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This paper presents a variational solution to the problem of the contact between an isotropic elastic halfspace and a rigid circular indentor, where the contact is achieved through a set of ligaments modeled by a continuously distributed layer of Winkler elements. The problem is of interest to the modeling of the ligament-type contact mechanics between a rigid cylinder and a substrate. The limiting solution for Boussinesq indentation is modified to take into consideration small but finite influences of the elastic stiffness of the ligaments forming the interface layer.

1. Introduction

The mechanics of contact between a component and a substrate is of interest to many areas of materials engineering and materials science. The classical definition of adhesive contact between two material regions assumes the complete compatibility of displacements between the two regions. Other forms of nonclassical contacts include interacting surfaces that exhibit limited adhesion, frictional constraints and slip. The developments, both fundamental and applied, in this area are too numerous to cite individually. We mention [Duvaut and Lions 1976; Selvadurai 1979; 2003; 2007; Gladwell 1980; Haslinger and Janovský 1983; Johnson 1985; Ciarlet 1988; Frémond 1988; Moreau et al. 1988; Curnier 1992; Klarbring 1993; Selvadurai and Boulon 1995] for further reviews of the topic. The idealization of the nature of adhesion is in itself a complex problem, where the fine structure and properties of the media/processes contributing to the generation of the adhesion need to be taken into consideration in developing a plausible model that can determine the onset of debonding [Plueddemann 1974; Anderson et al. 1977; de Lollis 1985; Pizzi and Mittal 1994; Mittal 1995]. Furthermore, depending on the nature of the interacting regions, the contact between the bodies in adhesive contact can in fact be induced at discrete regions at the micromechanical scale, which can contribute to the formation of a *structural model* of the adhesion zone as opposed to a *continuum model*. A model of this type was first introduced by Barenblatt [1959; 1962] in the discussion of brittle fracture and separation at material surfaces and the concepts were adopted and extended in [Dugdale 1960; Goodier 1968; Goodier and Field 1963; Goodier and Kanninen 1966; Kanninen 1970] in their studies of the ductile fracture problems, where cohesive forces of finite magnitude are present at the extremities of a decohesion zone. A key feature in these models is the structural or reduced continuum representation of the decohesion zone. The linear and nonlinear ligament models also allow for the interpretation of intermolecular and surface forces at adhesive zones [Tonck et al. 1988; Israelachvili 1992]. In this paper, we adopt the basic concepts expounded in the structural model of contact zone response and apply it to the modeling of a contact

Keywords: Winkler ligaments, contact problem, variational method, adhesive contact, compliance of indentor, rigid circular indentor, variational solution, elastic halfspace indentation.

between an isotropic elastic halfspace region and a rigid cylindrical indentor, which is achieved through a continuously distributed set of ligament connections. The term *bonded* or *adhesive* is avoided in the present discussion since these specifically refer to phenomena where complete continuity of displacements is established at the connecting zone. In particular, we restrict attention to the modeling of the interface as a series of Winkler elements, although the approach can be extended to include more advanced structural contacts represented by either Vlazov- and Reissner-type layers [Selvadurai 1979], which provide shear interaction between the Winkler elements, or the constrained elastic layer, where certain traction boundary conditions at the edges of the ligament zones are satisfied in an integral sense. A more appropriate terminology that describes this type of contact is *structural bonding*. An alternative to this approach is to consider the connecting layer as an elastic continuum itself. An example of such an application with relevance to nanorheological analysis of the contact between an elastic sphere and a plane separated by an interfacial elastic layer is given by Trifa et al. [2002] in connection with the compressive load transfer at a ligament zone. The Winkler ligament approach adopted here is perhaps not the most all-encompassing treatment of the contact process, but it allows the incorporation of the influences of a material characteristic that could be attributed to the zones that generate the bonding mechanism. In particular, the deformability characteristics of the substrate are accounted for in the modeling.

In this paper we consider the axisymmetric problem of the contact between a rigid cylinder and an isotropic elastic halfspace region, where the structural bonding zone corresponds to a series of closely spaced Winkler ligaments. The conventional approach to the solution of the resulting mixed boundary-value problem is to reduce the analysis to the solution of a Fredholm integral equation of the second kind, which can only be solved in an approximate fashion either by reducing it to a matrix equation or through the introduction of a series representation of the solution or through a variational technique itself. Here, we present a much simpler solution that is based on the application of a direct variational technique. The variational technique has been successfully applied to the study of the mechanics of contact between elastic continua and between structural elements and elastic continua [Kalker 1977; Selvadurai 1979; 1980; 1984; Karasudhi 1991]. This latter approach is a suitable approximation, in the sense that it yields results in closed form, which can be used to establish the influence of the idealized ligament zone in the load transfer mechanism between the rigid cylinder and the elastic halfspace as well as in the development of ligament adhesive stresses between the two regions.

2. The Winkler ligament contact problem

We consider the problem of a rigid circular cylinder of radius a and with a flat base, which is connected to an isotropic elastic halfspace region. The connectivity is provided by a set of Winkler elements that establishes continuity of displacements between the rigid cylinder and the elastic halfspace (Figure 1). The Winkler elements are characterized by a linear load-displacement relationship, although the analysis can be easily extended to include a nonlinear Winkler model with no provision for energy dissipation. The rigid cylinder is subjected to an axisymmetric force of magnitude P which induces rigid body displacement of the cylinder, a deformation of the set of Winkler ligaments and the displacements of the surface of the halfspace region.

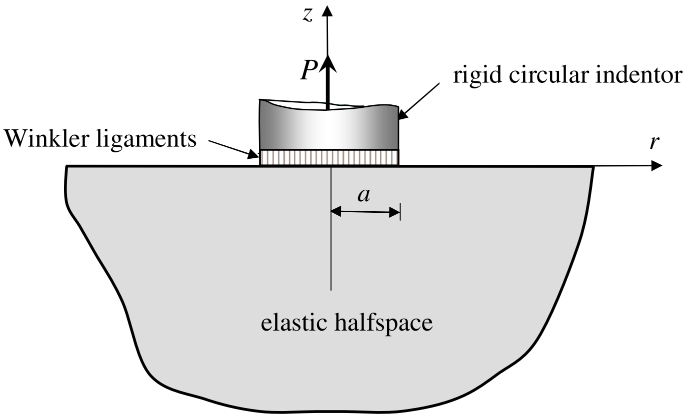

Figure 1. Contact problem for a rigid cylinder achieved through a layer of Winkler ligaments

In the variational approach adopted here, we assume that the vertical displacements of the surface of the halfspace region, within the contact region, can be approximated by a kinematically admissible displacement of the form

$$u_z^{\mathrm{HS}}(r,0)=a\left(C_1-C_2\left(\frac{r}{a}\right)^2\right);\quad r\in(0,a), \tag{1}$$

where C_1 and C_2 are arbitrary constants. Similarly, we assume that due to the loading of the rigid disc, the Winkler ligaments experience a displacement

$$u_z^{W}(r,0)=a\left(C_3+C_2\left(\frac{r}{a}\right)^2\right);\quad r\in(0,a), \tag{2}$$

where C_3 is an arbitrary constant. The displacement fields (1) and (2) satisfy the kinematic constraints necessary for the compatibility of the displacements between the elastic halfspace, the Winkler layer and the rigid disc. The prescribed displacement field in the system is therefore indeterminate to within the constants C_i $(i=1,2,3)$.

In order to apply the variational procedure, we need to evaluate the total energy of the system consistent with the displacement fields defined by Equations (1) and (2). The total energy of the systems consists of the strain energy of the halfspace region, the strain energy of the Winkler layer and the potential energy of the applied load. In order to calculate the strain energy of the halfspace region, we require the solution to a mixed boundary value problem where displacements are prescribed over the contact region. We consider the mixed boundary-value problem related to a halfspace region where the boundary plane $z=0$ is subjected to the boundary conditions

$$u_z^{\mathrm{HS}}(r,0)=w(r)\quad \text{for } r\in(0,a), \tag{3}$$

$$\sigma_{zz}(r,0)=0\quad \text{for } r\in(a,\infty), \tag{4}$$

$$\sigma_{rz}(r,0)=0\quad \text{for } r\in(0,\infty), \tag{5}$$

where u_z^{HS} is the axial displacement of the halfspace region and σ_{zz} and σ_{rz} are the stress components referred to the cylindrical polar coordinate system (r, θ, z). In addition, the displacements and stress fields should satisfy regularity conditions, which ensure that the displacement and stress fields decay uniformly to zero as $(r, z) \to \infty$. The solution of the mixed boundary-value problem defined by Equations (3)–(5) is standard and is given in the classical texts [Sneddon 1951; 1966; Green and Zerna 1968; Selvadurai 1979; 2000a; Gladwell 1980]. Following Green and Zerna [1968] it can be shown that when the shear traction vanishes on the plane $z = 0$, the solution to the elasticity problem can be expressed in terms of a single potential function $\phi(r, z)$, such that

$$2Gu_z(r, z) = z\frac{\partial^2\phi}{\partial u^2} - 2(1-\nu)\frac{\partial\phi}{\partial z}, \qquad \sigma_{zz}(r, z) = z\frac{\partial^3\phi}{\partial z^3} - \frac{\partial^2\phi}{\partial z^2}, \qquad \sigma_{rz}(r, z) = z\frac{\partial^2\phi}{\partial r\,\partial z},$$

where G and ν are the linear elastic shear modulus and Poisson's ratio for the halfspace material. Using results of potential theory we use a representation of $\partial\phi/\partial z$ in the form

$$\frac{\partial\phi}{\partial z} = \frac{1}{2}\int_{-a}^{a} \frac{g(t)dt}{\sqrt{\{r^2 + (z + it)^2\}}},$$

where $g(t)$ is an arbitrary function. For this representation in terms of the potential function, the boundary condition (5) is explicitly satisfied and the remaining boundary conditions (3) and (4) are equivalent to an Abel integral equation of the form

$$w(r) = \int_0^r \frac{g(t)dt}{\sqrt{r^2 - t^2}}$$

which can be solved [Sneddon 1966; Gladwell 1980; Gorenflo and Vessella 1991; Selvadurai 2000a] in the exact form

$$g(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{r\,w(r)dr}{\sqrt{t^2 - r^2}}.$$

The stresses within the contact region can be expressed as

$$\sigma_{zz}(r, 0) = -\frac{G}{1-\nu}\frac{1}{r}\frac{\partial}{\partial r}\int_r^a \frac{t\,g(t)dt}{\sqrt{t^2 - r^2}}.$$

Considering the assumed form of the displacement of the halfspace region within the contact zone, defined by (1), it can be shown that the induced stresses are given by

$$\sigma_{zz}(r, 0) = \frac{2G}{\pi(1-\nu)}\left[\frac{C_1}{\sqrt{a^2 - r^2}} + \frac{2C_2}{a^2}\left(\sqrt{a^2 - r^2} - \frac{r^2}{\sqrt{a^2 - r^2}}\right)\right]. \tag{6}$$

The strain energy of the halfspace region can be obtained by calculating the work done by the normal tractions in the contact zone, that is,

$$U_{\mathrm{HS}} = \frac{1}{2}\int_0^{2\pi}\int_0^a u_z^{\mathrm{HS}}(r, 0)\sigma_{zz}(r, 0)r\,dr\,d\theta.$$

Similarly, the strain energy of the Winkler ligament zone can be obtained from the result

$$U_W = \frac{1}{2}\int_0^{2\pi}\int_0^a k\big[u_z^W(r, 0)\big]^2 r\,dr\,d\theta,$$

where k is the stiffness of the Winkler ligament per unit area. The work of the applied force P is given by $W_P = -P\big[u_z^{\mathrm{HS}}(0,0) + u_z^{W}(0,0)\big]$. The total potential energy function for the system can be evaluated in the form

$$U = \frac{2Ga^3}{(1-\nu)}\big[C_1^2 - \tfrac{4}{3}C_1C_2 + \tfrac{4}{5}C_2^2\big] + \frac{\pi k a^4}{2}\big[\tfrac{1}{3}C_2^2 + C_2C_3 + C_3^2\big] - Pa[C_1 + C_3].$$

Considering the principle of minimum total potential energy for a conservative system, the arbitrary constants are determined from the conditions

$$\frac{\partial U}{\partial C_1} = \frac{\partial U}{\partial C_2} = \frac{\partial U}{\partial C_3} = 0$$

which gives the undetermined parameters C_1, C_2 and C_3. The constants take the forms

$$[C_1;\, C_2;\, C_3] = \frac{\bar{P}}{(16+15\Omega)}\left[3(7+5\Omega);\, \frac{15}{2};\, \frac{4}{\Omega}\right], \tag{7}$$

where $\bar{P} = P(1-\nu)/4Ga^2$ and $\Omega = \pi k a(1-\nu)/16G$. The formal variational solution for the contact problem associated with a set of Winkler ligaments is given by (1), (2), (6) and (7). Both the state of stress within the halfspace region and within the zone of Winkler ligaments can be determined from these results in conjunction with Boussinesq's solution for the loading of a halfspace region by a concentrated normal force [Timoshenko and Goodier 1970; Davis and Selvadurai 1996; Selvadurai 2001].

3. The role of the Winkler ligament zone

An inspection of the variational solution indicates that as the relative stiffness of the Winkler ligament zone (as defined by the parameter Ω) increases, the terms incorporating C_2 and C_3 will have a diminishing influence on the load transfer process. In the limit as $\Omega \to \infty$, $C_1 \to \bar{P}$ and the displacement of the rigid cylinder is given by $w(0) = P(1-\nu)/4Ga$, and the contact stress within the circular region is

$$\sigma_{zz}(r,0) = P/2\pi a\sqrt{a^2 - r^2},$$

which is Boussinesq's classical result for the indentation of a halfspace by a rigid circular indentor with a flat base. In terms of the contact problem, a ligament zone of high relative stiffness will invariably result in the development of a singular stress state at the boundary of the circular cylinder, which would represent a potential location for the development of delamination. For a finite value of the relative stiffness parameter Ω, the displacement of the rigid cylinder as well as the stresses in the ligament zone are influenced by the Winkler ligament stiffness k. Figure 2 illustrates the variation in the normalized displacement of the rigid disc $\bar{\Delta}$ (defined as $4G\Delta a/P(1-\nu)$, where Δ is the displacement of the rigid disc) as a function of the relative stiffness parameter Ω. As can be observed, the reduction to the case of the classical Boussinesq rigid punch problem is achieved for a value of $\Omega > 5$. The contact stress at the cylinder-Winkler ligament layer can similarly be evaluated in explicit form. From (6) and (7) we obtain

$$\bar{\sigma} = \frac{\sigma_{zz}(r,0)}{\sigma_0} = \frac{1}{2(15\Omega+16)}\left[\frac{(15\Omega+21)}{\sqrt{1-\rho^2}} + 15\left(\sqrt{1-\rho^2} - \frac{\rho^2}{\sqrt{1-\rho^2}}\right)\right]; \quad \rho \in (0,1), \tag{8}$$

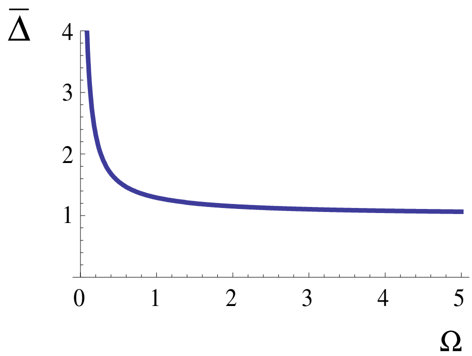

Figure 2. Influence of the relative stiffness parameter on the displacement of the bonded disc

where $\sigma_0 = P/\pi a^2$ and $\rho = r/a$. Figure 3 illustrates the variation in the contact stress as a function of the relative stiffness parameter Ω. As $\Omega \to 0$, the normal stresses exhibit a nonuniform distribution at the adhesive zone, but maintain the singular character, derived from the appropriate terms in (8). As $\Omega \to \infty$, the adhesive stresses reduce to the Boussinesq-type distribution, with singular behaviour as $\rho \to 1$. It is of interest to examine the influence of the relative stiffness parameter Ω in moderating the stress intensity factor at the boundary of the ligament zone, which can be compared with the critical stress intensity factor necessary to initiate brittle fracture at the boundary of the adhesion zone. Considering the definition of the Mode I stress intensity factor we have

$$K_I^a = \lim_{r \to a^-} [2(a-r)]^{1/2} \sigma_{zz}(r, 0). \tag{9}$$

Considering (8) and (9) we obtain

$$K_I^a = \frac{\sigma_0 \sqrt{a}}{2} \, \frac{15\Omega + 6}{15\Omega + 16}.$$

Again as $\Omega \to \infty$, we recover from the above equation the classical result for the stress intensity factor associated with the axisymmetric problem of an elastic medium of infinite extent with an intact region of radius a and subjected to a far-field stress that is equivalent to a total force P [Kassir and Sih 1968]. Also, as $\Omega \to 0$, the stress intensity factor approaches the value $K_I^a = 3\sigma_0\sqrt{a}/16$. This result is consistent with the observation made by Selvadurai [2000b] with regard to the Mode I stress intensity factor for a penny-shaped crack that is located at the interface of a functionally graded material where the elastic modulus exhibits a bounded exponential variation in the axial direction. In the case where the linear elastic shear modulus at the plane of the crack is lower than the finite value of the far-field shear modulus, the Mode I stress intensity factor is lower than the corresponding value applicable to the problem of a penny-shaped crack located in homogeneous elastic solid.

4. Concluding remarks

This paper presents a relatively elementary study of the mechanics of a Winkler ligament zone that forms the structural bonding between a rigid cylinder and an isotropic elastic substrate of semi-infinite extent. The variational approach presented here is an approximation to the more complex formulation that would involve a complete analysis of a Fredholm integral equation of the second kind, which will invariably

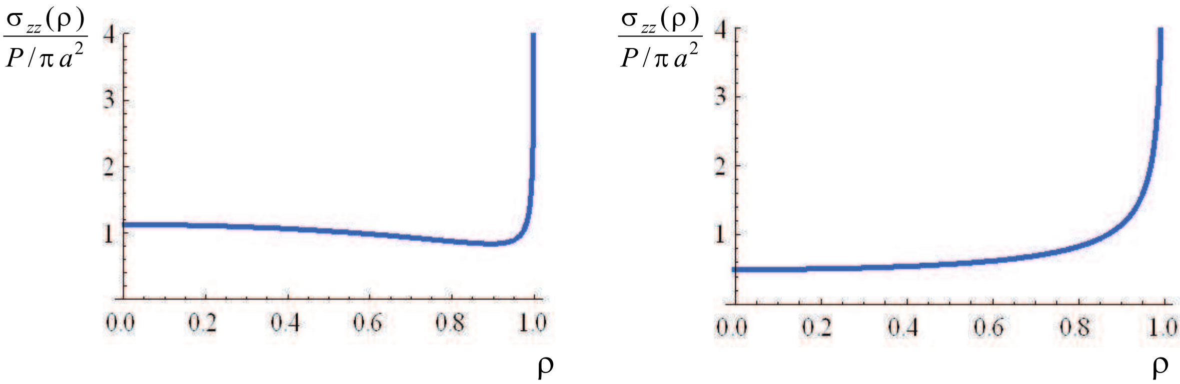

Figure 3. Adhesive stresses at the bonded zone [$\Omega \to 0$] (left) and [$\Omega \to \infty$] (right).

entail a numerical solution technique. The variational procedure provides a convenient approach for examining the particular influences of the Winkler ligament zone that provides the structural bonding between the rigid cylinder and the halfspace region. The displacement functions chosen satisfy the kinematic constraints and the range of the polynomial expressions used can be extended to include further terms. Such a treatment is perhaps unwarranted in view of the elementary nature of the modeling of the ligament zone as a continuous distribution of unconnected spring elements. The elementary analysis nonetheless illustrates trends that are important to the understanding of the mechanics of load transfer at ligament zones. The form of the displacement functions chosen for the variational treatment still maintains the singular behaviour of the stress states in the ligament zone for $\Omega \in (0, \infty)$, although such an interpretation should be viewed with some caution, since at the outset the stiffness of the ligament zone is assumed to be finite. In particular, it is noted that the presence of a ligament zone of low relative stiffness has a tendency to moderate the stress intensity factor at the boundary of the ligament zone. It should also be borne in mind that structural adherents with lower stiffness generally tend to possess lower resistance to fracture, indicative of low values of the critical stress intensity factors. Finally, the variational approach for the solution of contact problems of this nature would be most effective when the ligaments exhibit nonlinear force-displacement relationships. In such a case, the conventional integral equation approach leads to nonlinear forms that are not easily solved, except through the use of either perturbation schemes or a method of successive approximations. The variational approach with improved representations for the deflected shapes can lead to compact results in exact closed forms.

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