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# ON LARGE DEFORMATION GENERALIZED PLASTICITY

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*Dedicated to the memory of Juan C. Simo for his seminal contributions to solid and computational mechanics*

Large deformation generalized plasticity is presented in a covariant setting. For this purpose, the tensor analysis on manifolds is utilized and the manifold structure of the body as well of the ambient and the state space is postulated. On the basis of the multiplicative decomposition of the deformation gradient into elastic and plastic parts and the use of hyperelastic stress-strain relations, a large deformation elasto-plasticity model is proposed. Computational aspects and the predictions of the model under uniaxial and biaxial straining are also presented.

## 1. Introduction

Since the time of its initial introduction in [Lubliner 1974], generalized plasticity theory has been elaborated further within the large deformation analysis regime in order to deal with materials with a vanishing elastic domain [Lubliner 1975], the maximum plastic dissipation postulate [Lubliner 1986], and nonisothermal behavior [Lubliner 1987]. In these approaches the theory has been presented largely in an abstract manner dealing with issues appearing primarily in a referential setting. Moreover, even though constitutive models based on the generalized plasticity theory have been proposed and implemented numerically, within the context of the infinitesimal theory [Lubliner et al. 1993; Auricchio and Taylor 1995; Panoskaltsis et al. 1997], a model within the context of the finite theory has not been proposed yet.

The objective of this study is threefold: first, to present the theory in a covariant setting. For this purpose manifold structure is considered not only for the body of interest and the ambient space, but also for the state space, that is, the set of all realizable states over a material point. Accordingly, the motion of the body, which is considered as a time dependent mapping within the ambient space, is extended to a local dynamical process by considering the state space as a fiber over the body particles. In turn, the involvement of the standard pull-back/push-forward operations of the tensor analysis on manifolds [Marsden and Hughes 1994, p. 67] leads to the introduction of the convected Lie derivative [Simo and Marsden 1984], which eventually leads to a covariant formulation of the theory. It is noted that the covariant formulation leads to constitutive equations which are invariant under arbitrary spatial diffeomorphisms and the principle of objectivity — invariance under arbitrary spatial isometries — is trivially satisfied. This point of view has been exploited by Simo [1988] within the context of classical plasticity, and seems to have passed largely unnoticed within the literature. Unlike the presentation of

*Keywords:* large deformation plasticity, generalized plasticity, tensor analysis on manifolds, covariant formulation, Lie derivative, internal variables, metric of the elastically relaxed space, loading-unloading conditions, time integration algorithm, predictor-corrector algorithm.

[Simo 1988], where the covariance principle is applied after the problem kinematics have been specified, the present approach leaves the problem kinematics entirely unspecified.

Second, to propose a rather simple model on the basis of metal plasticity in order to make clear the covariance principle within the generalized plasticity context, in its most simple setting. The proposed model comprises the following components:

(i) decomposition of the total motion into a plastic motion in some relaxed space endowed with the structure of a Riemannian manifold, followed by an elastic motion, as suggested by Le and Stumpf [1993];

(ii) flow rule in terms of the Riemmanian metric of the relaxed space;

(iii) Von Mises loading surfaces with both isotropic and kinematic hardening;

(iv) hyperelastic constitutive equations for the characterization of the elastic response, as proposed in the work of Simo and Ortiz [1985].

Third, to present the computational implementation of a generalized plasticity model in a covariant formulation. Finally, we also present numerical simulations.

## 2. Constitutive theory

Following the erudite approach of Marsden and Hughes [1994] within the context of nonlinear elasticity, we consider both the body of interest and the ambient space as three-dimensional Riemannian manifolds. In particular, let $B$ be the reference configuration of the body of interest, which is modeled as a three-dimensional manifold with points labeled by $(X^1, X^2, X^3)$, and define a motion of $B$ as a time dependent mapping $\boldsymbol{x} : B \to b$, which is given by

$$x^1 = x^1(X^1, X^2, X^3, t), \qquad x^2 = x^2(X^1, X^2, X^3, t), \qquad x^3 = x^3(X^1, X^2, X^3, t), \tag{1}$$

and maps the points of the reference configuration $B$ onto the points $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$ of the current configuration $b$. The mapping Equation (1) is assumed to be one to one and twice differentiable, that is, an element of the Einstein group $\mathscr{E}$ [Dyson 1972]. The deformation gradient is defined as the tangent map of Equation (1),

$$\boldsymbol{F}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}},$$

with determinant $J = \det \boldsymbol{F}(\boldsymbol{X}, t) > 0$. Furthermore, let $\boldsymbol{G}$ and $\boldsymbol{g}$ be the covariant metrics of the reference configuration and the ambient space, respectively. Next we introduce the right Cauchy–Green tensor, defined as the pull-back of $\boldsymbol{g}$,

$$\boldsymbol{C} = \boldsymbol{x}^*(\boldsymbol{g}) = \boldsymbol{F}^{\boldsymbol{T}} \boldsymbol{g} \boldsymbol{F},$$

and the Finger deformation tensor, defined as the push-forward of $\boldsymbol{G}$,

$$\boldsymbol{b}^{-\boldsymbol{1}} = \boldsymbol{x}_*(\boldsymbol{G}) = \boldsymbol{F}^{-\boldsymbol{T}} \boldsymbol{G} \boldsymbol{F}^{\boldsymbol{T}}.$$

In general, the referential metric $\boldsymbol{G}$ is unknown, and several considerations must be made for its determination, including experimental procedures [Valanis and Panoskaltsis 2005]. By means of the adopted manifold structure and the consideration of the referential metric, several internal material structures, including directional densities, curved material structures, preformed materials, and prestressed reference configurations, can be accounted for by the proposed approach. For a dissipative material, like the elastic-plastic continuum to be discussed here, the referential metric is a function of the history of deformation [Valanis 1995]. The only case where the referential metric is constant in the course of deformation is that of an elastic material, like the one discussed in the covariant approaches of Simo et al. [1988] and Marsden and Hughes [1994].

Generalized plasticity is a local internal variable theory of rate independent behavior which is based primarily on the assumption that plastic deformation takes place on loading but not on unloading [Lubliner 1974; 1975]. In the absence of thermal effects, the material state at the point $\boldsymbol{X}$ with coordinates $(X^1, X^2, X^3)$ is assumed to be determined by the couple $(\boldsymbol{S}, \boldsymbol{Q})$, where $\boldsymbol{S}$ denotes the second Piola–Kirchhoff stress tensor and $\boldsymbol{Q}$ denotes the internal variable vector. The latter is assumed to be covariant in the sense that under the mapping Equation (1) it is transformed according to the general tensorial transformation law, as it is given, for instance, in [Marsden and Hughes 1994, p. 67]. On the basis of the previous discussion regarding the referential metric, it is concluded that the latter has to be included in $\boldsymbol{Q}$. The state space $S$ is assumed to be attached to the point $\boldsymbol{X}$ so that the set $\boldsymbol{X} \times S$ is a fiber of $\boldsymbol{X}$, and since this set is an open subset of $B \times S$, it is a local manifold. The dimension of this manifold is $6 + r$, where $r$ is the number of independent components of $\boldsymbol{Q}$.

A local process $\boldsymbol{\Psi}$ in $S$ is defined as a curve in $S$, that is, as a mapping $\boldsymbol{\Psi} : I \in \mathbb{R} \to S$, with $\boldsymbol{\Psi}(t) = (\boldsymbol{S}(t), \boldsymbol{Q}(t))$, where $t \in I$. The direction and the speed of the process are determined by the tangent vector $\dot{\boldsymbol{\Psi}} : S \to TS$, with $\dot{\boldsymbol{\Psi}} = (\dot{\boldsymbol{S}}, \dot{\boldsymbol{Q}})$, where $TS$ is the tangent space of $S$. Since $\dot{\boldsymbol{S}}$ is always known under stress control, the component $\dot{\boldsymbol{Q}}$ of $\dot{\boldsymbol{\Psi}}$ has to be determined. The latter is assumed to be given by rate equations of the form

$$\dot{\boldsymbol{Q}} = \boldsymbol{\Pi}(\boldsymbol{S}, \boldsymbol{C}, \boldsymbol{Q}, \dot{\boldsymbol{S}}), \tag{2}$$

where $\boldsymbol{\Pi} : S \times TS \to TS$ is a vector field in $TS$, which is considered as a tensorial function of the denoted arguments. We note the dependence of the function $\boldsymbol{\Pi}$ on the (convected) metric $\boldsymbol{C}$ in the reference configuration, which needs to be included not only for a covariant setting of the theory [Simo et al. 1988], but also to account for effects such as pressure dependence of the plastic response [Simo and Ortiz 1985]. Rate independence implies that Equation (2) is invariant under a change of the parameter $t$ by any monotonically increasing, continuously differentiable function $\chi(t)$ (see, for instance, [Lubliner 1987; Lucchesi and Podio-Guidugli 1992]). Then the necessary and sufficient condition for rate independence is that $\boldsymbol{\Pi}$ is homogeneous to the first degree [Lubliner 1986; 1987], that is

$$\boldsymbol{\Pi}(\boldsymbol{S}, \boldsymbol{C}, \boldsymbol{Q}, c \cdot \dot{\boldsymbol{S}}) = c \cdot \boldsymbol{\Pi}(\boldsymbol{S}, \boldsymbol{C}, \boldsymbol{Q}, \dot{\boldsymbol{S}}), \tag{3}$$

for any positive number $c$.

A local process is defined as elastic if it lies entirely in a six dimensional submanifold of $S$, the stress space defined by $\boldsymbol{Q} =$ constant, otherwise it is defined as plastic. The elastic range of a state is defined as a submanifold in stress space comprising the stresses that can be reached elastically from the current stress point [Pipkin and Rivlin 1965; Lucchesi and Podio-Guidugli 1992]. It is assumed further that the boundary of the elastic range is a five-dimensional manifold, the points of which have a coordinate neighborhood on it, which is attached to the interior in much the same way as a face of a cube is attached to the interior. The latter manifold may be defined as a loading surface [Eisenberg and Phillips 1971; Lubliner 1987]. In turn, a state within its elastic range may be defined as plastic if it lies on a loading surface and as elastic otherwise. On the basis of axioms and results from set theory and topology, Lubliner [1987] showed that the simplest function $\mathbf{\Pi}$ obeying the homogeneity condition Equation (3) consistent with the notion of the loading surface is

$$\mathbf{\Pi}(\boldsymbol{S}, \boldsymbol{C}, \boldsymbol{Q}, \dot{\boldsymbol{S}}) = \mathbf{\Lambda}(\boldsymbol{S}, \boldsymbol{C}, \boldsymbol{Q})\langle \boldsymbol{N} : \dot{\boldsymbol{S}}\rangle, \tag{4}$$

where $\boldsymbol{N}$ is the outward normal to the loading surface, assumed to be nonvanishing, and $\langle\cdot\rangle$ stands for the Macauley bracket, defined as

$$\langle x\rangle = \begin{cases} x, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

In view of Equations (2) and (4), the rate equations for the evolution of the internal variables can be written as

$$\dot{\boldsymbol{Q}} = \mathbf{\Lambda}(\boldsymbol{S}, \boldsymbol{C}, \boldsymbol{Q})\langle \boldsymbol{N} : \dot{\boldsymbol{S}}\rangle. \tag{5}$$

The inner product $\boldsymbol{N} : \dot{\boldsymbol{S}}$ of the tangent vector $\dot{\boldsymbol{S}}$ and the normal vector $\boldsymbol{N}$ (one form) is defined as the loading rate. The loading rate determines the velocity and the direction of the process from a plastic state, relative to its elastic range. If $\boldsymbol{N} : \dot{\boldsymbol{S}} < 0$, then the elastic range remains invariant under the flow of $\dot{\boldsymbol{S}}$ (see [Abraham et al. 1988, p. 257]) and the process is elastic. If $\boldsymbol{N} : \dot{\boldsymbol{S}} > 0$, then the elastic range in not invariant anymore, and a new plastic state at a new value of $\mathbf{Q}$ is initiated. The limiting case, where $\boldsymbol{N} : \dot{\boldsymbol{S}} = 0$, results in an elastic process and is defined as neutral loading. It is noted that the present formulation presupposes stability under stress control and is limited to work hardening materials. The incorporation of work softening phenomena within the theory can be conducted along the lines presented in this study in conjunction with some developments given in [Lubliner et al. 1993].

The manifold defined by the equation $\mathbf{\Lambda}(\boldsymbol{S}, \boldsymbol{C}, \boldsymbol{Q}) = \mathbf{0}$, which comprises all elastic states, may be called the elastic domain, and its boundary, which is assumed to be a submanifold, is called the yield hypersurface. The projection of the elastic domain on the manifold $\boldsymbol{Q} =$ constant is called the elastic domain at $\boldsymbol{Q}$. In general, the elastic domain at $\boldsymbol{Q}$ is a submanifold of the elastic range [Lubliner 1987]. The particular case where the two manifolds coincide corresponds to classical plasticity. In this case, the closure of the elastic domain $C$, which constitutes the whole state space $S$, is invariant under the action of the plastic flow. More specifically, if the yield hypersurface, which now comprises the totality of plastic states, is assumed to be given by the function $g : C \to \mathbb{R}$, with $g(\boldsymbol{S}, \boldsymbol{C}, \boldsymbol{Q}) = 0$, then the state space $C$ is defined, for any regular value $\lambda \leq 0$ of $g$, as the manifold $C = g^{-1}(\lambda)$. Then the invariance of the state space under the action of the plastic flow is equivalent to the invariance of $C$ under the action of the flow of the tangent vector to the process $\dot{\boldsymbol{\Psi}}$. The necessary and sufficient condition for the invariance of $C$ under the flow of $\dot{\boldsymbol{\Psi}}$ (see [Abraham et al. 1988, pp. 256–258]) is

$$\dot{\boldsymbol{\Psi}} : \mathrm{GRAD}\, g \leq 0. \tag{6}$$

The limiting case $\dot{\boldsymbol{\Psi}} : \mathrm{GRAD} g = 0$ corresponds to a process tangent to the yield hypersurface and constitutes the consistency condition of classical plasticity. This result can be generalized for the case in which the yield hypersurface is not a submanifold, but rather a piecewise smooth surface (see, for instance, [Hartman 1972]), and the multisurface plasticity formulation due to [Koiter 1953] is regained.

Another case of interest arises when the function $\boldsymbol{\Lambda}$ is a nonvanishing function of its arguments. If this is the case then there are no elastic states, and the elastic domain degenerates to a surface which may be called a quasiyield surface [Lubliner 1975].

With the help of Equation (5) we now define the general (stress-space) loading-unloading conditions explicitly as

$$\begin{cases} \boldsymbol{\Lambda}(\boldsymbol{S}, \boldsymbol{C}, \boldsymbol{Q}) = \boldsymbol{0}, & \text{elastic state,} \\ \boldsymbol{\Lambda}(\boldsymbol{S}, \boldsymbol{C}, \boldsymbol{Q}) \neq \boldsymbol{0}, & \text{and} \quad \begin{cases} \boldsymbol{N} : \dot{\boldsymbol{S}} < 0, & \text{elastic unloading,} \\ \boldsymbol{N} : \dot{\boldsymbol{S}} = 0, & \text{neutral loading,} \\ \boldsymbol{N} : \dot{\boldsymbol{S}} > 0, & \text{plastic loading.} \end{cases} \end{cases} \tag{7}$$

The equivalent development of the theory in the current configuration can be performed by considering the local vector bundle mapping (see [Abraham et al. 1988, p.167])

$$\boldsymbol{P} : B \times S \to b \times s \text{ with } \boldsymbol{P}(\boldsymbol{X}, \boldsymbol{S}, \boldsymbol{Q}, t) = (\boldsymbol{x}(\boldsymbol{X}, t), \boldsymbol{x}_*(\boldsymbol{S}), \boldsymbol{x}_*(\boldsymbol{Q})),$$

which, by keeping the point $\boldsymbol{X}$ fixed, may be identified as a (local) dynamical process. Accordingly, the state space $s$ at the point $\boldsymbol{x}$ with coordinates $(x^1, x^2, x^3)$ is composed by the Kirchhoff stress tensor $\boldsymbol{\tau} = \boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T$, and the push-forward of the internal variable vector, denoted as $\boldsymbol{q} = \boldsymbol{x}_*(\boldsymbol{Q})$.

Then, by the application of a push-forward operation to Equation (5) with the mapping Equation (1), we have in an equivalent spatial setting

$$L_v \boldsymbol{q} = \boldsymbol{\lambda}(\boldsymbol{\tau}, \boldsymbol{g}, \boldsymbol{q}, \boldsymbol{F}) \langle \boldsymbol{n} : L_v \boldsymbol{\tau} \rangle, \tag{8}$$

where $L_v(\cdot)$ stands for the Lie derivative, defined as the convected derivative relative to the current configuration (for instance, see [Simo and Marsden 1984; Le and Stumpf 1993]), $\boldsymbol{\lambda}$ is a vector field in $Ts$, and $\boldsymbol{n}$ is the outward normal to the loading surface in the current configuration. It is noted that the (invariant) loading rate $\boldsymbol{N} : \dot{\boldsymbol{S}}$ is transformed in the current configuration to $\boldsymbol{n} : L_v \boldsymbol{\tau}$ [Miehe 1998]. We further note the presence of the deformation gradient $\boldsymbol{F}$ among the arguments of $\boldsymbol{\lambda}$ due to the push-forward operation by which Equation (8) is derived from Equation (5). In view of Equation (8) we formulate the spatial version of the loading-unloading conditions as

$$\begin{cases} \boldsymbol{\lambda}(\boldsymbol{\tau}, \boldsymbol{g}, \boldsymbol{q}, \boldsymbol{F}) = \boldsymbol{0}, & \text{elastic state,} \\ \boldsymbol{\lambda}(\boldsymbol{\tau}, \boldsymbol{g}, \boldsymbol{q}, \boldsymbol{F}) \neq \boldsymbol{0}, & \text{and} \quad \begin{cases} \boldsymbol{n} : L_v \boldsymbol{\tau} < 0, & \text{elastic unloading,} \\ \boldsymbol{n} : L_v \boldsymbol{\tau} = 0, & \text{neutral loading,} \\ \boldsymbol{n} : L_v \boldsymbol{\tau} > 0, & \text{plastic loading.} \end{cases} \end{cases} \tag{9}$$

Equation (5), along with the mathematical expression for the loading surfaces, which are assumed to be given as a single parameter family of the form $\Phi(\boldsymbol{S}, \boldsymbol{C}, \boldsymbol{Q}) = K$, constitute the simplest version of generalized plasticity in the reference configuration. An equivalent spatial setting consists of Equation (8) and an expression for the loading surfaces of the form $\varphi(\boldsymbol{\tau}, \boldsymbol{g}, \boldsymbol{q}, \boldsymbol{F}) = k$. In order to develop a generalized plasticity model, we have to specify:

(i) the kinematic assumptions regarding the geometry of deformation,

(ii) the stress-deformation relations,

(iii) the kind and the number of the internal variables.

These are specified in the forthcoming section, where a rather simple model is proposed.

## 3. A model problem

Consistent with the covariant formulation employed, the basic kinematical assumption consists of a decomposition of the motion into a plastic motion in some relaxed space $B_e$, considered as a Riemannian manifold, followed by an elastic motion [Le and Stumpf 1993]. In particular the plastic motion, which will be termed as plastic flow, is defined as the time dependent mapping $\boldsymbol{x^p} : B \to \boldsymbol{B_e}$, which is given as

$$x^{p1} = x^{p1}(X^1, X^2, X^3, t), \qquad x^{p2} = x^{p2}(X^1, X^2, X^3, t), \qquad x^{p3} = x^{p3}(X^1, X^2, X^3, t).$$

Then the total motion Equation (1) can be decomposed as

$$\boldsymbol{x} = \boldsymbol{x^e} \circ \boldsymbol{x^p}, \tag{10}$$

where the motion $\boldsymbol{x^e} : B_e \to b$, with

$$x^{e1} = x^{e1}(x^{p1}, x^{p2}, x^{p3}, t), \qquad x^{e2} = x^{e2}(x^{p1}, x^{p2}, x^{p3}, t), \qquad x^{e3} = x^{e3}(x^{p1}, x^{p2}, x^{p3}, t),$$

constitutes the elastic deformation.

The decomposition Equation (10) of the motion leads to the multiplicative decomposition of the deformation gradient into elastic and plastic parts, $\boldsymbol{F} = \boldsymbol{F^e} \boldsymbol{F^p}$. Such decomposition has been considered by Lee [1969], Mandel [1972], Lubliner [1986], and Simo [1988], among others. Following Le and Stumpf [1993], we introduce the model in the (elastically) relaxed space. A similar approach has been also favored by Lee [1969], Mandel [1972], Dashner [1986], and Dafalias [1998], among others. Accordingly, the state variables are assumed to be the (contravariant) stress tensor $\boldsymbol{S_e}$, defined as the push-forward of the second Piola–Kirchhoff stress tensor by the plastic motion $\boldsymbol{S_e} = \boldsymbol{F^p} \boldsymbol{S} \boldsymbol{F^{p^T}}$, and the internal variable vector, which is assumed to be composed of the Riemannian metric in $B_e$, $\boldsymbol{G_e}$ and an additional internal variable vector $\boldsymbol{Q_e}$. The selection of the metric of the (elastically) relaxed space as a primary state variable is natural and relies on the fact that it is precisely the tensor $\boldsymbol{G_e}$ and the internal variables which determine the continuously evolving geometry of the relaxed space under the action of the plastic flow. This approach has its origins in the work of Le and Stumpf [1993], is consistent with the necessity of the selection of a measure of plastic deformation as an independent variable according to Naghdi [1990], and takes into account the fact that the referential metric varies in the course of plastic deformation, according to Valanis [1995].

Motivated by classical metal plasticity we introduce a von Mises type of expression for the loading surfaces with both isotropic and kinematic hardening,

$$\Phi(\boldsymbol{S_e}, \boldsymbol{C_e}, \alpha, \boldsymbol{H_e}) = \sqrt{(S_e^{\alpha\beta} - H_e^{\alpha\beta})(S_e^{\gamma\delta} - H_e^{\gamma\delta}) C_{e\alpha\gamma} C_{e\beta\delta} - \frac{1}{3}(S_e^{\alpha\beta} C_{e\alpha\beta})^2} - \sqrt{\frac{2}{3}}(\sigma_y + K\alpha), \tag{11}$$

where $\boldsymbol{C_e}$ is the (convected) metric for the relaxed space, defined as the pull-back of the spatial metric by the elastic deformation $\boldsymbol{C_e} = \boldsymbol{F^{e^T}} \boldsymbol{g} \boldsymbol{F^e}$, where $\alpha$ is a scalar internal variable which controls the size of the loading surfaces, $\boldsymbol{H_e}$ is a deviatoric stress tensor (that is, $\mathrm{tr}(\boldsymbol{H_e}\boldsymbol{C_e}) = 0$), usually termed back stress which controls the location of the loading surfaces, and $\sigma_y$ and $K$ are two model parameters designating the uniaxial yield stress and the (isotropic) hardening modulus, respectively.

The evolution of the plastic flow (flow rule) is considered to be normal to the loading surfaces as per

$$L_{V^P}\boldsymbol{G}_{\boldsymbol{e}}^{-1} = h\boldsymbol{N}_{\boldsymbol{e}}\langle \boldsymbol{N}_{\boldsymbol{e}} : L_{V^P}\boldsymbol{S}_{\boldsymbol{e}}\rangle, \tag{12}$$

where $L_{V^P}(\cdot)$ is the (convected) Lie derivative along the velocity of the plastic flow; the velocity may be defined as [Le and Stumpf 1993]

$$\boldsymbol{V}^p = \bar{\boldsymbol{V}}^P \circ \boldsymbol{x}^{p-1}, \text{ where } \bar{V}^{P\alpha} = \frac{\partial x^{p\alpha}}{\partial t}(\boldsymbol{X}, t)|_{X=\text{constant}}.$$

$\boldsymbol{N}_{\boldsymbol{e}}$ is the normal vector to the loading surfaces which, in view of Equation (11) [Simo and Ortiz 1985; Simo 1988], is given as

$$\boldsymbol{N}_e = \frac{\partial \Phi}{\partial S_e} = \frac{(\boldsymbol{S}_{\boldsymbol{e}} - \boldsymbol{H}_e) - \frac{1}{3}\{(\boldsymbol{S}_{\boldsymbol{e}} - \boldsymbol{H}_e) : \boldsymbol{C}_{\boldsymbol{e}}\}\boldsymbol{C}_e^{-1}}{\|(\boldsymbol{S}_e - \boldsymbol{H}_e) - \frac{1}{3}\{(\boldsymbol{S}_{\boldsymbol{e}} - \boldsymbol{H}_e) : \boldsymbol{C}_{\boldsymbol{e}}\}\boldsymbol{C}_e^{-1}\|},$$

where $\|\cdot\|$ is the Euclidean norm and $h$ is a scalar function of the state variables which enforces the defining property of a plastic state. Accordingly, the value of $h$ must be positive at any plastic state and zero at any elastic one. It should be noted that in taking the derivative of the loading function with respect to the stress tensor $\boldsymbol{S}_e$, the quantities $\boldsymbol{S}_{\boldsymbol{e}}$ and $\boldsymbol{C}_e$ are treated as independent variables. The relation between the stress tensor $\boldsymbol{S}_e$ and the metric $\boldsymbol{C}_e$, as it is expressed in Equations (24) and (25), is an a posteriori fact related to the choice of the constitutive equations (in this case hyperelastic).

It is emphasized that in the particular case in which the relaxed space is considered as flat, or almost flat, and by noting that

$$L_{V^P}\boldsymbol{G}_{\boldsymbol{e}}^{-1} = \dot{\boldsymbol{G}}_{\boldsymbol{e}}^{-1} - \boldsymbol{L}_{\boldsymbol{P}}\boldsymbol{G}_{\boldsymbol{e}}^{-1} - \boldsymbol{G}_{\boldsymbol{e}}^{-1}\boldsymbol{L}_{\boldsymbol{P}}^T,$$

where $\boldsymbol{L}_{\boldsymbol{p}} = \dot{\boldsymbol{F}}^{\boldsymbol{P}}\boldsymbol{F}^{\boldsymbol{P}-1}$ is the (true) plastic velocity gradient, the flow rule can be stated as

$$\text{Sym}\left[\boldsymbol{G}_{\boldsymbol{e}}^{-1}\boldsymbol{L}_{\boldsymbol{p}}\right] = -\tfrac{1}{2}h\boldsymbol{N}_{\boldsymbol{e}}\langle \boldsymbol{N}_{\boldsymbol{e}} : L_{V^P}\boldsymbol{S}_{\boldsymbol{e}}\rangle,$$

where Sym $[\cdot]$ stands for the symmetric part of its argument. For the particular case in which the relaxed space is considered as Euclidean, that is, $\boldsymbol{G}_{\boldsymbol{e}}^{-1} = \boldsymbol{1}$, a flow rule in terms of Sym $\left[\boldsymbol{L}_{\boldsymbol{p}}\right]$ is derived. If this is the case, the adopted flow rule resembles the associative flow rule derived by Simo [1988], based on the maximum plastic dissipation postulate, within the context of classical plasticity

$$\text{Sym}\left[\boldsymbol{L}_{\boldsymbol{p}}\right] = \dot{\gamma}\boldsymbol{C}_{\boldsymbol{e}}\boldsymbol{N}_{\boldsymbol{e}}, \tag{13}$$

where $\dot{\gamma}$ is the consistency parameter, which can be determined by means of the consistency condition (Equation (6)). It is noted that the flow rule adopted herein (12), differs from the associated flow rule (13) by a factor equal to $\boldsymbol{C}_{\boldsymbol{e}}$ and a multiplicative scalar.

In accordance with the classical theory we propose the following evolution equations for the remaining internal variables [Simo and Hughes 1997, p. 90]:

$$\dot{\alpha} = \sqrt{\tfrac{2}{3}}h\langle \boldsymbol{N}_{\boldsymbol{e}} : L_{V^P}\boldsymbol{S}_{\boldsymbol{e}}\rangle, \tag{14}$$

$$L_{V^P}\boldsymbol{H}_{\boldsymbol{e}} = \tfrac{2}{3}Hh\boldsymbol{N}_{\boldsymbol{e}}\langle \boldsymbol{N}_{\boldsymbol{e}} : L_{V^P}\boldsymbol{S}_{\boldsymbol{e}}\rangle, \tag{15}$$

where $H$ is the (kinematic) hardening modulus.

The referential setting of the model can be determined by applying a pull-back operation to Equations (11), (12), (14), and (15) by the plastic flow as

$$\Phi(\boldsymbol{S}, \boldsymbol{C}, \alpha, \boldsymbol{H}) = \sqrt{(S^{IJ} - H^{IJ})(S^{KL} - H^{KL})C_{IK}C_{JL} - \frac{1}{3}(S^{KL}C_{KL})^2} - \sqrt{\frac{2}{3}}(\sigma_y + K\alpha), \tag{16}$$

$$\dot{\boldsymbol{C}}^{\boldsymbol{p}-1} = h\boldsymbol{N}\langle\boldsymbol{N} : \dot{\boldsymbol{S}}\rangle, \tag{17}$$

$$\dot{\alpha} = \sqrt{\tfrac{2}{3}}h\langle\boldsymbol{N} : \dot{\boldsymbol{S}}\rangle, \tag{18}$$

$$\dot{\boldsymbol{H}} = \tfrac{2}{3}Hh\boldsymbol{N}\langle\boldsymbol{N} : \dot{\boldsymbol{S}}\rangle, \tag{19}$$

where $\boldsymbol{C^P}$, $\boldsymbol{H}$, and $\boldsymbol{N}$ are the pull-backs in the reference configuration of the tensors $\boldsymbol{G_e}$, $\boldsymbol{H_e}$, and $\boldsymbol{N_e}$, respectively, by the plastic flow. It is concluded that $\boldsymbol{C^P}$, besides being the primary measure of plastic deformation (see Equation (17)), also plays the role of the aforementioned referential metric.

It is noted that unlike the theoretical presentation, which was developed primarily in the reference configuration, the model is developed primarily in the relaxed space. Thus the relaxed space, as well as any other configuration of the body, can also serve as a reference configuration. This point of view enables us to visualize the deeper inside of the notion of spatial covariance, according to which all configurations of the body are practically indistinguishable and the equation forming is a matter of observation. This statement is an interpretation within the generalized plasticity context of the comment by Dyson [1972], "Einstein based his theory on the principle that God did not attach any preferred labels to the points of space-time." As a result, once the equations describing the state of the body are known in some configuration, they are known in any configuration by employing the covariant transformation laws. An application the equivalent setting of the model in the current configuration can be derived by a push-forward operation to Equations (16), (17), (18), and (19) by the total motion as

$$\varphi(\boldsymbol{\tau}, \boldsymbol{g}, a, \boldsymbol{h}) = \sqrt{(\tau^{ij} - h^{ij})(\tau^{kl} - h^{kl})g_{ik}g_{jl} - \tfrac{1}{3}(\tau^{kl}g_{kl})^2} - \sqrt{\tfrac{2}{3}}(\sigma_y + Ka), \tag{20}$$

$$L_v\boldsymbol{b^e} = h\boldsymbol{n}\langle\boldsymbol{n} : L_v\boldsymbol{\tau}\rangle, \tag{21}$$

$$\dot{\alpha} = \sqrt{\tfrac{2}{3}}h\langle\boldsymbol{n} : L_v\boldsymbol{\tau}\rangle, \tag{22}$$

$$L_v\boldsymbol{h} = h\tfrac{2}{3}H\boldsymbol{n}\langle\boldsymbol{n} : L_v\boldsymbol{\tau}\rangle, \tag{23}$$

where $\boldsymbol{b^e}$, $\boldsymbol{h}$, and $\boldsymbol{n}$ are the push forwards into the current configuration of the referential tensors $\boldsymbol{C^p}$, $\boldsymbol{H}$, and $\boldsymbol{N}$, respectively.

To this end, it is emphasized that the presented covariant approach has been discussed, on the basis of physical grounds, by Dafalias [1998] (see also [Dafalias 1993; 2001]). In particular, it is argued that a rate equation for the evolution of a tensorial internal variable in terms of the convected Lie derivative embodies only the evolutionary characteristics of this internal variable and not its (possible) orientational characteristics related to the material substructure, which must be accounted by the constitutive model. In order to accomplish this goal, Dafalias [1998] adopts a flow rule in the form (see also [Dafalias 1993]) $\mathrm{Sym}\left[\boldsymbol{L_p}\right] = \dot{\gamma}\boldsymbol{N_e}(\boldsymbol{T}, a, \boldsymbol{H_e})$, where $\boldsymbol{T}$ is a stress tensor defined in the relaxed space in terms of the

Cauchy stress tensor $\boldsymbol{\sigma}$ as $\boldsymbol{T} = \det(\boldsymbol{F}^e)\boldsymbol{F}^{e-1}\boldsymbol{\sigma}\boldsymbol{F}^{e-T}$, while for the evolution of the back stress tensor proposes an equation in terms of a corotational derivative as

$$\hat{\boldsymbol{H}}_e = \dot{\boldsymbol{H}}_e - \boldsymbol{\omega}\boldsymbol{H}_e + \boldsymbol{H}_e\boldsymbol{\omega} = \dot{\gamma}\boldsymbol{M}(\boldsymbol{T}, \alpha, \boldsymbol{H}_e),$$

where $\boldsymbol{\omega}$ is defined as the constitutive spin and is related to the aforementioned orientational characteristics inherited to the back stress tensor due to the material substructure and $\boldsymbol{M}$ is a tensorial function, which, due to invariance requirements, is considered as isotropic. The determination of $\boldsymbol{\omega}$ lies crucially on the fact that, in the absence of plastic deformation ($\dot{\gamma} = 0$), one has $\hat{\boldsymbol{H}}_e = \boldsymbol{0}$, and $\boldsymbol{H}_e$ just spins by $\boldsymbol{\omega}$. By noting further that, in the absence of plastic deformation, the relaxed space spins by the antisymmetric part of the plastic velocity gradient $\boldsymbol{L}_p$, ant$\left[\boldsymbol{L}_p\right]$, it is proposed that $\boldsymbol{\omega}$ can be determined by an expression of the form

$$\text{ant}\left[\boldsymbol{L}_p\right] = \boldsymbol{\omega} + \boldsymbol{W}^p = \boldsymbol{\omega} + \dot{\gamma}\boldsymbol{\Omega}(\boldsymbol{T}, \alpha, \boldsymbol{H}_e),$$

where $\boldsymbol{W}^p$ is defined as the plastic spin and $\boldsymbol{\Omega}$ is an isotropic function of the state variables. From the authors' point of view, noting that both the convected and the corotational mode of evolution for the internal variables are different manifestations of the Lie derivative concept [Marsden and Hughes 1994, p.100], the adequate form of evolution has to be decided on the basis of the experimentally observed behavior. It should become clear that, unlike the Lie derivative concept, which is a purely kinematical one, the constitutive and plastic spin concepts require the existence of a substructure whose kinematics may be different from those of the continuum. These issues, together with a possible extension of the proposed covariant formulation, in order to account for crystal plasticity and crystal defects, are a subject of our ongoing research. Finally, the stress response is assumed to be hyperelastic, governed by an isotropic strain energy function proposed within the context of nonlinear elasticity by Ciarlet [1988] and utilized in a somewhat different format within the context of classical plasticity in [Simo and Hughes 1997, p. 258],

$$W = \lambda\frac{I_{e3} - 1}{4} - \left(\frac{\lambda}{2} + \mu\right)\ln\sqrt{I_{e3}} + \tfrac{1}{2}\mu(I_{e1} - 3),$$

where $I_{e1} = \text{tr}(\boldsymbol{C}_e\boldsymbol{G}_e^{-1})$ and $I_{e3} = \det(\boldsymbol{C}_e\boldsymbol{G}_e^{-1})$ are the first and third invariants of $\boldsymbol{C}_e\boldsymbol{G}_e^{-1}$, and $\lambda$ and $\mu$ are material parameters to be interpreted as Lamé constants. Then the stress response in the relaxed space is determined by

$$\boldsymbol{S}_e = 2\frac{\partial W}{\partial \boldsymbol{C}_e}, \tag{24}$$

which yields

$$\boldsymbol{S}_e = \lambda\frac{I_{e3} - 1}{2}\boldsymbol{C}_e^{-1} + \mu(\boldsymbol{G}_e^{-1} - \boldsymbol{C}_e^{-1}). \tag{25}$$

By employing once more the standard pull-back and push-forward operations, Equation (25) may be equivalently written in the forms

$$\boldsymbol{S} = \lambda\frac{I_3 - 1}{2}\boldsymbol{C}^{-1} + \mu(\boldsymbol{C}^{p-1} - \boldsymbol{C}^{-1}), \qquad \boldsymbol{\tau} = \lambda\frac{i_3 - 1}{2}\boldsymbol{g}^{-1} + \mu(\boldsymbol{b}^e - \boldsymbol{g}^{-1}), \tag{26}$$

where $I_3$ and $i_3$ are the third invariants of the tensors $\boldsymbol{C}\boldsymbol{C}^{p-1}$ and $\boldsymbol{g}\boldsymbol{b}^e$, respectively.

## 4. Computational aspects

The numerical implementation of a generalized plasticity based model relies crucially on the fact that, unlike the classical elastoplastic case, the internal variables are no longer constrained to lie within the closure of the elastic domain. Accordingly, unlike the classical elastoplastic case where the evolution equations define a unilaterally constrained problem of evolution, in the case of generalized plasticity the evolution equations form a differential system, which must obey the continuous form of the loading-unloading conditions (see Equations (7) and (9)) [Panoskaltsis et al. 1997].

As a result it is concluded that, from a theoretical point of view, by means of the continuous form of the loading-unloading conditions one has a complete characterization not only of the current state of the material (elastic or plastic), but also, in the case the material state is plastic, of the type of the applied loading process (elastic unloading, neutral loading, plastic loading). From a computational point of view, the crucial requirement for the numerical implementation of an elastoplastic model, simply consists of the unambiguous knowledge of whether plastic loading takes place. For the classical elastoplastic case this requirement is provided by the introduction of the Kuhn–Tucker conditions of the theory of optimization, which, as it is noted in [Simo and Hughes 1997, p.84], imply the generalization of the loading-unloading criteria of the strain-space plasticity as they are given, for instance, in [Naghdi 1990]. Unlike this case, in our case the aforementioned requirement can be provided directly from the stress-space loading-unloading conditions by means of the (algorithmic) parameters $\boldsymbol{Z} = \boldsymbol{\Lambda}\langle \boldsymbol{N} : \dot{\boldsymbol{S}}\rangle$ and $z = \boldsymbol{\lambda}\langle \boldsymbol{n} : L_v \boldsymbol{\tau}\rangle$. By use of these parameters, and in view of the basic evolution equations (see Equations (5) and (8)), we state the algorithmic loading-unloading conditions as if $\boldsymbol{Z} = \boldsymbol{0}$ then $\dot{\boldsymbol{Q}} = \boldsymbol{0}$, if $\boldsymbol{Z} \neq \boldsymbol{0}$ then $\dot{\boldsymbol{Q}} \neq \boldsymbol{0}$, or, equivalently, if $z = \boldsymbol{0}$, then $L_v \boldsymbol{q} = \boldsymbol{0}$, if $z \neq \boldsymbol{0}$ then $L_v \boldsymbol{q} \neq 0$.

From now on, our analysis will be focused on the rather simple model proposed in Section 3. The concepts which will be presented on the basis of this model can be extended, with some computational cost, in more sophisticated models encompassing nonconstant elasticities, nonnormality flow rules, multiple hardening mechanisms, and damage.

The time integration procedure may in principle be formulated equivalently with respect to the reference or the current configurations. Since we deal with large scale plastic flow, the kinematics of the problem, together with the principle of covariance, suggest that a numerical formulation in terms of the Kirchhoff stress and its convected derivative (see Equations (20), (21), (22), (23), and (26)) is more fundamental. Further, in the current configuration the spatial metric usually has a diagonal form, which makes the computations simpler than those in the reference configuration, where the (convected) metric $\boldsymbol{C}$ is fully populated [Miehe 1998]. The details of the implementation procedure follow.

Let $I = [0, T]$, the time interval of interest. It is assumed that at time $t_n \in I$, the configuration of the body of interest $b_n$, defined as $b_n = \{\boldsymbol{x}_n = \boldsymbol{x}_n(\boldsymbol{X}) \mid \boldsymbol{X} \in B\}$, along with the state variables $\{\boldsymbol{x}_n, \boldsymbol{\tau}_n, \boldsymbol{b}^e_n, \alpha_n, \boldsymbol{h}_n\}$, are the known data at time $t_n$.

Assume a time increment $\Delta t_n$, which drives the time to $t_{n+1} = t_n + \Delta t_n$, and the body configuration to

$$b_{n+1} = \{\boldsymbol{x}_{n+1} = \boldsymbol{x}_{n+1}(\boldsymbol{X}) \mid \boldsymbol{X} \in B\},$$

where

$$\boldsymbol{x}_{n+1}(\boldsymbol{X}) = \boldsymbol{x}_n(\boldsymbol{X}) + \boldsymbol{U}(\boldsymbol{X}) = \boldsymbol{x}_n(\boldsymbol{X}) + \boldsymbol{u}(\boldsymbol{x}_n(\boldsymbol{X})),$$

and $\boldsymbol{u}$ is the incremental displacement field, which is assumed to be given. Then the algorithmic problem at hand is to update the stress tensor and the internal variables to the time step $t_{n+1}$ in a manner consistent with the continuous Equations (20), (21), (22), (23), and (26). To this end the continuous equations will be discretized by the backward Euler scheme which is first order accurate and unconditionally stable. Because of the presence of Lie derivatives within the continuous equations, adequate approximations for these objects are derived on the basis of their defining property and the general tensorial transformation law. In particular, the defining relation for the Lie derivative of a tensor $\boldsymbol{q}$ of type $\binom{r}{s}$ in the $b_{n+1}$ configuration is

$$L_v \boldsymbol{q}_{n+1} = \boldsymbol{x}_{n+1*}\left(\frac{\partial}{\partial t}\boldsymbol{x}_{n+1}{}^{*}(\boldsymbol{q})\right). \tag{27}$$

By performing a pull-back operation, Equation (27) can be written consecutively as

$$\boldsymbol{x}^{*} L_v \boldsymbol{q}_{n+1} = \frac{\partial}{\partial t}(\boldsymbol{x}^{*}(\boldsymbol{q}_{n+1})) = \dot{\boldsymbol{Q}}_{n+1} \doteq \frac{1}{\Delta t_n}(\boldsymbol{Q}_{n+1} - \boldsymbol{Q}_n),$$

which in turn may be written in component form on the basis of the general tensorial transformation law as

$$\left[\frac{\partial X^{I_1}}{\partial x^{i_1}_{n+1}} \cdots \frac{\partial X^{I_r}}{\partial x^{i_r}_{n+1}} \frac{\partial x^{j_1}_{n+1}}{\partial X^{J_1}} \cdots \frac{\partial x^{j_s}_{n+1}}{\partial X^{J_s}}\right] L_v (q^{i_1\cdots i_r}{}_{j_1\cdots j_s})_{n+1} = \frac{1}{\Delta t_n}\left[(Q^{I_1\cdots I_r}{}_{J_1\cdots J_S})_{n+1} - (Q^{I_1\cdots I_r}{}_{J_1\cdots J_S})_n\right]$$

$$= \frac{1}{\Delta t_n}\left[\frac{\partial X^{I_1}}{\partial x^{k_1}_{n+1}} \cdots \frac{\partial X^{I_r}}{\partial x^{k_r}_{n+1}} \frac{\partial x^{l_1}_{n+1}}{\partial X^{J_1}} \cdots \frac{\partial x^{l_s}_{n+1}}{\partial X^{J_s}} (q^{k_1\cdots k_r}{}_{l_1\cdots l_s})_{n+1} - \frac{\partial X^{I_1}}{\partial x^{k_1}_{n}} \cdots \frac{\partial X^{I_r}}{\partial x^{k_r}_{n}} \frac{\partial x^{l_1}_{n}}{\partial X^{J_1}} \cdots \frac{\partial x^{l_s}_{n}}{\partial X^{J_s}} (q^{k_1\cdots k_r}{}_{l_1\cdots l_s})_{n}\right],$$

from which $L_v (q^{i_1\cdots i_r}{}_{j_1\cdots j_s})_{n+1}$ can be determined as

$$L_v (q^{i_1\cdots i_r}{}_{j_1\cdots j_s})_{n+1} = \frac{1}{\Delta t_n}\left[(q^{i_1\cdots i_r}{}_{j_1\cdots j_s})_{n+1} - \frac{\partial x^{i_1}_{n+1}}{\partial x^{k_1}_{n}} \cdots \frac{\partial x^{i_r}_{n+1}}{\partial x^{k_r}_{n}} \frac{\partial x^{l_1}_{n}}{\partial x^{j_1}_{n+1}} \cdots \frac{\partial x^{l_s}_{n}}{\partial x^{j_s}_{n+1}} (q^{k_1\cdots k_r}{}_{l_1\cdots l_s})_{n}\right], \tag{28}$$

where the tensor, with components

$$(f^{i}{}_{j})_{n+1} = \frac{\partial x^{i}_{n+1}}{\partial x^{j}_{n}} = \frac{\partial x^{i}_{n+1}}{\partial X^{I}} \frac{\partial X^{I}}{\partial x^{j}_{n}} = (F^{i}_{I})_{n+1}((F^{-1})^{I}_{j})_n,$$

is defined as the relative deformation gradient with respect to the configuration $b_{n+1}$ [Simo and Hughes 1997, p. 279]. It is interesting to note that since any objective derivative of a tensorial quantity $\boldsymbol{q}$ differs for its convected Lie derivative by terms depending on $\boldsymbol{q}$ and the Lie derivative of the spatial metric $\boldsymbol{g}$ (see [Atluri 1984] and [Marsden and Hughes 1994, p. 100]), (28) can be used as a basis for the objective approximation of other objective derivatives, which may be used in place of the convected derivative, used herein.

By means of Equation (28) a (covariant) approximation for a contravariant $\binom{0}{2}$ tensor $\boldsymbol{q}$, like the tensors $\boldsymbol{\tau}$, $\boldsymbol{b}^{e}$, and $\boldsymbol{h}$, participating in the proposed model is derived as

$$L_v (q^{ij})_{n+1} = \frac{1}{\Delta t_n}\left[(q^{ij})_{n+1} - \frac{\partial x^{i}_{n+1}}{\partial x^{k}_{n}} \frac{\partial x^{j}_{n+1}}{\partial x^{l}_{n}} (q^{kl})_n\right],$$

or equivalently

$$L_v \boldsymbol{q}_{n+1} = \frac{1}{\Delta t_n}\big(\boldsymbol{q}_{n+1} - \boldsymbol{f}_{n+1}\boldsymbol{q}_{n+1}\boldsymbol{f}^T_{n+1}\big).$$

Accordingly, the time discrete counterparts of Equations (21), (22), (23) and (26) are

$$\frac{1}{\Delta t_n}(\boldsymbol{b}^{\boldsymbol{e}}_{n+1} - \boldsymbol{f}_{n+1}\boldsymbol{b}^{\boldsymbol{e}}_n\boldsymbol{f}^T_{n+1}) = h_{n+1}\boldsymbol{n}_{n+1}l_{n+1}, \tag{29}$$

$$\frac{1}{\Delta t_n}(\alpha_{n+1} - \alpha_n) = \sqrt{\frac{2}{3}}h_{n+1}l_{n+1}, \tag{30}$$

$$\frac{1}{\Delta t_n}(\boldsymbol{h} - \boldsymbol{f}_{n+1}\boldsymbol{h}_n\boldsymbol{f}^T_{n+1}) = \frac{2}{3}Hh_{n+1}\boldsymbol{n}_{n+1}l_{n+1}, \tag{31}$$

$$\boldsymbol{\tau}_{n+1} = \lambda\frac{\det(\boldsymbol{g}_{n+1}\boldsymbol{b}^{\boldsymbol{e}}_{n+1}) - 1}{2}\boldsymbol{g}^{-1}_{n+1} + \mu(\boldsymbol{b}^{\boldsymbol{e}}_{n+1} - \boldsymbol{g}^{-1}_{n+1}), \tag{32}$$

where

$$\begin{aligned}
\boldsymbol{g}_{n+1} &= \boldsymbol{g}(\boldsymbol{x}_{n+1}),\\
h_{n+1} &= h(\boldsymbol{g}_{n+1}, \boldsymbol{\tau}_{n+1}, \boldsymbol{b}^{\boldsymbol{e}}_{n+1}\boldsymbol{h}_{n+1}),\\
l_{n+1} &= \left\langle \boldsymbol{n} : \frac{1}{\Delta t_n}(\boldsymbol{\tau}_{n+1} - \boldsymbol{f}_{n+1}\boldsymbol{\tau}_n\boldsymbol{f}^T_{n+1})\right\rangle
\end{aligned}$$

and

$$\boldsymbol{n}_{n+1} = \frac{(\boldsymbol{\tau}_{n+1} - \boldsymbol{h}_{n+1}) - \frac{1}{3}\{(\boldsymbol{\tau}_{n+1} - \boldsymbol{h}_{n+1}) : \boldsymbol{g}_{n+1}\}\boldsymbol{g}^{-1}_{n+1}}{\|(\boldsymbol{\tau}_{n+1} - \boldsymbol{h}_{n+1}) - \frac{1}{3}\{(\boldsymbol{\tau}_{n+1} - \boldsymbol{h}_{n+1}) : \boldsymbol{g}_{n+1}\}\boldsymbol{g}^{-1}_{n+1}\|},$$

are quantities expressed in terms of the basic variables, subjected to the time discrete counterpart of the algorithmic loading-unloading conditions, which can be written as:

$$\text{If } z_{n+1} = 0, \text{ then } \begin{cases} \boldsymbol{b}^{\boldsymbol{e}}_{n+1} &= \boldsymbol{f}_{n+1}\boldsymbol{b}^{\boldsymbol{e}}_n\boldsymbol{f}^T_{n+1},\\ \alpha_{n+1} &= \alpha_n,\\ \boldsymbol{h}_{n+1} &= \boldsymbol{f}_{n+1}\boldsymbol{h}_n\boldsymbol{f}^T_{n+1}, \end{cases} \tag{33}$$

and

$$\text{If } z_{n+1} \neq 0, \text{ then } \begin{cases} \boldsymbol{b}^{\boldsymbol{e}}_{n+1} &\neq \boldsymbol{f}_{n+1}\boldsymbol{b}^{\boldsymbol{e}}_n\boldsymbol{f}^T_{n+1},\\ \alpha_{n+1} &\neq \alpha_n,\\ \boldsymbol{h}_{n+1} &\neq \boldsymbol{f}_{n+1}\boldsymbol{h}_n\boldsymbol{f}^T_{n+1}, \end{cases} \tag{34}$$

where $z_{n+1} = h_{n+1}l_{n+1}$. It is observed that Equations (29), (30), (31) and (32), subjected to the time discrete algorithmic loading-unloading conditions of Equation (33), form a system of four equations in four unknowns ($\boldsymbol{b}^{\boldsymbol{e}}_{n+1}$, $\alpha_{n+1}$, $\boldsymbol{h}_{n+1}$, and $\boldsymbol{\tau}_{n+1}$). The solution of this system can be performed by a predictor-corrector algorithm like the one presented by Panoskaltsis et al. [1997], in conjunction with some developments proposed within the context of large deformation computational plasticity, by Simo and Ortiz [1985] and Auricchio and Taylor [1999]. It is noted that, unlike the classical elastoplastic case, the consistency condition and accordingly the consistency parameter are absent from the model governing equations. Due to this absence the resulting system is simpler than in the classical elastoplastic case and more computer power is preserved.

## 5. Numerical simulations

The predictions of the model introduced in Section 3 will be illustrated by considering two problems of large scale plastic flow, namely a simple shear test and the biaxial extension of a material block. The model will be implemented numerically by following our development in Section 4. The model parameters are $\lambda = 330$, $\mu = 150$, $\sigma_y = 20$, $K = 15$, and $H = 0$.

The function $h$ is set as

$$h = \frac{\langle \Phi \rangle}{\beta |\Phi|} \qquad \text{and} \qquad \frac{1}{\beta}$$

for $\Phi = 0$, where $\beta$ is a model parameter.

In this case the elastic domain is the manifold defined by the set

$$D(\boldsymbol{C}, \boldsymbol{S}, \alpha, \boldsymbol{H}) = \{(\boldsymbol{C}, \boldsymbol{S}, \alpha, \boldsymbol{H}) / \Phi(\boldsymbol{C}, \boldsymbol{S}, \alpha, \boldsymbol{H}) < 0\},$$

while the elastic range $E$ is defined at any material state, by noting the one to one correspondence which exists between the end point of a vector of constant origin and the vector itself, as

$$E(\boldsymbol{C}^*, \boldsymbol{S}^*)|_{(\alpha, \boldsymbol{H})=\text{constant}} = \Big\{(\boldsymbol{C}^*, \boldsymbol{S}^*) \backslash \boldsymbol{C}^* = \boldsymbol{C} + \dot{\boldsymbol{C}},\ \boldsymbol{S}^* = \boldsymbol{S} + \dot{\boldsymbol{S}},\ \text{ if } \Phi(\boldsymbol{C}, \boldsymbol{S}, \alpha, \boldsymbol{H})|_{(\alpha, \boldsymbol{H})=\text{constant}} < 0 \text{ or } \boldsymbol{N} : \dot{\boldsymbol{S}} \leq 0\Big\}.$$

The elastic domain at the state in question is defined as

$$D(\boldsymbol{C}^*, \boldsymbol{S}^*)|_{(\alpha, \boldsymbol{H})=\text{constant}} = \Big\{(\boldsymbol{C}^*, \boldsymbol{S}^*) \backslash \Phi(\boldsymbol{C}^*, \boldsymbol{S}^*, \alpha, \boldsymbol{H})|_{(\alpha, \boldsymbol{H})=\text{constant}} < 0\Big\},$$

which is clearly a submanifold of $E(\boldsymbol{C}^*, \boldsymbol{S}^*)|_{(\alpha, \boldsymbol{H})=\text{constant}}$.

The limit $\beta \to 0$ corresponds to classical plasticity. In this case the initial loading surface defined by $\Phi = 0$ coincides with the yield surface of classical plasticity [Eisenberg and Phillips 1971], while the limit $h_{\beta \to 0}$ is determined by the consistency condition (6).

The particular case where $h$ is considered as a positive function of the state variables (for example, constant, exponential, hyperbolic) corresponds to a model with a quasiyield surface. In this model every state is a plastic state, plastic loading appears from the initiation of loading, and every reloading process, following (elastic) unloading, results in plastic response.

The simple shear problem [Gurtin 1981, p. 115] is defined by

$$x^1 = X^1 + \gamma X^2, \qquad x^2 = X^2, \qquad x^3 = X^3,$$

where $\gamma$ is the shearing parameter. This problem has been used extensively as a testing problem (see, for example, [Lee et al. 1983; Dafalias 1983; Haupt and Tsakmakis 1986; Atluri 1984]) within the context of large deformation plasticity.

The predictions of the model for different values of the parameter $\beta$ are shown in Figure 1. We note that for large values of $\beta$ the predicted response is identical to that of a perfectly plastic material. Furthermore, the oscillating behavior, which is reported in [Atluri 1984] in finite shear, even in the case of classical isotropic hardening plasticity, does not appear.

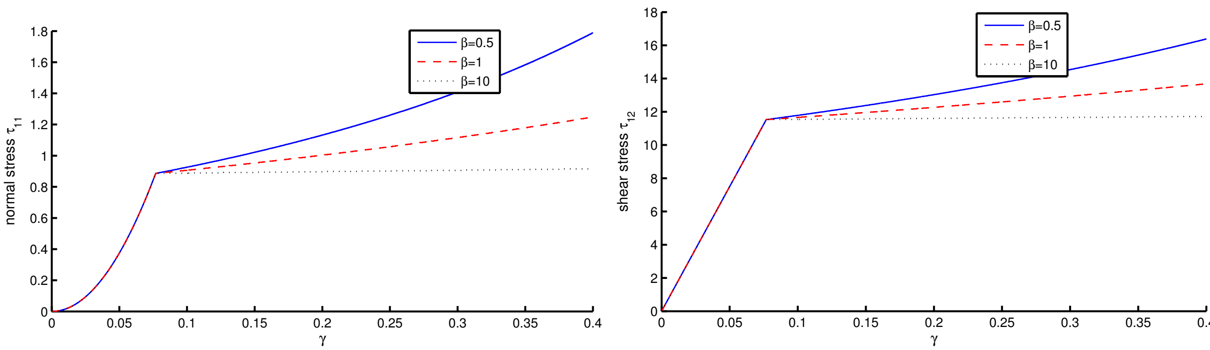


**Figure 1.** Normal (left) and shear (right) stresses versus the shearing parameter.

The second problem is the biaxial extension of a material block. The straining occurs along the $X^1$ and $X^2$ axes while the block is assumed to be fixed along the $X^3$ direction. This problem is defined as

$$x^1 = (1+\lambda)X^1, \qquad x^2 = (1+\omega)X^2, \qquad x^3 = X^3,$$

where $\lambda$ and $\omega$ are the straining parameters. The predictions for the normal stresses for different interrelations of the straining parameters are shown in Figure 2.

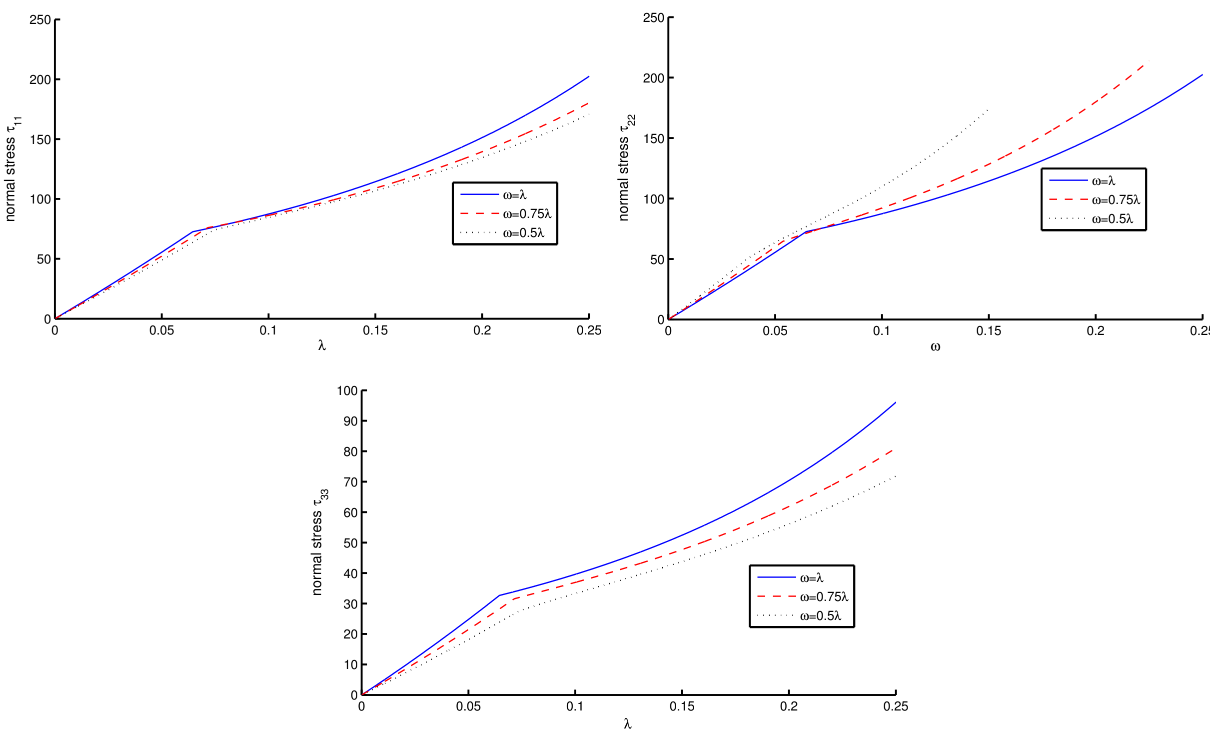


**Figure 2.** Biaxial extension of a material block: normal stresses versus straining parameters.

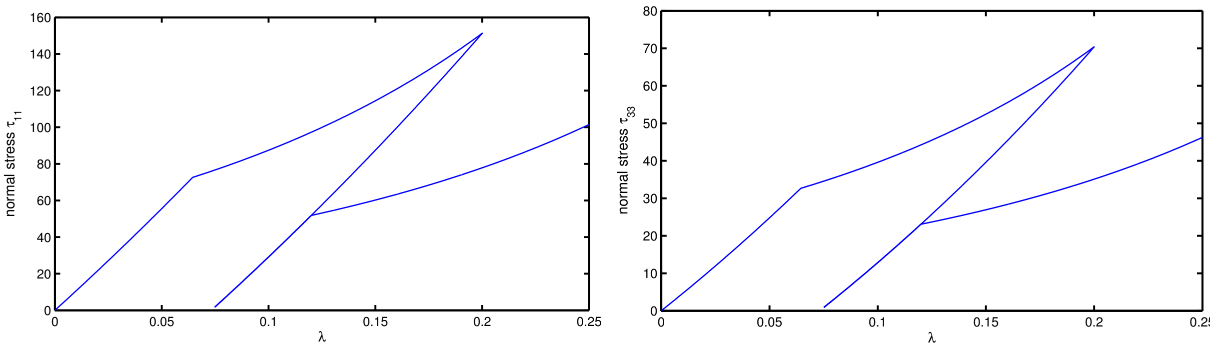


**Figure 3.** Biaxial extension of a material block: initial loading, unloading, and reloading.

A second loading history comprising loading, unloading from a plastic state, and reloading is given for $\lambda = \omega$ in Figure 3. We note that, consistently with a generalized plasticity based model, during reloading, after unloading from a plastic state, plastic behavior appears before attaining the state where the unloading began.

## 6. Conclusions

One of the main contributions of this paper is the presentation of the (stress space) covariant formulation of rate independent generalized plasticity. For this purpose, the manifold structure of the body, as well as of the ambient and the state space, is postulated. In the course of the development of the theory, and based on geometry of manifolds, the consistency condition of classical plasticity is derived. A rather simple model is proposed in order to emphasize the covariant presentation of the generalized plasticity concept in its most simple setting. The model is developed in the (elastically) relaxed space. By employing the pull-back and push-forward operations the model is also derived in the reference and current configurations, respectively. A time integration algorithm, in the current configuration, is developed in detail. Appropriate algorithmic approximations of the Lie derivatives of the tensorial quantities entering the algorithm are derived. Also, algorithmic loading-unloading conditions are derived. The proposed model is tested numerically in the solution of two problems of large scale plastic flow.

Further research directions comprise the derivation of more sophisticated models that include rate, thermal, and anisotropic effects for the accurate description of solid behavior, as well as the development of the additional necessary computational tools for the implementation of those models within the context of the finite element method.

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