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LOUIS MILTON BROCK

A rigid cylinder rolls at constant speed on a thermoelastic half-space under a compressive load. Heat flow across the contact zone is neglected, and the zone has a central region of perfect contact and two edge regions of frictionless slip. A robust asymptotic inversion of the exact transform solution to a related unmixed boundary value problem allows the mixed-mixed problem of rolling contact to be solved analytically. The solution is compared with that for perfect rolling contact. Both show variations in contact zone size and temperature change with rolling speed and load. Distinctions exist however: slip zones preclude oscillatory solution behavior and are much smaller than zones of oscillation. Moreover, perfect rolling contact may exaggerate the difference between imposed and effective angular velocity due to surface deformation.

Introduction

Models for rolling contact that involve elastic bodies have been developed within the framework of contact mechanics [Muskhelishvili 1975; Gladwell 1980; Johnson 1987; Hills and Barber 1993; Hills et al. 1993] and empirical observation [Bayer 1994; Blau 1996]. The more basic models are generally quasistatic, assume Hertzian contact, and are isothermal. However, more recent studies consider, variously, thermoelastic contact and inertial effects [Hills and Barber 1986; Georgiadis and Barber 1993; Pauk and Yevtushenko 1997; Barber 1999; Jang 2000; Pauk and Zastrau 2002; Andersson et al. 2005; Jang 2005]. In addition, studies of the mathematically-similar problem of the interface crack [Hills and Barber 1993; Hills et al. 1993] address issues that also arise in contact. Two key issues are the oscillatory solution behavior that occurs when perfect contact is modeled and a Hertzian contact zone stress distribution is not assumed, and the role of critical speed in contact zone formation.

In light of these issues, Brock [2004a; 2004b] considered a rigid cylinder of infinite length rolling at constant speed over a thermoelastic half-space. A dynamic steady state of plane strain was assumed, and robust asymptotic solutions to the mixed-mixed problem were obtained analytically. These exhibited clear variations with rolling speed and increases in contact zone temperature. The increase was prominent when the compressive force on the cylinder was large enough to produce contact zone compressive stress that neared values critical for yield.

The solutions also exhibited the aforementioned oscillation near contact zone edges. This is, of course, typical of mixed-mixed problems [Muskhelishvili 1975], but the behavior violates the assumption of nontensile contact zone stress. On the other hand, violation is confined to edge zones that are orders

Keywords: rolling contact, slip zones, perfect contact, effective angular velocity, coupled thermoelasticity, mixed-mixed problem.

of magnitude smaller than the zone itself. These microzones can be interpreted to mean that slip must occur at the contact zone edges.

For additional insight into their effects, this article imposes slip zones in the mixed-mixed problem for rolling contact. For simplicity, friction is ignored, and the resultant force that keeps the cylinder in contact with the surface acts through the cylinder axis. Jang [2000; 2005] has presented results for the basic problem of transient thermoelastic contact. However, to allow comparison with [Brock 2004a], coupled thermoelasticity governs here, but a dynamic steady state of plain strain is considered. Similarly, while it will be seen that a thermoelastic Rayleigh speed produces critical solution behavior, rolling speed is subcritical. Insight into behavior at supercritical speed can be found in work by Georgiadis and Barber [1993] and Brock and Georgiadis [2000].

The study begins in the next section with the problem formulation. Subsequently, exact expressions for the integral transform solution to a related unmixed problem are presented, and robust approximate inversions extracted. An analytical result for coupled singular integral equations provides a candidate solution. The solution itself follows by enforcing auxiliary conditions for rolling contact. Numerical calculations of contact zone parameters are then compared with those presented in [Brock 2004a]. Although similar, the two calculation sets illustrate distinctive behaviors. In particular, slip zones are orders of magnitude smaller than zones of oscillation seen in the no-slip model.

Problem formulation

Consider a linear isotropic thermoelastic half-space defined by the Cartesian coordinates (x, y, z) as the region $y > 0$. It is initially at rest at uniform (absolute) temperature T_0 when a rigid cylinder of infinite length and radius r is pressed into the surface with constant force (per unit of cylinder length) F and rolled in the positive x-direction with constant subcritical speed v. It will be shown that this speed corresponds to a thermoelastic Rayleigh speed. The process creates a zone of perfect contact between the cylinder and the half-space that is bordered by two zones of slip (frictionless sliding contact). The cylinder geometry is independent of coordinate z, so that the process is one of plane strain. Because (r, F, v) are constant, it is also assumed that a dynamic steady state is achieved in which the perfect contact and slip zones maintain constant widths.

It is convenient, then, to locate the Cartesian system origin below the cylinder axis $(x = 0)$ and translate it with the same speed v. The boundary conditions governing the half-space surface $y = 0$ can then be written as [Brock 2004a]

$$\partial_y \theta = 0 \quad (\text{all } x), \qquad (\sigma_{xy}, \sigma_y) = 0 \quad (x \notin C), \tag{1a}$$

$$\sigma_{xy} = 0, \qquad \partial_x u_y = -\frac{x}{r} \quad (x \in C_\pm), \tag{1b}$$

$$\partial_x u_x = -\frac{\dot{U}_0}{v_r} - \frac{x^2}{2r^2}, \qquad \partial_x u_y = -\frac{x}{r} \quad (x \in C_0). \tag{1c}$$

Here (u_x, u_y, θ) are the (x, y)-displacements and change in temperature from T_0, and $(\sigma_x, \sigma_y, \sigma_z, \sigma_{xy})$ are the tractions in plane strain. These quantities depend only on (x, y), and (∂_x, ∂_y) signify (x, y)-differentiation. Constant $\dot{U}_0$ is the unknown tangential velocity of the contact zone directly below the cylinder axis $(x = 0)$. This is the point of maximum depression and has no normal velocity. Region C

is the contact zone defined by $L_- < x < L_+$. Region C_0, defined by $l_- < x < l_+$, is the zone of perfect contact in C. Regions $C_\pm$ are slip zones defined by $l_+ < x < L_+$ and $L_- < x < l_-$. Lengths $(L_\pm, l_\pm)$ are unknown, but one can assume that

$$L_- < l_- < 0 < l_+ < L_+, \qquad l < L \ll r, \tag{2}$$

$$L = L_+ - L_-, \qquad l = l_+ - l_-. \tag{3}$$

The first condition in Equation (1a) imposes an assumption that heat flux through C is negligible. For a half-space that obeys the Fourier model of coupled thermoelasticity, governing equations for $y > 0$ can be written as [Brock and Georgiadis 2000; Brock 2004a]

$$\partial_x\sigma_x + \partial_y\sigma_{yx} = c^2\partial_x^2 u_x, \qquad \partial_x\sigma_{xy} + \partial_y\sigma_y = c^2\partial_x^2 u_y, \tag{4a}$$

$$h\nabla^2\theta + c\partial_x\left(\frac{\varepsilon}{\alpha_v}\Delta + \theta\right) = 0, \tag{4b}$$

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{bmatrix} = \mu\begin{bmatrix} m+1 & m-1 & 1\\ m-1 & m+1 & 1\\ m-1 & m-1 & 1\end{bmatrix}\begin{bmatrix}\partial_x u_x\\ \partial_y u_y\\ -\alpha_v\theta\end{bmatrix}, \tag{4c}$$

$$(\sigma_{xy}, \sigma_{yx}) = \mu(\partial_y u_x + \partial_x u_y). \tag{4d}$$

Equations (4b) and (4c) can be used to rewrite Equation (4a) in partly coupled form

$$(a\nabla^2 - c^2\partial_x^2)\Delta - \alpha_v\nabla^2\theta = 0, \qquad (\nabla^2 - c^2\partial_x^2)\varpi_z = 0. \tag{5}$$

Here (Δ, ϖ_z) is the dilatation and rotation. Constants $(m, a, h, \varepsilon, c)$ are given by

$$m = \frac{1}{1-2\nu}, \qquad a = m+1, \qquad h = \frac{k}{c_v\sqrt{\mu\rho}}, \qquad \varepsilon = \frac{\mu T_0}{\rho c_v}\alpha_v^2, \tag{6a}$$

$$c = \frac{v}{v_r}, \qquad v_r = \sqrt{\frac{\mu}{\rho}}. \tag{6b}$$

Quantities (h, ε, v_r) are the thermoelastic characteristic length, dimensionless coupling constant and rotational wave speed; $(\mu, \rho, \nu, k, \alpha_v, c_v)$ are, respectively, shear modulus, mass density, Poisson's ratio, conductivity, volumetric thermal expansion coefficient, and specific heat at constant strain. Various sources [Sokolnikoff 1956; Chadwick 1960; Achenbach 1973; Davis 1998; Brock 1999] indicate that Equation (6a) gives

$$\varepsilon \approx O(10^{-2}), \qquad h \approx O(10^{-8})\,\mathrm{m}. \tag{7}$$

In addition to satisfying Equations (1)–(7), field quantities $(u_x, u_y, \nabla u_x, \nabla u_y, \theta)$ should be continuous for $y \geq 0$ and bounded above as $\sqrt{x^2+y^2} \to \infty$. Smooth separation of the rolling cylinder and half-space surface requires, in particular, that $(\nabla u_x, \nabla u_y)$ are finite at the zone edge $x = (L_\pm, l_\pm)$. Contact is also governed by the constraints

$$\sigma_y \leq 0 \qquad (x \in C), \tag{8a}$$

$$\int_C \sigma_y dx = -F, \qquad \int_C x\sigma_y dx = 0, \qquad \int_{C_0} \sigma_{xy} dx = 0. \tag{8b}$$

The unilateral constraint, Equation (8a), guarantees a nontensile contact zone stress. The first two conditions in Equation (8b) specify that F is the resultant force on the cylinder and that its line of action is through the cylinder axis. The last condition specifies that the cylinder axis does not accelerate. The boundedness-continuity requirements, Equation (8a) and Equation (8b) give additional formulas necessary to find the contact zone parameters $(U_0', L_\pm, l_\pm)$.

Related problem: unmixed boundary conditions

The rolling contact problem is addressed by first considering the related problem of a half-space governed by Equations –(7) and the same boundedness-continuity conditions. The constraint, Equation (8), is relaxed, however, and Equation (1) for $y = 0$ is replaced by the unmixed conditions

$$\partial_y\theta = 0, \qquad \sigma_y = \sigma(x), \qquad \sigma_{xy} = \tau(x), \tag{9a}$$

$$\sigma(x) \equiv 0 \quad (x \notin C), \qquad \tau(x) \equiv 0 \quad (x \notin C_0). \tag{9b}$$

The traction σ is continuous for $x \in C$ and vanishes at $x = L_\pm$, while the traction τ is continuous for $x \in C_0$ and vanishes at $x = l_\pm$. By following [Brock 2004a], an exact solution for the bilateral Laplace transform in x for this related problem can be obtained, and analytical expressions for the inverse can be derived that are valid for $\sqrt{x^2 + y^2} \gg h$. In view of Equation (7), these expressions are robust and are given for all $y \geq 0$ by

$$\begin{aligned}
\partial_x u_x &= -\frac{Ay}{\pi R}\int_C \frac{\sigma}{\mu}\left[\frac{T}{(t-x)^2 + A^2y^2} + \frac{2B^2}{(t-x)^2 + B^2y^2}\right]dt \\
&\qquad\qquad - \frac{B}{\pi R}\int_{C_0} \frac{\tau}{\mu}\left[\frac{2(t-x)}{(t-x)^2 + A^2y^2} + \frac{T(t-x)}{(t-x)^2 + B^2y^2}\right]dt, \\
\partial_x u_y &= \frac{By}{\pi R}\int_{C_0} \frac{\tau}{\mu}\left[\frac{2A^2}{(t-x)^2 + A^2y^2} + \frac{T}{(t-x)^2 + B^2y^2}\right]dt \\
&\qquad\qquad - \frac{A}{\pi R}\int_C \frac{\sigma}{\mu}\left[\frac{T(t-x)}{(t-x)^2 + A^2y^2} + \frac{2(t-x)}{(t-x)^2 + B^2y^2}\right]dt, \\
\theta &= \frac{\varepsilon c^2}{\alpha_v a_\varepsilon}\frac{1}{\pi R}\left[T\int_C \frac{\sigma}{\mu}\frac{Ay}{(t-x)^2 + A^2y^2}dt + 2B\int_{C_0} \frac{\tau}{\mu}\frac{t-x}{(t-x)^2 + B^2y^2}dt\right].
\end{aligned} \tag{10}$$

In Equation (10), the quantities (A, B, T, R) are defined as

$$A = \sqrt{1 - \frac{c^2}{a_\varepsilon}}, \qquad B = \sqrt{1 - c^2}, \qquad T = c^2 - 2, \qquad R = 4AB - T^2, \tag{11a}$$

$$a_\varepsilon = a + \varepsilon. \tag{11b}$$

The quantity R is a form of the classical [Achenbach 1973] Rayleigh function in the dimensionless rolling speed c. It has roots $c = \pm c_R$ $(0 < c_R < 1)$ and these correspond to the Rayleigh speed $v_R = c_R v_r$. Equation (10) exhibits critical behavior as $R \to 0$, so rolling speed in this study is restricted to

$$0 < v < v_R \quad (0 < c < c_R). \tag{12}$$

The critical nature of the Rayleigh speed is well-established in isothermal elastodynamic contact [Craggs and Roberts 1967; Robinson and Thompson 1974; Georgiadis and Barber 1993]. Results for thermoelastic sliding contact with friction for any constant sliding speed are found in [Brock and Georgiadis 2000].

Candidate solution

In view of Equation (9), (σ, τ) correspond to the contact zone traction, and Equation (1a) and the first condition in Equation (1b) are automatically satisfied by Equation (10). Enforcing Equation (1c) and the second condition in Equation (1b), and using Equation (10), give integral equations for (σ, τ):

$$-\frac{c^2 A}{\pi R}(vp)\int_C \frac{\sigma}{\mu}\frac{dt}{t-x} = -\frac{x}{r} \qquad (x \in C_\pm), \tag{13a}$$

$$\frac{N}{R}\frac{\tau}{\mu} - \frac{c^2 A}{\pi R}(vp)\int_C \frac{\sigma}{\mu}\frac{dt}{t-x} = -\frac{x}{r} \qquad (x \in C_0), \qquad N = T + 2AB, \tag{13b}$$

$$-\frac{N}{R}\frac{\sigma}{\mu} - \frac{c^2 B}{\pi R}(vp)\int_{C_0} \frac{\tau}{\mu}\frac{dt}{t-x} = -\frac{\dot{U}_0}{v_r} - \frac{x^2}{2r^2} \qquad (x \in C_0). \tag{13c}$$

Here, (vp) signifies the Cauchy principal value, and use is made of the Dirac relation [Carrier and Pearson 1988],

$$\frac{\eta}{\xi^2+\eta^2} \to \pi\delta(\xi) \quad (\eta \to 0+). \tag{14}$$

To address Equation (13), we introduce the trial functions

$$\frac{\sigma}{\mu} = G_\pm \cos\pi\upsilon_\pm + \frac{Q_+}{\pi}\Sigma_\pm \sin\pi_\pm, \qquad (x \in C_\pm), \tag{15a}$$

$$\frac{\sigma}{\mu} = G_0 \cos\pi\upsilon_0 + \frac{Q_0}{\pi}\Sigma_0 \sin\pi\upsilon_0, \qquad (x \in C_0), \tag{15b}$$

$$\Sigma_\pm = S_0 + S_\mp + (vp)S_\pm, \qquad (x \in C_\pm), \tag{15c}$$

$$\Sigma_0 = S_- + S_+ + (vp)S_0, \qquad (x \in C_0). \tag{15d}$$

Here, $(\upsilon_0, \upsilon_\pm)$ are real-valued constants of magnitude $|\upsilon_0, \upsilon_\pm| < 1$, $(G_0, G_\pm)$ are unknown functions, and

$$\begin{aligned} S_- &= \int_{C_-}\frac{G_-}{Q_-}\frac{dt}{t-x}, & Q_- &= \left(\frac{l_- - x}{x - L_-}\right)^{\upsilon_-}\left(\frac{l_+ - x}{l_- - x}\right)^{\upsilon_0}\left(\frac{L_+ - x}{l_+ - x}\right)^{\upsilon_+}, \\ S_0 &= \int_{C_0}\frac{G_0}{Q_0}\frac{dt}{t-x}, & Q_0 &= \left(\frac{x - l_-}{x - L_-}\right)^{\upsilon_-}\left(\frac{l_+ - x}{x - l_-}\right)^{\upsilon_0}\left(\frac{L_+ - x}{l_+ - x}\right)^{\upsilon_+}, \\ S_+ &= \int_{C_+}\frac{G_+}{Q_+}\frac{dt}{t-x}, & Q_+ &= \left(\frac{x - l_-}{x - L_-}\right)^{\upsilon_-}\left(\frac{x - l_+}{x - l_-}\right)^{\upsilon_0}\left(\frac{L_+ - x}{x - l_+}\right)^{\upsilon_+}. \end{aligned} \tag{16}$$

The trial functions have the property that

$$\frac{1}{\pi}(vp)\int_C \frac{\sigma}{\mu}\frac{dt}{t-x} = -G_\pm \sin\pi\upsilon_\pm + \frac{Q_\pm}{\pi}\Sigma_\pm\cos\pi\upsilon_\pm \quad (x\in C_\pm), \tag{17a}$$

$$\frac{1}{\pi}(vp)\int_C \frac{\sigma}{\mu}\frac{dt}{t-x} = -G_0 \sin\pi_0 + \frac{Q_0}{\pi}\Sigma_0\cos\pi\upsilon_0 \quad (x\in C_0). \tag{17b}$$

Substitution of Equation (15a) and Equation (15c) into Equation (13a), in view of Equation (17a), leads to the result

$$G_\pm = \pm\frac{R}{c^2 A}\frac{x}{r}, \qquad \upsilon_\pm = \mp\frac{1}{2} \quad (x\in C_\pm). \tag{18}$$

In a similar manner, substitution of Equation (15b) into (13b) and (13c), in view of (17b), produces coupled equations for (G_0, τ) that can be solved to give

$$\frac{\tau}{\mu} = \frac{R}{c^2 B}\frac{Q}{\pi}(vp)\int_{C_0}\left(U_0' - \frac{t^2}{2r^2}\right)\frac{dt}{Q(t-x)} \quad (x\in C_0), \tag{19a}$$

$$G_0 = Q_0\frac{Q}{\pi}(vp)\int_{C_0}\frac{P_0}{Q}\frac{dt}{t-x}, \qquad \upsilon_0 = 0 \quad (x\in C_0). \tag{19b}$$

Equation (19) defines the quantities

$$Q = \sqrt{\frac{l_+ - x}{x - l_-}}, \qquad P_0 = \frac{1}{c^2 A Q_0}\left(-R\frac{x}{r} - \frac{N}{2}\frac{\tau}{\mu}\right) + \frac{1}{\pi}(S_+ + S_-). \tag{20}$$

Obtaining Equation (10) and Equations (18)–(20) completes construction of a candidate solution for the rolling problem. The solution itself must be bounded and continuous for $x \in C$ and satisfy auxiliary condition (8).

Rolling contact solution

Equation (19a) is bounded at $x = l_-$, and the last two conditions in (8b) are satisfied if

$$L_\pm = \pm\frac{L}{2}, \qquad l_\pm = \pm\frac{l}{2}, \qquad \dot{U}_0 = -\frac{l^2 v_r}{16r^2}. \tag{21}$$

In view of Equations (18) and (19b), Equations (15a) and (15b) are bounded for all $x \in C$, and the first condition in (8b) is satisfied when

$$\pi + \frac{N}{2c^2 B}\frac{L}{2r}[K(\lambda) - E(\lambda)] = 0, \qquad \lambda = \frac{l}{L}, \tag{22a}$$

$$\frac{\pi R}{c^2 A r^2}\left[\frac{Ll}{2} + \frac{5}{32}(3L + l)(L - l)\right] = \frac{F}{\mu r}. \tag{22b}$$

Here (K, E) are complete elliptic integrals of the first and second kind of modulus λ. The solution to Equation (22) will give the contact zone length parameters (l, L), whereupon $(L_\pm, l_\pm, \acute{U}_0)$ can be obtained from Equation (21).

Contact zone fields

Use of Equations (19b), (22), and a standard table [Gradshteyn and Ryzhik 1980] in Equations (15a), (15b), and (19a), gives the contact zone traction

$$\frac{\tau}{\mu} = \frac{-R}{4c^2 B}\frac{x}{r^2}\sqrt{l^2 - 4x^2} \quad (x \in C_0), \tag{23a}$$

$$\frac{\sigma}{\mu} = \frac{RN}{8\pi c^4 AB}\frac{1}{r^2 L}|4x^2 - l^2|\sqrt{L^2 - 4x^2}\left[K(\lambda) - \Pi\left(\frac{l^2}{4x^2}, \lambda\right)\right] \quad (x \in C). \tag{23b}$$

In a similar manner, the temperature change θ_C in the contact zone is

$$\theta_C = \frac{\varepsilon}{\alpha_v a_\varepsilon}\left[\frac{1}{8r^2}(8x^2 - l^2) - \frac{\sigma}{\mu}\right] \quad (x \in C_0), \tag{24a}$$

$$\theta_C = \frac{\varepsilon}{\alpha_v a_\varepsilon}\left[\frac{1}{8r^2}(2x - \sqrt{4x^2 - 1})^2 + \frac{c^2 T}{R}\frac{\sigma}{\mu}\right] \quad (c \in C_\pm). \tag{24b}$$

Here Π is the complete elliptic integral of the third kind of modulus λ and parameter $l^2/4x^2$. In light of the property that $N \le 0,\ T < 0,\ R \ge 0$ for Equation (12), Equation (23b) satisfies the unilateral constraint Equation (8a), and Equation (24) gives positive values. It can be shown that the maximum (compressive) normal traction σ^* occurs at $x = 0$ and is given by

$$\frac{\sigma^*}{\mu} = -\frac{R}{c^2 A}\frac{l^2}{2Lr}\frac{K(\lambda)}{K(\lambda) - E(\lambda)}. \tag{25}$$

A useful measure of the thermal response of the contact zone is the average temperature change $\tilde{\theta}_C$. Integration of Equation (24) gives this quantity as

$$\tilde{\theta}_C = \frac{\varepsilon}{2\alpha_v a_\varepsilon r^2}\left[\frac{1}{B}\left(\frac{L^2}{3} - \frac{l^2}{2}\right) - \frac{\pi T r}{AL}\left(Ll + \frac{5}{16}(3L + l)(L - l)\right)\right]. \tag{26}$$

Calculations: comparison with no-slip rolling contact results

Insight into the solution behavior can be gained by providing calculations for key results and comparing them with those for the model of rolling contact with no slip. Expressions that correspond to Equations (21)–(26) for that model can be found in [Brock 2004a] and are presented in the Appendix. The symbols are altered to match those employed here. The properties for what was referred to as 4340 steel were used for calculations. Under the updated classification schemes [Davis 1998], the properties are close to those for ASTM-A36 structural steel and, in any event, are also used here for purposes of comparison:

$$\begin{aligned} \nu &= \tfrac{1}{3}, & \rho &= 7834\,\text{kg/m}^3, \\ \mu &= 75\,\text{GPa}, & v_r &= 3094\,\text{m/s}, \\ v_R &= 2887\,\text{m/s}, & k &= 34.6\,\text{W/m}\cdot{}^\circ\text{K}, \\ \alpha_v &= 89.6(10^{-6})1/{}^\circ\text{K}, & c_v &= 448\,\text{J/kg}\cdot{}^\circ\text{K}. \end{aligned}$$

For rolling contact with no slip, infinitely rapid oscillations in contact zone traction occur in regions at zone edges defined by Equation (A.6) in the Appendix. Their existence may imply [Johnson 1987] that slip should in fact occur. Thus $|x|$ defined by Equation (A.6) plays the role of $l/2$, and a parameter corresponding to that in Equation (22a) can be obtained:

$$\lambda = \tanh \frac{\pi}{4\omega}. \tag{27}$$

Calculations in [Brock 2004a] show that Equation (27) behaves as $\lambda \approx 1-$, and so λ in both Equation (22a) and Equation (27) is written in terms of a dimensionless exponent χ:

$$\lambda = 1 - 10^{-\chi}. \tag{28}$$

Thus χ and the dimensionless contact zone width, L/r, for the two models can be obtained from Equations (A.1), (A.6), and (22). These, in turn, give (l, L). Values of $(\chi, L/r)$ are given in Table 1 for the dimensionless subcritical rolling speed c. Similarly, values for the tangential velocity $\dot{U}_0$ of the contact zone at its maximum depression point, maximum normal stress σ^*, and average change in contact zone temperature $\tilde{\theta}_C$, are given in Table 2. The dimensionless normal force used in both tables is

$$\frac{F}{\mu r} = 10^{-6}.$$

Values of χ in Table 1 indicate that the ratio $\frac{1}{2}(1 - l/L)$ of oscillation zone to contact zone widths are orders of magnitude smaller than unity, but that the ratio of slip zone to contact zone widths is orders of magnitude smaller yet. In the latter case, Equation (2) and a computer algorithm [Abramowitz and Stegun 1972] for (K, E) allow Equations (22) to be treated essentially as polynomials in $(\chi, L/r)$. Table 1 also shows that the ratios increase (χ decreases) markedly with dimensionless rolling speed c. Values of L/r

$\frac{F}{\mu r} = 10^{-6}$				
	Rolling contact with slip		Rolling contact (no slip)	
c	χ	L/r	χ	L/r
0.1	63,444	0.000653	8.121	0.000128
0.2	61,784	0.00066	7.811	0.00013
0.3	58,557	0.0006732	7.442	0.000133
0.4	54,115	0.000694	6.926	0.000136
0.5	48,456	0.000725	6.246	0.000142
0.6	41,575	0.000774	5.408	0.000151
0.7	33,424	0.000857	4.407	0.000165
0.8	23,734	0.00103	3.203	0.000192
0.9	10,745	0.001793	1.594	0.000291

Table 1. Dimensionless exponent χ and contact zone width L/r.

for the two cases in Table 1 are more comparable in magnitude, but those for perfect contact with slip are larger.

Table 2 indicates that the magnitude of σ^* for rolling contact (no slip) is greater than that for rolling contact with slip. Similarly, $\dot{U}_0$ can be orders of magnitude larger when slip does not occur, and the difference grows with increasing c. Both velocities are in the direction opposite to that of cylinder travel. Both models exhibit nominal increases in $\tilde{\theta}_C$. The increase for rolling contact with no slip is greater at low ($c \to 0$) rolling speed; that for rolling contact with slip is greater as rolling speed becomes critical ($c \to c_R$).

In ideal (rigid-rigid) rolling contact by a cylinder of radius r over a stationary plane surface, the single contact point (line parallel to the cylinder axis) has no velocity, so that the angular velocity is v/r, where again v is the translational speed of the cylinder axis. In this study, the corresponding point translates parallel to the deformable surface with velocity $\dot{U}_0$. Thus, the effective angular velocity $\dot{\Theta}$ and its percentage difference $\delta\dot{\Theta}$ with v/r are, respectively,

$$\dot{\Theta} = \frac{1}{r}(v - \dot{U}_0), \qquad \delta\dot{\Theta} = -\frac{\dot{U}_0}{v}\,(100\%). \tag{29}$$

The percentage difference is given in Table 3 for the data used in Tables 1 and 2. The values are all positive and small. However, it is well-known [Johnson 1987; Hills and Barber 1993; Hills et al. 1993] that this effective angular velocity behavior produces measured travel distances for rolling bodies that are less than the distance predicted from the number revolutions performed. Although both are small, the percentage changes for rolling contact with perfect contact in Table 3 are orders of magnitude larger than those for the slip case. That is, the more artificial no-slip rolling contact model may serve to exaggerate the difference between imposed and effective angular velocity.

$\frac{F}{\mu r} = 10^{-6}$						
	Rolling contact with slip			Rolling contact (no slip)		
c	$\dot{U}_0$ (m/s)	σ^* (GPa)	$\tilde{\theta}_C$ (°K)	$\dot{U}_0$ (m/s)	σ^* (GPa)	$\tilde{\theta}_C$ (°K)
0.1	−8.24-E4	−0.03657	0.284	−3.20-E3	−0.07460	1.0858
0.2	−8.42-E4	−0.03618	0.2837	−6.71-E3	−0.07374	1.0691
0.3	−8.76-E4	−0.03547	0.2839	−0.01068	−0.07274	1.0450
0.4	−9.31-E4	−0.03431	0.2844	−0.01537	−0.07040	1.0291
0.5	−1.017-E3	−0.03291	0.2861	−0.02164	−0.06790	0.9788
0.6	−1.159-E3	−0.03083	0.2904	−0.03067	−0.06441	0.9204
0.7	−1.421-E3	−0.02785	0.3014	−0.04519	−0.05483	0.8423
0.8	−2.050-E3	−0.02319	0.3333	−0.07482	−0.05157	0.7239
0.9	−6.220-E3	−0.01331	0.5209	−0.20368	−0.03694	0.4776

Table 2. Tangential speed $\dot{U}_0$, maximum normal traction σ^*, average temperature change $\tilde{\theta}_C$. Note: $\pm M - EN \equiv \pm M(10^{-N})$.

$\frac{F}{\mu r} = 10^{-6}$				
	Rolling contact with slip		Rolling contact (no slip)	
c	$\dot{U}_0$ (m/s)	$\delta\dot{\Theta}(\%)$	$\dot{U}_0$(m/s)	$\delta\dot{\Theta}(\%)$
0.1	−8.24-E4	0.000266	−3.20-E3	0.001035
0.2	−8.42-E4	0.000136	−6.71-E3	0.001084
0.3	−8.76-E4	0.000094	−0.01068	0.001151
0.4	−9.31-E4	0.000075	−0.01537	0.001242
0.5	−1.117-E3	0.000066	−0.02164	0.001399
0.6	−1.159-E3	0.000062	−0.03067	0.001652
0.7	−1.421-E3	0.000066	−0.4519	0.002087
0.8	−2.050-E3	0.000083	−0.07482	0.003023
0.9	−6.220-E3	0.000223	−0.20368	0.007315

Table 3. Tangential speed $\dot{U}_0$, difference $\delta\dot{\Theta}$ between effective and imposed angular velocity. Note: $\pm M - EN \equiv \pm M(10^{-N})$.

Calculations for rolling contact with slip

Tables 4 and 5 give values of $(\chi, L/r, \dot{U}_0, \sigma^*, \tilde{\theta}_C)$ for rolling contact with slip when the dimensionless applied normal forces are, respectively,

$$\frac{F}{\mu r} = 10^{-5}, \qquad \frac{F}{\mu r} = 5(10^{-5}).$$

These values show that increasing F decreases χ but increases the magnitudes of contact zone parameters $(L/r, \dot{U}_0, \sigma^*, \tilde{\theta}_C)$. The increase involving $\dot{U}_0$ is essentially linear, those involving $(L/r, \sigma^*, \tilde{\theta}_C)$ are less than linear, and the decrease in χ is greater than linear. Tables 1, 2, 4 and 5 also indicate that parameters $(L/r, \dot{U}_0)$ increase in magnitude with increasing c while parameter χ decreases. Parameter $\tilde{\theta}_C$ however, decreases for small c, reaches a minimum and then increases with increasing c. The variation with c for σ^* is itself sensitive to F: Tables 2 and 4 exhibit decreases in the magnitude with increasing c, but Table 5 shows that the magnitude of σ^* actually increases for small c, reaches a peak, and then decreases. That is, for small c, $(\sigma^*, \tilde{\theta}_C)$ vary inversely with each other. It should be noted that the maximum magnitude of σ^* displayed in Table 5 is close to that for plastic yield under uniaxial loading [Davis 1998], and that changes in $\tilde{\theta}_C$ exhibited in Table 5 are nominal but not trivial.

General comments

The observations above are based on a two-dimensional dynamic steady-state analysis of two idealized models for rolling contact. The one presented here allows slip zones at a contact zone edge, the one considered in [Brock 2004a] involved only perfect contact. In both models, the rolling cylinder is rigid, the resultant force on it is purely compressive and is directed through the cylinder axis, and heat flow

$\frac{F}{\mu r} = 10^{-5}$					
c	χ	L/r	$\dot{U}_0$ (m/s)	σ^* (GPa)	$\tilde{\theta}_C$(°K)
0.1	20,063	2.0644-E3	−8.24-E4	−0.11565	0.8980
0.2	19,538	2.0866-E3	−8.42-E4	−0.11442	0.8973
0.3	18,518	2.1286-E3	−8.76-E4	−0.11216	0.8976
0.4	17,113	2.1942-E3	−9.31-E4	−0.10881	0.8994
0.5	15,323	2.2938-E3	−1.017-E3	−0.10409	0.9047
0.6	13,147	2.4484-E3	−1.159-E3	−0.09751	0.9184
0.7	10,570	2.7104-E3	−1.421-E3	−0.08809	0.9530
0.8	7506	3.2560-E3	−2.050-E3	−0.07332	1.0536
0.9	3398	5.6712-E3	−6.219-E3	−0.04211	1.6472

Table 4. Dimensionless exponent χ and contact zone width L/r, tangential speed $\dot{U}_0$, maximum normal traction σ^*, average temperature change $\tilde{\theta}_C$. Note: $\pm M - EN \equiv \pm M(10^{-N})$.

across the contact zone is neglected. In the model treated here, slip is frictionless and occurs only in two edge zones.

Nevertheless, the observations are based on solutions that are generated from the mixed-mixed problems that arise in rolling contact. The solutions and calculations based on them exhibit four basic features. The first is, of course, that solution oscillation does not occur when slip zones exist at the contact zone edges. The second feature is that variation in size, average temperature change and maximum compressive traction of the contact zone with parameters rolling speed and resultant compressive force is

$\frac{F}{\mu r} = 5(10^{-5})$					
c	χ	L/r	$\dot{U}_0$ (m/s)	σ^* (GPa)	$\tilde{\theta}_C$(°K)
0.1	8973	4.616-E3	−4.120-E3	−0.23311	2.0079
0.2	8738	4.666-E3	−4.210-E3	−0.25588	2.0064
0.3	8282	4.760-E3	−4.381-E3	−0.25082	2.0072
0.4	7653	4.907-E3	−4.655-E3	−0.24332	2.0111
0.5	6853	5.129-E3	−5.087-E3	−0.23277	2.0230
0.6	5880	5.475-E3	−5.797-E3	−0.21807	2.0536
0.7	4727	6.061-E3	−7.103-E3	−0.19700	2.1310
0.8	3357	7.281-E3	−0.01026	−0.16401	2.3558
0.9	1520	0.012681	−0.03110	−0.09421	3.6833

Table 5. Dimensionless exponent χ and contact zone width L/r, tangential speed $\dot{U}_0$, maximum normal traction σ^*, average temperature change $\tilde{\theta}_C$. Note: $\pm M - EN \equiv \pm M(10^{-N})$.

important for both models. The third feature is that slip zones form essentially where the separation of the cylinder and elastic body occurs. Their widths are orders of magnitude smaller even than the extremely small slip zone widths implied by the oscillation zones in rolling contact without slip. This phenomenon may well arise from the continuity of traction required everywhere in the contact zone. As seen in Equation (23), this requirement enforces zero traction at points $|x| = L/2$ of the cylinder/half-space separation but also gives zero traction at points $|x| = l/2$ of the perfect contact-slip transition. Finally, the perfect rolling contact model may overstate the increase in effective angular velocity of rolling above the rigid-rigid limit.

It is hoped that the results of this article, while limited in various aspects in comparison to the newer contact analyses listed at the outset, does allow insight into aspects of rapid contact behavior. These results are now forming the basis of dynamic studies that include thermal relaxation effects, heat conduction across a contact zone, and both dry and viscous friction.

Appendix

For the case of perfect contact over all C [Brock 2004a], the contact zone parameters $(L, L_\pm, \dot{U}_0)$ are given by

$$L_\pm = \pm\frac{L}{2}, \qquad \dot{U}_0 = \frac{L}{4}\left[\sqrt{\frac{B}{A}}\omega + (4\omega^2 - 1)\frac{L}{16r}\right]v_r,$$
$$\left(\frac{L}{\kappa r}\right)^3 + 3\left(\frac{L}{\kappa r}\right)^2 - 4\frac{F}{F_0} = 0, \qquad \frac{F_0}{\mu r} = \frac{\pi B R}{6c^2 A^2}\left(4 + \frac{1}{\omega^2}\right). \tag{A.1}$$

The contact zone traction is

$$\frac{\tau}{\mu} = -\frac{P(x)}{4Br^2}\sqrt{L^2 - 4x^2}\sin\left(\phi - \omega \ln\frac{L-2x}{L+2x}\right) \qquad (x \in C), \tag{A.2a}$$

$$\frac{\sigma}{\mu} = -\frac{P(x)}{4\sqrt{AB}r^2}\sqrt{L^2 - 4x^2}\cos\left(\phi - \omega \ln\frac{L-2x}{L+2x}\right) \qquad (x \in C). \tag{A.2b}$$

In Equations (A.1) and (A.2), the terms $(P, \phi, \omega, \kappa)$ are given by

$$P(x) = \sqrt{\frac{R}{1-AB}}\sqrt{(2Br + \omega AL)^2 + 4ABx^2}, \qquad \tan\phi = \frac{2\sqrt{AB}x}{2Br + \omega AL},$$
$$\omega = \frac{1}{2\pi}\ln\frac{c^2\sqrt{AB} + N}{c^2\sqrt{AB} - N}, \qquad \kappa = \frac{\sqrt{B}}{\omega\sqrt{A}}.$$

Here, (R, N, A, B) are defined by Equations (11a) and (13b), and for the parameter $\omega > 0$ in Equation (12). The contact zone temperature change θ_C can be extracted from [Brock 2004a] as

$$\theta_C = \frac{\varepsilon}{\alpha_v a_\varepsilon}\left[\frac{x^2}{r^2} + \omega\sqrt{\frac{B}{A}}\frac{L}{2r} - \frac{1}{8}(1 - 4\omega^2)\frac{L^2}{r^2} - \frac{\sigma}{\mu}\right] \quad (x \in C). \tag{A.3}$$

The maximum value of (A.2b) occurs at $x = 0$ and is given by

$$\frac{\sigma^*}{\mu} = -\sqrt{\frac{R}{1-AB}}\left(\sqrt{\frac{B}{A}} + \frac{\omega L}{2r}\right)\frac{L}{2r}. \tag{A.4}$$

The average of (A.3) is obtained as

$$\tilde{\theta}_C = \frac{\varepsilon}{\alpha_v a_\varepsilon}\left[\frac{F}{\mu L} + 2\frac{\omega\sqrt{B}}{\sqrt{A}}\frac{L}{r} + \frac{1}{24}(12\omega^2 - 1)\frac{L^2}{r^2}\right]. \tag{A.5}$$

The oscillatory behavior exhibited by (A.2) as $|x| \to 0$ implies that the condition of nontensile contact stress does not hold everywhere in regions at the edges of C defined by

$$\tanh\frac{\pi}{4\omega} < \left|\frac{2x}{L}\right| < 1. \tag{A.6}$$

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