102.

1962; 1969; Zak 1968; 1969]. This discovery establishes that the electronic energy spectrum consists of a series of energies $E_1, E_2, \ldots$ such that the nearest-neighbor level spacing is constant and proportional to the intensity of the external field. This set of energies forms the so called Wannier–Stark ladder. These amazing characteristics contrast with what occurs in a nonelectrified crystal, where the electrons travel through the whole periodic structure (Bloch waves) and where the constructive and destructive interferences give rise to an energy band spectrum, which consists of bands of allowed energy and regions where no values of the energy are permissible, forbidden bands or gaps [Brillouin 1946]. Thus, according to Bloch and Wannier, when the electric field modifies the periodic potential, the band structure is destroyed and states are localized.

These predictions were very important since the band structure is the basis of electronic devices. However, when Bloch and Wannier made their predictions it was impossible to test them experimentally. On the one hand, the BO are difficult to observe because the electrons lose their coherence in times shorter than the expected period of the oscillations. On the other hand, the Wannier–Stark ladders are difficult to observe because the width of the levels is larger than the separation between levels. Obviously both effects are related since a short lifetime of a state implies a wider associated energy level. The first confirmation of the Bloch–Wannier model came from the observations of the WSLR that appeared first in numerical experiments [Rabinovitch 1977; Banavar and Coon 1978], and thereafter from the laboratory [Méndez et al. 1988]. This was around 20 years after the prediction of Wannier. Later on, in 1992, the BO were also observed [Feldmann et al. 1992; Leo et al. 1992; Dekorsy et al. 1994; Lyssenko et al. 1998]. This occurred when the semiconductor superlattices were built [Esaki and Tsu 1970], since in these systems the period of BO is shorter. Actually, there exists considerable literature on the WSLR and the BO, and now it is recognized that the original ideas of Bloch and Wannier are essentially correct. The BO are due to the fact that when an electron in the crystal is accelerated by the electric field, its velocity is increased until it reaches the end of the Brillouin zone, where it is dispersed and its velocity decreases. This effect continues until the velocity is equal to zero and changes sign returning to the original position inside the crystal. Then, the electron is again accelerated by the field and the cycle is repeated. Under these circumstances the wave functions are localized in the zones where the oscillatory movement occurs. We should mention, however, that there always exists a probability that the electron tunnels to other regions of the crystal (Zener tunneling). Whenever this probability is small, the BO can be present.

The origin of the WSLR can be understood as follows. Consider the time independent Schrödinger's equation for an electron of charge e in a one-dimensional crystal in the presence of a static external electric field F,

$$-\frac{1}{2}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi, \tag{1}$$

where we are using a system of units in which the electron mass and Plank's constant are set to 1. The function $V(x)$ is the potential acting on the electron, E is the energy of the electron and $e|\psi|^2$ is the charge density. In our case $V(x)$ has the form $V(x) = V_p(x) + eFx$, where $V_p(x)$ is the periodic potential due to the atoms of the crystal, that is $V_p(x) = V_p(x + np)$, p being the period and n an arbitrary integer. When $F = 0$, the potential is periodic and the system will show a band structure [Brillouin 1946]. However, when $F \neq 0$, the periodicity is broken and the potential acquires the crucial property $V(x + np) = V(x) + neFp$. Making the change of variable $x = x' + np$ and using the above property of $V(x)$, Equation (1) becomes

$$-\frac{1}{2}\frac{d^2\varphi(x')}{dx'^2}+V(x'+np)\varphi(x')=E\varphi(x') \quad \Longrightarrow \quad -\frac{1}{2}\frac{d^2\varphi(x')}{dx'^2}+V(x')\varphi(x')=(E-neFp)\varphi(x'), \quad (2)$$

where $\varphi(x') = \psi(x' + np)$. Comparing Equations (1) and (2), we see that if E is an eigenvalue, then $E - neFp$ is also an eigenvalue. The difference between two consecutive eigenvalues is eFp and the Wannier–Stark ladder is formed. We should mention, however, that the simple mathematical derivation just described above is rather subtle [Zak 1968; 1969; Wannier 1969; Rabinovitch and Zak 1971] and a rigorous description is very difficult. For this reason, as mentioned before, the existence of the Wannier–Stark ladder in quantum mechanics was a controversial matter for many years. Actually, one finds that the energies forming a ladder are indeed a set of resonances embedded in a continuous energy spectrum. This is why the Wannier–Stark ladders are called WSLR.

Several systems whose behavior is governed by classical physics and which present analogous phenomena to the band structure, BO, and WSLR, have been studied up to now. For example, an electromagnetic wave traveling through a structure with a dielectric function that varies periodically will exhibit a band structure, which in turn gives rise to photonic crystals. Applications of photonic crystals in light flow control have been described by Joannopoulos et al. [1995]. In an elastic structure with a specific impedance that varies periodically, the transmission spectra of elastic waves will also show a band structure [Esquivel-Sirvent and Cocoletzi 1994]. For theoretical studies of the BO and WSLR analogies see [Monsivais et al. 1990; Mateos and Monsivais 1994; de Sterke et al. 1998; Monsivais et al. 2003] and references therein. However, there are relatively few experimental studies [Sapienza et al. 2003; Agarwal et al. 2004], and it is just in this context that this paper is placed. We show experimentally that it is possible to find a WSLR-like structure in the spectrum of frequencies associated with torsional waves of special beams. We also use a numerical model whose predictions are in excellent agreement with our measurements. We have used our numerical model to show that the separation between the frequencies depends linearly on the parameter that plays the role of the static external electric field. However, we will see that the analogies are not exact.

2. The physical system

The system analyzed in this paper is a special elastic beam described below. It is well known that in any elastic system there exist several types of waves. However, for the case of beams, and in the range of frequencies and wave lengths used in this work, it can be assumed that only three types of vibrations exist: compressional, torsional and flexural; see Figure 1. For the present study we have considered only torsional vibrations.

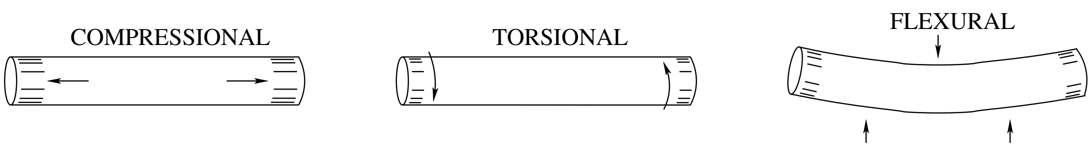

Figure 1. Three types of vibrations in a beam: compressional, torsional and flexural.

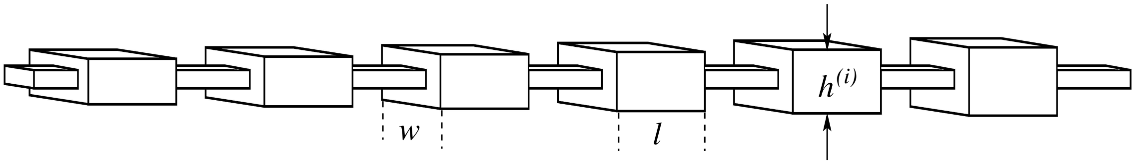

Figure 2. Beam used to obtain Wannier–Stark ladder resonances. Cuboids have same length $l = 5\,\text{cm}$ and width $w = 1.905\,\text{cm}$, but different heights. $v^{(i)} = (1 + i\gamma)v$, $i = 1, 2, \ldots, 15$ with $v = 2027.3$ m/s and $\gamma = 0.02786$. The width, height, and length of the small cuboids are $w' = 5\,\text{mm}$, $h' = 5\,\text{mm}$, and $l' = 6\,\text{mm}$, respectively.

The elastic system is depicted in Figure 2. It was constructed by machining a solid aluminum piece whose original shape was a beam with rectangular cross section. The result of the machining is a set on N cuboids or subbeams of constant width w and constant length l. They have different heights $h^{(i)}$, with $w, h^{(i)} \ll l$ for $i = 1, 2, \ldots, N$. These cuboids are separated by other small cuboids of dimensions $w', l', h' \ll l$, where $w' = h'$. We have observed that the behavior of machined systems is different from the behavior of similar systems constructed by welding their different parts. This property can be used to carry out nondestructive tests on the systems.

We now discuss the rule used to assign the values of the heights $h^{(i)}$. The procedure is different from the formulation used by Wannier just discussed above because the torsional waves are described by an equation different from the one describing the electrons in an electrified crystal. To design the beam we are guided by a qualitative analysis of what could be called an independent beam model in which each body oscillates independently form the rest.

It is well known that torsional waves are described by the equation [Graff 1975]

$$\frac{\partial^2 \theta}{\partial x^2} - \left(\frac{1}{v}\right)^2 \frac{\partial^2 \theta}{\partial t^2} = 0,$$

where v is the velocity of the waves and $\theta = \theta(x, t)$ the angle of rotation of the cross section at point x and time t. The x-axis lies on the axis of the beam. We now apply this equation to the torsional normal modes of the i-th cuboid, for $i = 1, 2, \ldots, N$, with free ends. If we denote by n_n the number of nodes of this mode and by $\omega^{(i)}_{n_n}$ its angular frequency, the above equation becomes

$$\frac{\partial^2 \theta}{\partial x^2} + \left(k^{(i)}_{n_n}\right)^2 \theta = 0,$$

where $k^{(i)}_{n_n}$ is the wave number of the mode given by $k^{(i)}_{n_n} = \omega^{(i)}_{n_n}/v^{(i)} = 2\pi/\lambda^{(i)}_{n_n}$, where $\lambda^{(i)}_{n_n}$ is the wave length and $v^{(i)}$ the velocity of the waves in the i-th cuboid. It is clear that the length l of a cuboid with free ends is related to $\lambda^{(i)}_{n_n}$ via $l = \lambda^{(i)}_{n_n} n_n/2$, which implies that the angular frequency $\omega^{(i)}_{n_n}$ is given by

$$\omega^{(i)}_{n_n} = \pi v^{(i)} n_n / l. \tag{3}$$

To obtain a set of equidistant frequencies (for a given value of n_n) we look for a set of velocities $\{v^{(i)}\}$ such that $v^{(i)} = (1 + i\gamma)v$, where v is an arbitrary constant velocity. The parameter γ is dimensionless.

We then obtain

$$\omega_{n_n}^{(i)} = \frac{\pi v(1+i\gamma)}{l} n_n, \qquad \Delta_{n_n} \equiv \Delta\omega_{n_n}^{(i)} = \frac{\pi v\gamma}{l} n_n, \tag{4}$$

for the frequencies and their differences (for a given value of n_n) $\Delta\omega_{n_n}^{(i)} = \omega_{n_n}^{(i)} - \omega_{n_n}^{(i-1)}$. Since the latter are independent of i, we have dropped the index i and defined Δ_{n_n}. In this model of independent beams, we therefore obtain sets of equidistant frequencies for each value of n_n. The required set of velocities $\{v^{(i)}\}$ can be obtained by taking appropriate values for the heights $h^{(i)}$ as described below. We should mention that this procedure is not possible for cylindrical bars, since the torsional wave velocity in cylindrical bars is independent of the radius.

There is, however, another possibility to have equidistant frequencies. It consists in taking different lengths $l^{(i)}$ for the different subbeams in Equation (3), with $l^{(i)} = l/(1+i\gamma)$, but this possibility will not be considered here. The corresponding analysis has been published recently [Gutiérrez et al. 2006].

To obtain the velocities $\{v^{(i)}\}$ we have used the expression derived by Navier [1827] for the torsional velocity in the i-th cuboid

$$v^{(i)} = \sqrt{\frac{G}{\rho}}\sqrt{\frac{\alpha^{(i)}}{I^{(i)}}},$$

where $I^{(i)} = (h^{(i)}w^3 + [h^{(i)}]^3 w)/12$ is the moment of inertia with respect to the axis of the system, ρ is the density, G is the shear modulus, and $\alpha^{(i)}$ is given by

$$\alpha^{(i)} = \frac{256}{\pi^6}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{1}{(2m+1)^2(2p+1)^2}\;\frac{h^{(i)}w}{[(2m+1)/h^{(i)}]^2 + [(2p+1)/w]^2}.$$

We can solve these equations to obtain the values of $h^{(i)}$ such that $v^{(i)}$ take the required value $(1+i\gamma)v$. Figure 3 shows a plot of v as a function of h for particular values of the parameters w and $\sqrt{G/\rho}$. This figure also shows a comparison with experimental results.

We now return to discuss the properties of the whole beam constructed by machining a solid piece as shown in Figure 2. We have used the above procedure to calculate the N heights $\{h^{(i)}\}$ of the cuboids or subbeams forming the beam. When the parameter γ is equal to zero, a locally periodic beam is formed. This kind of locally periodic beams shows a discrete band spectrum [Morales et al. 2002]. If we break the periodicity by setting $\gamma \neq 0$, a completely different spectrum occurs. The discrete band structure disappears and the new spectrum resembles the WSLR. We see from Equation (4) that γ here plays the role of the electric field F for the quantum mechanical ladders.

Before presenting the calculations of the normal modes for this system and showing numerical and experimental results, let us make a qualitative analysis to see what type of spectrum could be expected from the independent beam model. At the lowest frequencies, the wavelength λ is of the same order of magnitude as $L \approx lN$ and the whole beam is excited. When ω increases and λ becomes of the order of l, the length of the subbeams, the state equivalent to its lowest normal mode is excited. This occurs at the first subbeam since it has the smallest velocity $v^{(1)} = (1+\gamma)v$; furthermore, this corresponds to the subbeam with the smallest $h^{(i)}$; see Figure 3. The rest of the subbeams are out of resonance, so the amplitude decreases as one moves away from subbeam 1. Therefore, the state is localized around the latter. In some sense this was to be expected since we are disturbing a periodic structure to obtain a disordered one-dimensional system, which always shows localized wave amplitudes. Increasing the exciting frequency by Δ_1, the subbeam with velocity $v^{(2)} = (1 + 2\gamma)v$, that is, subbeam 2, will now be excited and the rest will be out of resonance. The amplitude of the vibrations therefore decreases as the distance from subbeam 2 increases. The wave amplitude is again localized but now around subbeam 2; it has a similar shape as the wave amplitude that subbeam 1 had previously. The same arguments apply when subbeam i of velocity $v^{(i)} = (1 + i\gamma)v$ is excited.

What we have done is to produce a finite Wannier–Stark ladder, that is, N localized states with constant difference in frequency given by Equation (4). However, more ladders exist since normal modes with two or more nodes can also be excited in each subbeam. For instance, taking $n_n = 2$ in Equation (4), a second ladder is obtained. This ladder is different from the first one because the frequency difference is now twice the one of the lower ladder, as can be seen from Equation (4). The states are again localized and all have similar shape. A third ladder exists with $\Delta_3 = 3\Delta_1$ and so on for the other values of n_n. The difference between the quantum-mechanical WSLR [Thommen et al. 2004] and the ladders discussed here is that in the latter the spacing between resonances is not the same for different ladders.

To measure normal modes frequencies and amplitudes we have used an Electromagnetic Acoustic Transducer (EMAT) which has been recently developed [Morales et al. 2001; 2002]. This EMAT is versatile and operates at low frequencies. It can selectively excite or detect compressional, flexural or torsional vibrations. We have also calculated these quantities using the transfer matrix method for torsional waves with the following boundary conditions between the different sections of the beam

$$\theta^{(i)}(x)\Big|_{x=x_i} = \theta^{(i+1)}(x)\Big|_{x=x_i}, \qquad \beta^{(i)} \frac{d\theta^{(i)}(x)}{dx}\Bigg|_{x=x_i} = \beta^{(i+1)} \frac{d\theta^{(i+1)}(x)}{dx}\Bigg|_{x=x_i},$$

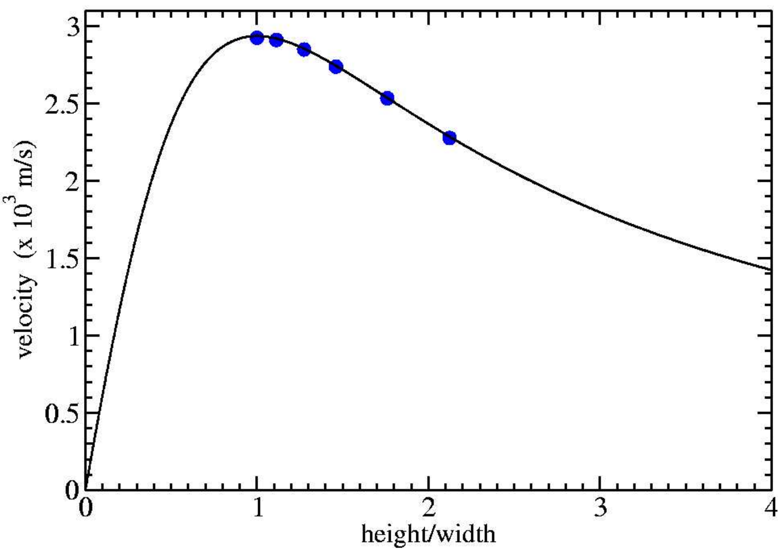

Figure 3. Navier prediction for velocity v as a function of height h for cuboids (continuous line). Experimental values (points) fit the prediction. Here $\sqrt{G/\rho} = 3190$ m/s and $w = 1.905$ cm.

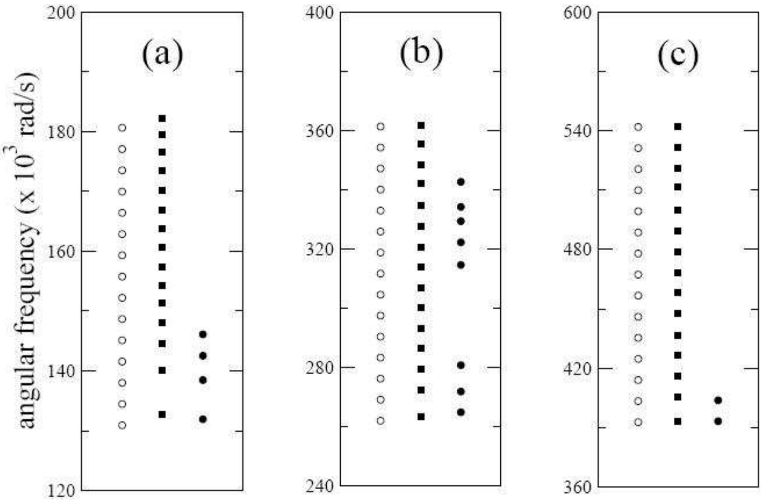

Figure 4. Normal mode angular frequencies of beam in Figure 2 yielding the elastic Wannier–Stark effect: (a) $n_n = 1$, (b) $n_n = 2$, (c) $n_n = 3$. In each figure, left column shows frequencies from the independent beam model, middle column — from the transfer matrix model, right column — measured in the laboratory.

where $\beta^{(i)}$ is the square of the cross-section area of the i-th beam. Free end boundary conditions were used as discussed by Morales et al. [2002]. Our calculation shows explicitly that the frequency difference Δ_{n_n} is proportional to the parameter γ. Furthermore, as mentioned above, for $\gamma = 0$ a discrete band spectrum appears, and as γ grows the levels of each band separate to form the WSLR. In Figure 4 we show the theoretical normal mode frequencies and the values obtained from the independent beam model of the system shown in Figure 2 for $\gamma = 0.02786$. We see that this model provides a rather good first approximation. As also shown in this figure, the experimental values are very well reproduced by the theoretical values.

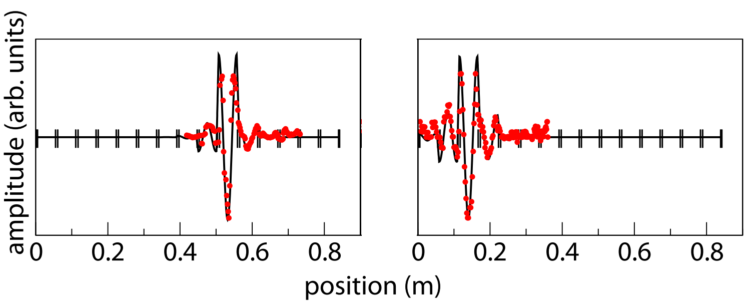

Figure 5. Two wave amplitudes for torsional waves of beam in Figure 2 associated with the second ladder $n_n = 2$, the left one localized on the tenth subbeam, the right one on the third. Double small vertical lines along beam axis indicate the position of small cuboids. Points are experimental values, while solid line gives calculated values.

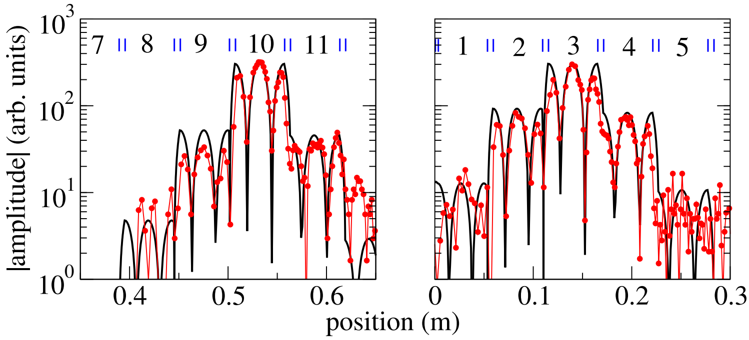

Figure 6. Logarithmic plot of the absolute value of wave amplitudes shown in Figure 5. Points are experimental values, while solid line gives calculated values.

One can clearly see from Figure 4 that the states form a set of WSLR. The first band composed by extended modes is not displayed in this graph. Notice that the frequencies in the extremes of each ladder do not have the same frequency difference as those in the middle of the ladder. This is due to a border effect in the wave amplitudes localized near the free ends [Gutiérrez et al. 2006].

In Figure 5 we show as an example the comparison between theoretical and experimental wave amplitudes for two states of the second WSLR. These are localized around particular subbeams. For example, in Figure 5 (left) the tenth subbeam resonates while in Figure 5 (right) the third subbeam resonates. Both wave amplitudes have the same form and, as expected, the amplitudes show two nodes at the resonating subbeams. Note that we again have excellent agreement between experiment and theory, where we have used the one-dimensional transfer method in spite of the fact that w and $h^{(i)}$ are not much smaller than l as required by the method. We have also calculated the wave amplitudes for states of the first ladder. Localization is again observed, and the amplitudes show one node at the resonating subbeams.

In Figure 6 we show the theoretical and experimental values of the logarithm of the wave amplitude corresponding to the states of Figure 5. The plots show that the envelopes of the amplitudes of the normal modes are exponentially localized.

3. Conclusions

In this paper we have constructed an elastic analogue of the WSLR in the torsional frequency spectra of some special beams. Starting with the independent beam model, we find the appropriate geometry of the bars in order to have the WSLR. The geometry is related to the cross section of the subbeams forming the whole beam, as indicated by the Navier formula, which we tested experimentally for the first time. In contrast. In contrast with the optical analogue of [Sapienza et al. 2003; Agarwal et al. 2004], we have observed the WSLR directly. Furthermore, we have measured the wave amplitudes, including phases, which show exponential localization. We also observed higher WSLR. Our numerical studies are in close agreement with experimental results. The elastic WSLR have potential applications in the design of systems with localized vibrations.

References

[Agarwal et al. 2004] V. Agarwal, J. A. del Río, G. Malpuech, M. Zamfirescu, A. Kavokin, D. Coquillat, D. Scalbert, M. Vladimirova, and B. Gil, "Photon Bloch oscillations in porous silicon optical superlattices", *Phys. Rev. Lett.* **92** (2004), #097401.

[Banavar and Coon 1978] J. R. Banavar and D. D. Coon, "Widths and spacing of Stark ladder levels", *Phys. Rev. B* **17**:10 (1978), 3744–3749.

[Bloch 1928] F. Bloch, "Über die Quantenmechanik der Electronen in Kristallgittern", *Z. Phys.* **52** (1928), 555–600.

[Brillouin 1946] L. Brillouin, *Waveguide propagation in periodic structures; electric filters and crystal lattices*, McGraw Hill, New York, 1946. Reprinted Dover, New York, 1953.

[Dekorsy et al. 1994] T. Dekorsy, P. Leisching, K. Köhler, and H. Kurz, "Electro-optic detection of Bloch oscillations", *Phys. Rev. B* **50**:11 (1994), 8106–8109.

[Esaki and Tsu 1970] L. Esaki and R. Tsu, "Superlattice and negative differential conductivity in semiconductors", *IBM J. Res. Dev.* **14**:1 (1970), 61–65.

[Esquivel-Sirvent and Cocoletzi 1994] R. Esquivel-Sirvent and G. H. Cocoletzi, "Band structure for the propagation of elastic waves in superlattices", *J. Acoust. Soc. Am.* **95**:1 (1994), 86–90.

[Feldmann et al. 1992] J. Feldmann, K. Leo, J. Shah, D. A. B. Miller, J. E. Cunningham, T. Meier, G. von Plessen, A. Schulze, P. Thomas, and S. Schmitt-Rink, "Optical investigation of Bloch oscillations in a semiconductor superlattice", *Phys. Rev. B* **46**:11 (1992), 7252–7255.

[Graff 1975] K. F. Graff, *Wave motion in elastic solids*, Clarendon, Oxford, 1975. Reprinted Dover, New York, 1991.

[Gutiérrez et al. 2006] L. Gutiérrez, A. D. de Anda, J. Flores, R. A. Méndez-Sánchez, G. Monsivais, and A. Morales, "Wannier–Stark ladders in one-dimensional elastic systems", *Phys. Rev. Lett.* **97** (2006), #114301.

[Hart and Emin 1988] C. F. Hart and D. Emin, "Time evolution of a Bloch electron in a constant electric field", *Phys. Rev. B* **37**:11 (1988), 6100–6104.

[James 1949] H. M. James, "Electronic states in perturbed periodic systems", *Phys. Rev.* **76**:11 (1949), 1611–1624.

[Joannopoulos et al. 1995] J. D. Joannopoulos, R. D. Meade, and J. N. Winn, *Photonic crystals: molding the flow of light*, Princeton University Press, Princeton, NJ, 1995.

[Leo et al. 1992] K. Leo, P. H. Bolivar, F. Brüggemann, R. Schwedler, and K. Köhler, "Observation of Bloch oscillations in a semiconductor superlattice", *Solid State Commun.* **84**:10 (1992), 943–946.

[Lyssenko et al. 1998] V. G. Lyssenko, M. Sudzius, F. Loser, and G. Valusis, "Bloch oscillations in semiconductor superlattices", *Adv. Solid State Phys.* **38** (1998), 257–268.

[Mateos and Monsivais 1994] J. L. Mateos and G. Monsivais, "Stark-ladder resonances in elastic waves", *Physica A* **207**:1-3 (1994), 445–451.

[Méndez et al. 1988] E. E. Méndez, F. Agulló-Rueda, and J. M. Hong, "Stark localization in GaAs-GaAlAs superlattices under an electric field", *Phys. Rev. Lett.* **60**:23 (1988), 2426–2429.

[Monsivais et al. 1990] G. Monsivais, M. del Castillo-Mussot, and F. Claro, "Stark-ladder resonances in the propagation of electromagnetic waves", *Phys. Rev. Lett.* **64**:12 (1990), 1433–1436.

[Monsivais et al. 2003] G. Monsivais, R. Rodríguez-Ramos, R. Esquivel-Sirvent, and L. Fernández-Álvarez, "Stark-ladder resonances in piezoelectric composites", *Phys. Rev. B* **68**:17 (2003), #174109.

[Morales et al. 2001] A. Morales, L. Gutiérrez, and J. Flores, "Improved eddy current driver-detector for elastic vibrations", *Am. J. Phys.* **69**:4 (2001), 517–522.

[Morales et al. 2002] A. Morales, J. Flores, L. Gutiérrez, and R. A. Méndez-Sánchez, "Compressional and torsional wave amplitudes in rods with periodic structures", *J. Acoust. Soc. Am.* **112**:5 (2002), 1961–1967.

[Navier 1827] M. Navier, "Mémoire sur lois de l'équilibre et du mouvement des corps solides élastiques", *Mém. Acad. Roy. Sci. Inst. France* **7** (1827), 375–393.

[Rabinovitch 1977] A. Rabinovitch, "Stark ladders in finite crystals!", *Phys. Lett. A* **59**:6 (1977), 475–477.

[Rabinovitch and Zak 1971] A. Rabinovitch and J. Zak, "Electrons in crystals in a finite-range electric field", *Phys. Rev. B* **4**:8 (1971), 2358–2370.

[Sapienza et al. 2003] R. Sapienza, P. Costantino, D. Wiersma, M. Ghulinyan, C. J. Oton, and L. Pavesi, "Optical analogue of electronic Bloch oscillations", *Phys. Rev. Lett.* **91**:26 (2003), #263902.

[Sheng 1995] P. Sheng, *Introduction to wave scattering,localization, and mesoscopic phenomena*, Academic Press, New York, 1995.

[Soukoulis 1996] C. M. Soukoulis (editor), *Photonic band gap materials* (Elounda, Greece, 1995), NATO ASI series E **315**, Kluwer, Dordrecht, 1996.

[de Sterke et al. 1998] C. M. de Sterke, J. N. Bright, P. A. Krug, and T. E. Hammon, "Observation of an optical Wannier–Stark ladder", *Phys. Rev. E* **57**:2 (1998), 2365–2370.

[Thommen et al. 2004] Q. Thommen, J. C. Garreau, and V. Zehnlé, "Quantum interference in a driven washboard potential", *Am. J. Phys.* **72**:8 (2004), 1017–1025.

[Wannier 1955] G. H. Wannier, "Possibility of a Zener effect", *Phys. Rev.* **100**:4 (1955), 1227–1227.

[Wannier 1962] G. H. Wannier, "Dynamics of band electrons in electric and magnetic fields", *Rev. Mod. Phys.* **34**:4 (1962), 645–655.

[Wannier 1969] G. H. Wannier, "Stark ladder in solids? A reply", *Phys. Rev.* **181**:3 (1969), 1364–1365.

[Zak 1968] J. Zak, "Stark ladder in Solids?", *Phys. Rev. Lett.* **20**:26 (1968), 1477–1481.

[Zak 1969] J. Zak, "Stark ladder in solids? A reply to a reply", *Phys. Rev.* **181**:3 (1969), 1366–1367.

[Zener 1934] C. Zener, "A theory of the electrical breakdown of solid dielectrics", *P. Roy. Soc. Lond. A Mat.* **145**:855 (1934), 523–529.

Received 31 Jul 2006. Accepted 8 May 2007.

GUILLERMO MONSIVAIS: monsi@fisica.unam.mx
Instituto de Fisica, Universidad Nacional Autonoma de Mexico, Apdo. Postal 20-364, Mexico 01000 DF, Mexico

RAFAEL A. MÉNDEZ-SÁNCHEZ: mendez@ce.fis.unam.mx
Instituto de Ciencias Fisicas, Universidad Nacional Autonoma de Mexico, P.O. Box 48-3, 62251 Cuernavaca Mor., Mexico

ALFREDO DÍAZ DE ANDA: ada@ce.fis.unam.mx
Instituto de Ciencias Fisicas, Universidad Nacional Autonoma de Mexico, P.O. Box 48-3, 62251 Cuernavaca Mor., Mexico

JORGE FLORES: flores@fisica.unam.mx
Instituto de Fisica, Universidad Nacional Autonoma de Mexico, Apdo. Postal 20-364, Mexico 01000, DF, Mexico

LUIS GUTIÉRREZ: luisg@ce.fis.unam.mx
Instituto de Ciencias Fisicas, Universidad Nacional Autonoma de Mexico, P.O. Box 48-3, 62251 Cuernavaca Mor., Mexico

ALEJANDRO MORALES: mori@fis.unam.mx
Instituto de Ciencias Fisicas, Universidad Nacional Autonoma de Mexico, P.O. Box 48-3, 62251 Cuernavaca Mor., Mexico