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INTERFACIAL CRACK KINKING SUBJECTED TO CONTACT EFFECTS

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We investigate the problem of a kinking crack at a bimaterial interface when the two surfaces are in contact near the crack tip. Using a potential function and the dislocation technique, we relate, by a singular integral equation, the stress intensity factors (SIF) at the kinking crack tip to the SIF before crack kinking. We use Gauss–Chebyshev integration formulas to solve this integral equation numerically. We evaluate the kinking angles from a bimaterial interface under conditions of contact using the maximum energy release rate criterion and compare these angles with our experiments and those in the literature. The interfacial crack is demonstrated by simulation and experiments to kink into the more compliant material at an angle of about 80°.

1. Introduction

Interfacial fracture mechanics has been studied for several decades, and treatments have steadily grown more effective as applications have increased [Sih 1977; Muskhelishvili 1977; Sanford 1997a; 1997b; Hutchinson and Suo 1992]. Using continuum fracture mechanics, there are three candidate criteria for predicting the growth direction for an interfacial crack: maximum loop stress [Wang 1994; Li et al. 2004], maximum energy release rate [Mulville et al. 1978; Sun and Jih 1987], and zero mode II intensity stress factor ($K_{II} = 0$) [Sih 1977; Hutchinson and Suo 1992; Sanford 1997b; Banks-Sills and Ashkenazi 2000]. The maximum loop stress criterion was demonstrated to be ineffective and inaccurate [Xie et al. 2005]. The criteria of maximum energy release rate and zero mode II intensity stress factor were shown to predict the same crack propagation direction in most loading conditions [Hutchinson and Suo 1992]. It is thus reasonable to use the maximum energy release rate criterion to predict the direction the crack propagates. Because this rate can be completely expressed in terms of local stress intensity factors [Rice 1988], it is important, in bimaterial systems with complex geometry, to find how the local SIF relates to the loading SIF.

Some have tried using finite element methods (FEM) to predict the kinking angles out of an interface [Leblond and Frelat 2001; Leblond and Frelat 2004]. However, calculating the energy release rate for all possible kinking angles is very tedious and may be inaccurate due to the singular characteristics of stress and strain at a crack tip. With help of a complex variable representation, the dislocation method provides a powerful tool for solving crack problems in both homogeneous solids and nonhomogeneous layered materials [Atkinson 1966; Lo 1978; Hayashi and Nemat-Nasser 1981; Hills et al. 1996]. Therefore, we adopt the dislocation technique here to find the relation between the SIF before and after kinking, under general loading conditions. We then apply the solution to find how cracks kink from a interface between two materials in contact.

Keywords: complex variables, composites, contact effect, interfacial crack, kinking crack, singular integration.

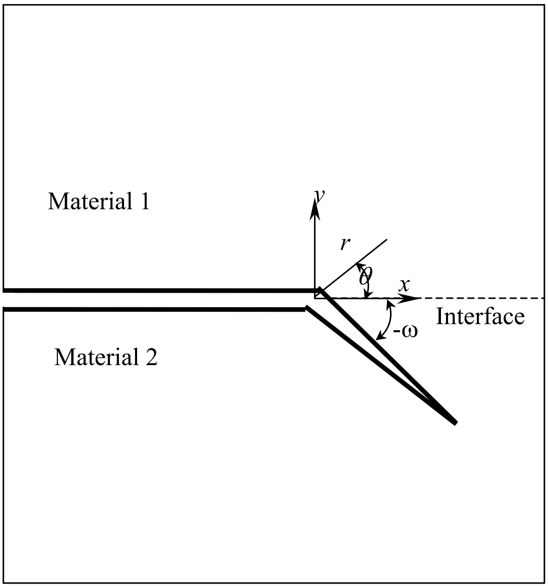

Figure 1. Geometry of kinked crack.

When the surfaces near an interfacial crack tip are in contact, the conditions around the tip locally become pure mode II, at least before the crack kinks [Comninou 1977; Comninou and Dundurs 1979a; 1979b; Fan et al. 1998; Gautesen and Dundurs 1988]. Equivalently, the effect of contact can be included by applying a pure shear loading at infinity or by setting the loading phase angle to 90° [Leblond and Frelat 2001; 2004].

2. Problem formulations

Consider a two-dimensional infinite bimaterial system, with a semiinfinite edge crack on the interface and a significantly smaller kinked crack in the more compliant material labeled 2; see Figure 1. Global loading of this geometry causes a primary semiinfinite interfacial crack to propagate into material 2. As the linear fracture solution may cause the crack to exhibit an inadmissible flank penetration into the bimaterial interface, the primary semiinfinite interfacial crack before kinking is represented by a contact model [Comninou 1977; Comninou and Dundurs 1979a; 1979b] in which a small contact zone exists around the crack tip; see Figure 2.

In either the linear fracture model or the contact model, the stress intensity factors for the interfacial crack can be expressed in a complex variable [Rice 1988; Hutchinson and Suo 1992]

$$K = K_I + iK_{II} = |K|e^{i\psi_0}, \tag{1}$$

where $i = \sqrt{-1}$, and where K_I and K_{II} are the mode I and II SIF initially applied at infinity. $|K|$ and ψ_0 are the magnitude and phase angle of the SIF.

When a contact zone occurs around the crack tip, the complex SIF is determined by K_{II} with a diminished K_I [Whitcomb 1981; Sun and Jih 1987]. Due to the contact between the flanks of the

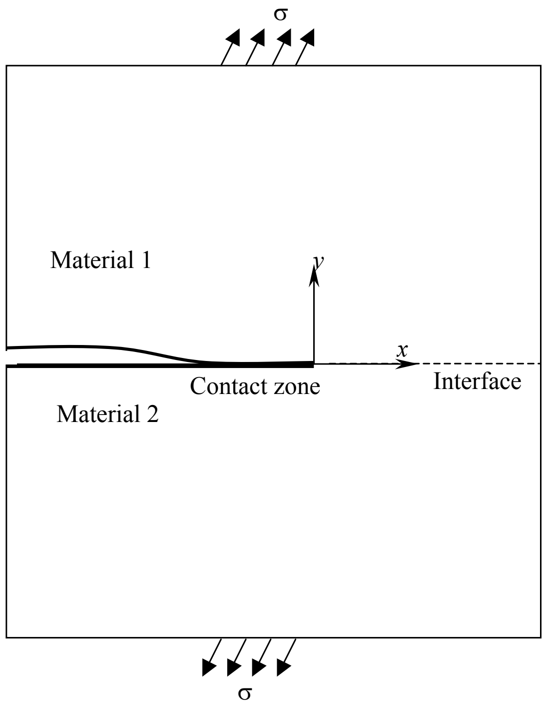

Figure 2. Contact model of primary semiinfinite interfacial crack.

interfacial crack, the mode I SIF disappears, that is, $K_I = 0$ or $\psi_0 = 90°$ [Comninou 1977; Leblond and Frelat 2001; Leblond and Frelat 2004].

After the primary interfacial crack kinks into one side, the stress intensity factor after kinking K' can be similarly expressed in a complex variable

$$K' = K_1 + iK_2,$$

where K_1 and K_2 are the postkinking mode I and II SIF.

Because the crack tip is confined to a one side after kinking — that is, in a single homogeneous and isotropic medium — the kinked crack problem can be solved by superposition using the basic linear fracture solution for an edge dislocation in material 2 interacting with an initial semiinfinite interfacial crack. As illustrated in Figure 3, the actual kinked crack tip is represented by a virtual dislocation along the crack tip.

The stresses at point $z = te^{-i\omega}$ caused by a discrete edge dislocation at $z_0 = \eta e^{-i\omega}$ can be expressed in complex form [Lo 1978; Hayashi and Nemat-Nasser 1981; Wang 1994] as

$$\sigma_{\theta\theta}(t) + i\sigma_{r\theta}(t) = 2\bar{B}e^{-i\omega}(t-\eta)^{-1} + BH_1(t,\eta) + \bar{B}H_2(t,\eta),$$

where

$$B = \frac{\mu_2}{i\pi(\kappa_2+1)}e^{i\theta}(u_r + iv_\theta) \tag{2}$$

is the Burgers vector. The functions H_1 and H_2 are specified in Appendix A.

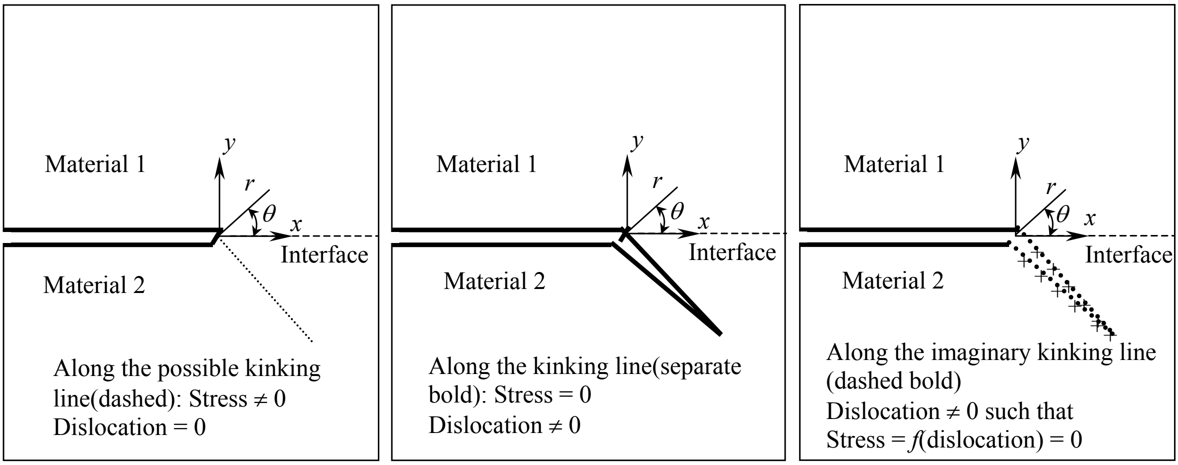

Figure 3. Modeling a kinked crack by dislocation. From left: without kinking; after kinking, in the traditional representation with a crack; after kinking, in the dislocation representation without a crack.

After the kinked crack is represented by a distributed dislocation $B(\eta)$, where η is the distance of a discrete dislocation from the initial crack tip, the crack-dislocation relation becomes an integral equation specifying that the net traction is zero on the kinking line (see Figure 3):

$$2e^{-i\omega}\int_0^1 \overline{B}(\eta)(t-\eta)^{-1}d\eta+\int_0^1 B(\eta)H_1(t,\eta)d\eta+\int_0^1 \overline{B}(\eta)H_2(t,\eta)d\eta=-(\sigma_{\theta\theta}(t)+i\sigma_{r\theta}(t)). \quad (3)$$

Because $s=2\eta-1$ and $s_0=2t-1$, Equation (3) can be equivalently transformed into

$$2e^{-i\omega}\int_{-1}^1 \overline{B}(s)(s_0-s)^{-1}ds+\int_{-1}^1 B(s)\frac{H_1(s_0,s)}{2}ds+\int_{-1}^1 \overline{B}(s)\frac{H_2(s_0,s)}{2}ds=-(\sigma_{\theta\theta}(s_0)+i\sigma_{r\theta}(s_0)), \quad (4)$$

where s is new integration variable, s_0 is the point along the kinking crack from the initial crack tip to the propagated crack tip, and $B(s)$ is the distributed dislocation along the kinking crack.

This integral equation can be solved numerically using a method developed by [Erdogan and Gupta 1972; Sih 1977]. First, the unknown equivalent dislocation function $B(s)$ is factored into a singularity term and a bounded term $P(s)$ [He and Hutchinson 1989]:

$$B(s)=(1-s^2)^{-1/2}P(s).$$

Then Equation (4) becomes a set of linear equations

$$\sum_{i=1}^{n}\frac{1}{n}\overline{P}(s_i)\left[\frac{2\pi e^{-i\varpi}}{s_{0k}-s_i}+\frac{\pi}{2}H_2(s_{0k},s_i)\right]+\sum_{i=1}^{n}\frac{1}{n}P(s_i)\left[\frac{\pi}{2}H_1(s_{0k},s_i)\right]=-(\sigma_{\theta\theta}(s_{0k})+i\sigma_{r\theta}(s_{0k})), \quad (5)$$

where n is the number of sampling point and the highest of order for numerical approximation,

$$s_i = \cos\Big(\frac{2i-1}{2n}\pi\Big) \text{ for } i = 1, 2, \dots, n, \quad \text{and} \quad s_{0k} = \cos\Big(\frac{k}{n}\pi\Big) \text{ for } k = 1, 2, \dots, n-1,$$

The function $P(s)$ can be expended in Chebyshev polynomials $T_j(s)$ of the first kind as

$$P(s) = \sum_{j=1}^{n} C_j T_{j-1}(s).$$

Consequently, Equation (5) becomes the set of linear equations

$$\sum_{j=1}^{n} \{C_j E_j(s_{0k}) + \bar{C}_j F_j(s_{0k})\} = -(\sigma_{\theta\theta}(s_{0k}) + i\sigma_{r\theta}(s_{0k})),$$

where $E_j(s_{0k})$ and $F_j(s_{0k})$ are the integrated functions that describe the distributed dislocation of stress at the point s_{0k}. They can be written as

$$E_j(s_{0k}) = \int_{-1}^{1} B(s) \frac{H_1(s_{0k}, s)}{2} ds,$$

$$F_j(s_{0k}) = 2e^{-i\omega} \int_{-1}^{1} \bar{B}(s)(s_{0k} - s)^{-1} ds + \int_{-1}^{1} \bar{B}(s) \frac{H_2(s_{0k}, s)}{2} ds.$$

Because there are n unknowns but only $n-1$ equations in Equation (5), an additional equation needs be introduced. He and Hutchinson [1989] chose

$$P(-1) = \sum_{j=1}^{n} C_j T_{j-1}(-1) = 0.$$

After the dislocation function is numerically determined from the linear equations, the stress intensity factor after kinking can be calculated, according to [Lo 1978; He and Hutchinson 1989], from

$$K' = K_1 + iK_2 = (2\pi)^{3/2} e^{-i\varpi} \lim_{\eta \to 0} \{(1-\eta)^{1/2} \bar{B}(\eta)\} = (2\pi)^{3/2} e^{-i\varpi} \bar{P}(1).$$

This equation relates the SIF after kinking to that before, because as $P(1)$ is connected to the initially applied SIF K through Equation (5).

In plane strain, the initial energy release rate G of the interfacial crack is related to the initial SIF K as [He and Hutchinson 1989]

$$G_0 = \Big(\frac{1-\nu_1}{\mu_1} + \frac{1-\nu_2}{\mu_2}\Big)\Big(\frac{K\bar{K}}{4\cosh^2 \pi\varepsilon}\Big),$$

where μ_1 and ν_1 are the shear modulus and Poisson's ratio of material 1, and μ_2 and ν_2 are the shear modulus and Poisson's ratio of material 2. The material mismatch index ε is defined as

$$\varepsilon = \frac{1}{2\pi} \ln\Big(\frac{1-\beta}{1+\beta}\Big).$$

α and β are Dundur's material parameters, defined in plane strain as

$$\alpha = \frac{\mu_1(1-\nu_2)-\mu_2(1-\nu_1)}{\mu_1(1-\nu_2)+\mu_2(1-\nu_1)} \quad \text{and} \quad \beta = \frac{1}{2}\frac{\mu_1(1-2\nu_2)-\mu_2(1-2\nu_1)}{\mu_1(1-\nu_2)+\mu_2(1-\nu_1)}. \tag{6}$$

As the crack propagates in a homogeneous medium after kinking, the energy release rate G is given by [He and Hutchinson 1989]

$$G = \frac{1-\nu_2}{2\mu_2}(K_1^2+K_2^2),$$

where μ_2 and ν_2 are the shear modulus and Poisson's ratio of material 2 and where K_1 and K_2 are the postkinking SIFs of mode I and II.

We will next evaluate, for different material combinations, the relative energy release rate G/G_0, and we will predict the kinking angle by the criterion of maximum energy release rate.

3. Numerical solution and results

3.1. ***Validation of program.*** The numerical simulation is in MATLAB. Before implementing the numerical analysis, we validate the program by comparing our results with [He and Hutchinson 1989, Table 1] for $\alpha = 0$, $\beta = 0$ and $\alpha = 0.56$, $\beta = 0.12$. In both cases, the kinking angle is set to 45° and the initial SIF K is set to 1; see the results in Table 1.

Comparing our results with those of [He and Hutchinson 1989], we find that the real parts of $P(1)$ agree very well. The imaginary parts of $P(1)$ are close in value, but ours takes a negative sign. Our calculation is further validated by Cotterell and Rice's equation [1980].

From Table 1, we also see that the values of $P(1)$ differ by less than 0.1 percent in going from $N = 40$ to $N = 100$. Thus, for the rest of this paper, we will use $N = 40$.

3.2. ***Numerical simulation results.*** Figure 4 plots how relative energy release rate varies with angle, for different material combinations. In the simulation, the Poisson's ratios of both materials are assumed to be same, and the ratio of material Young's moduli E_1/E_2 are chosen as 1, 2, 5, 10 and 100. The figure demonstrates that in each case the relative release energy rate reaches its maximum at some angle.

$\alpha=\beta=0$		
N	$P_R(1)$	$P_m(1)$
4	0.04923	−0.02096
8	0.04976	−0.02108
12	0.04989	−0.02107
16	0.04994	−0.02106
20	0.04997	−0.02105
40	0.05001	−0.02102

$\alpha=0.56,\ \beta=0.12$		
N	$P_R(1)$	$P_m(1)$
4	0.04149	−0.01805
8	0.04183	−0.01871
12	0.04193	−0.01879
20	0.04199	−0.01883
40	0.04201	−0.01885
100	0.04202	−0.01885

Table 1. Calculation of $P(1)$ for $\omega = 45°$ and $K = 1$.

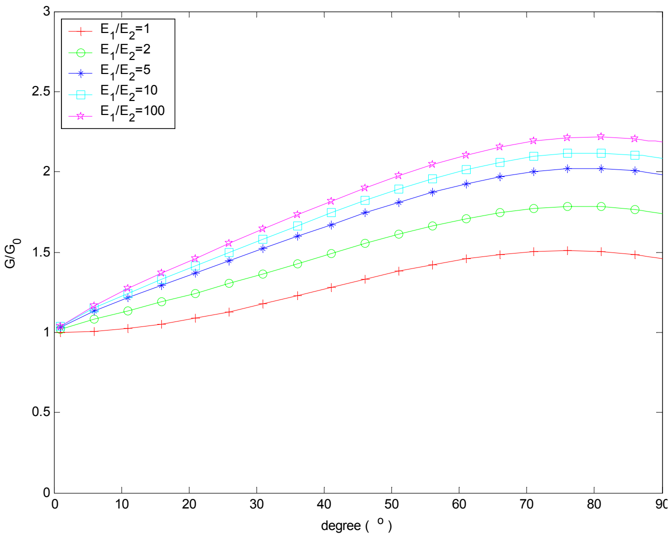

Figure 4. Energy release rate versus kinking angle.

Adopting the criterion of maximum energy release rate, we can find the kinking angle for each case. See the results in Table 2, which also lists, for comparison, the kinking angle evaluated from the criterion $K_{II} = 0$.

4. Comparison with experiments and discussion

For the homogeneous limit ($\alpha = \beta = 0$), here, $E_1/E_2 = 1$, we compute the kinking angle at 77.1° using the criterion $K_{II} = 0$; see Table 2. This result agrees well with the 77.3° obtained by correlating of initial and local SIF by quadratures [Bilby and Cardew 1975] and the 77.8° obtained by FEM analysis [Leblond and Frelat 2001; 2004], both of which use the criterion $K_{II} = 0$.

Pelegri and Chen [2000] performed experiments on cross-ply laminated composites (IM7/5260) using a Mixed Mode Bending (MMB) test facility. The kinking angles were found at different loading ratios. The experimental results show that the kinking angle is 80° when the loading ratio of shear to tension is 4:1. From the fractographic images [Gilchrist and Svensson 1995; Partridge and Singh 1995; Gilchrist et al. 1996] of loading ratio experiments similar to those of Pelegri, the interfacial cracks show contact between the crack flanks at a loading ratio of 4:1. Here, the program is implemented to compute the kinking angle for those experiments. Considering the contact effect in our simulation, we predict the kinking angle of 80.4° when the loading ratio of 4:1, which matches well the experimental result.

E_1/E_2	1	2	5	10	20	50	100
Kinking angle by maximum G/G_0	75.7°	77.9°	79.3°	79.8°	80.0°	80.2°	80.2°
Kinking angle by $K_2 = 0$	77.1°	81.1°	84.3°	85.5°	86.1°	86.5°	86.7°

Table 2. Kinking angles in the presence of contact, assuming $\nu_1 = \nu_2$.

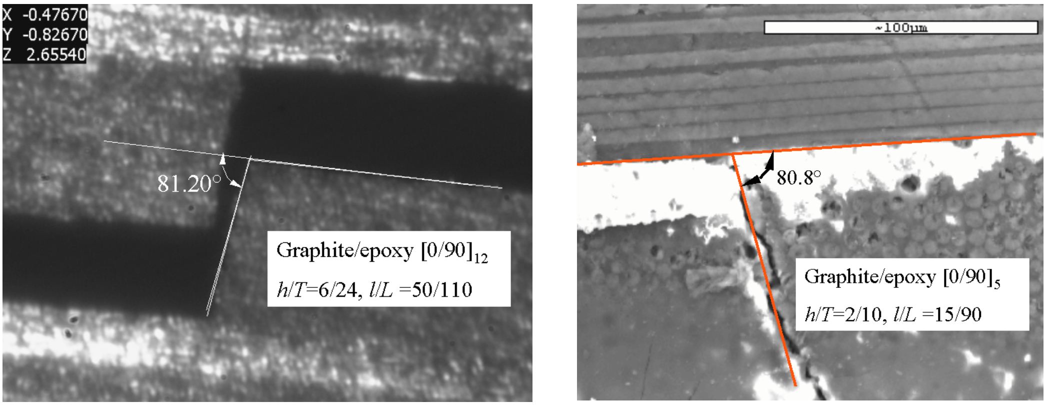

Figure 5. At left, the kinking angle for the cantilever bending experiment and, at right, for the microscopic three-point experiment.

We experiment with cantilever bending and microscopic three-point bending on cross-ply laminated composite (IM7/G8548) [Shan and Pelegri 2003]. Figure 5 shows images of the kinking crack in these two cases. In the cantilever bending experiment, the cantilever beam can withstand the shear resultant force and moment after its local buckling, so that the inner delamination tip is in local compressive and mode II dominant conditions, and the two surfaces are in contact near the tip. The dominance of mode II and the presence of contact are also verified by finite element analysis [Shan and Pelegri 2003], which Figure 6 shows as a large ratio of K_{II}/K_I. This paper's program computes the kinking angle to be 80.4° for the mode II dominant condition, which fits well with the experimental result of 81.2°; see Figure 5 at left. Figure 5, at right, illustrates a microscopic experiment by three-point bending; the results are also dominated by mode II and the presence of contact. The kinking angle of 80.8° agrees well with our numerics.

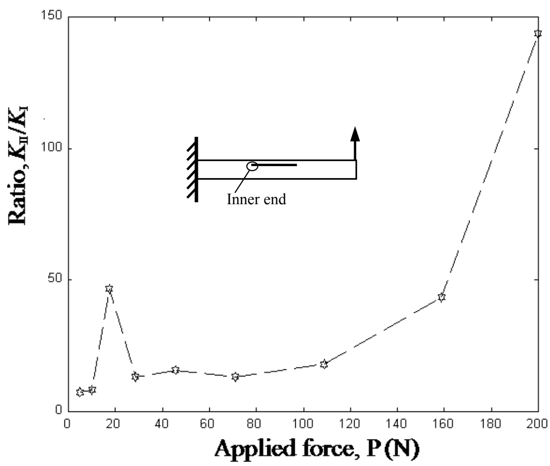

Figure 6. Ratio of stress intensity factors in mode II to mode I by FEM.

5. Conclusions

We determined the kinking angle for an interfacial crack under the effects of contact by a numerical analysis using a complex SIF, a dislocation technique, and singular integration. The contact effects are important for evaluating the competition between the interlayer and intralayered cracks in laminate materials and composites.

When contact is present, our numerical analysis finds the kinking angle out of the interface into the more compliant material to be 75.7° for homogeneous layered materials and around 80.2° for a wide range of nonhomogeneous material combinations. According to the analytical, FEM, and experimental results, this angle is independent of the structural geometry, the loading type (be it static, fatigue, or dynamic), and loading history and has little relation to material elastic constants. Furthermore, the progressive crack tends to get trapped on the interface regardless of the loading and architectural configuration for the laminate composites. This it because the crack is eventually forced against interface and accordingly meet more resistance to its further propagation.

Our future work will focus on the crack growth law under effects of contact and friction and on how the crack progresses from pure mode I opening to mixed mode to pure mode II shearing. The size and pattern of heckles formed during as the crack propagates may serve to measure this phenomenon.

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Appendix A. Derivation of the functions H_1 and H_2 in the dislocation method

In terms of the Muskhelishvili [1977] potentials, the stresses and displacements in two-dimensional infinite bimaterial system may be expressed as

$$(\sigma_{yy} - i\sigma_{xy})_j = \Phi_j(z) + \overline{\Phi_j(z)} + z\overline{\Phi'_j(z)} + \overline{\Psi_j(z)}, \tag{A.1}$$

$$(\sigma_{yy} + \sigma_{xx})_j = 2\,(\Phi_j(z) + \overline{\Phi_j(z)}), \tag{A.2}$$

$$2\mu_j\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right)_j = \kappa_j\Phi_j(z) - (\overline{\Phi_j(z)} + z\overline{\Phi'_j(z)} + \overline{\Psi_j(z)}), \tag{A.3}$$

where the subscript $j = 1, 2$ means "in the region j" and where Φ_j and Ψ_j are the potentials. Also, the complex z is $x + iy$, the prime takes derivatives in z, the overbar denotes complex conjugation, $\kappa = 3 - 4\nu$ for plane strain, $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress, and ν is Poisson's ratio.

Mukai et al. [1990] introduced two additional jump potentials Ω_S and Ω_D to solve the branching of interfacial crack of finite length within an infinite large body. Similarly to Mukai's method, Suo [1989] obtained the Muskhelishvili potentials for an infinite large body with a semiinfinite interfacial crack from

$$\Omega_S = \begin{cases} \Phi_1(z) - [\bar{\Phi}_2(z) + z\bar{\Phi}'_2(z) + \bar{\Psi}_2(z)], & z \in S_1, \\ \Phi_2(z) - [\bar{\Phi}_1(z) + z\bar{\Phi}'_1(z) + \bar{\Psi}_1(z)], & z \in S_2, \end{cases} \tag{A.4}$$

$$\Omega_D = \begin{cases} \dfrac{\kappa_1}{2\mu_1}\Phi_1(z) + \dfrac{1}{2\mu_2}[\bar{\Phi}_2(z) + z\bar{\Phi}'_2(z) + \bar{\Psi}_2(z)], & z \in S_1, \\ \dfrac{\kappa_2}{2\mu_2}\Phi_2(z) + \dfrac{1}{2\mu_1}[\bar{\Phi}_1(z) + z\bar{\Phi}'_1(z) + \bar{\Psi}_1(z)], & z \in S_2, \end{cases} \tag{A.5}$$

$$\Phi_1(z) = Q_1\left[\frac{1}{2\mu_2}\Omega_{s1}(z) + \Omega_{D1}(z)\right], \tag{A.6}$$

$$\Psi_1(z) = Q_2\left[\frac{-\kappa_2}{2\mu_2}\bar{\Omega}_{S2} + \bar{\Omega}_{D2}\right] - \Phi_1(z) - z\Phi'_1(z), \tag{A.7}$$

$$\Phi_2(z) = Q_2\left[\frac{1}{2\mu_1}\Omega_{S2}(z) + \Omega_{D2}(z)\right], \tag{A.8}$$

$$\Psi_2(z) = Q_1\left[\frac{-\kappa_1}{2\mu_1}\bar{\Omega}_{S1} + \bar{\Omega}_{D1}\right] - \Phi_2(z) - z\Phi'_2(z), \tag{A.9}$$

where

$$Q_1 = \frac{2\mu_1\mu_2}{\mu_1 + \kappa_1\mu_2}, \qquad Q_2 = \frac{2\mu_1\mu_2}{\mu_2 + \kappa_2\mu_1},$$

and $\overline{f}(z) \equiv \overline{f(\bar{z})}$. If $f(z)$ is analytic for z in region S, then $\overline{f}(z)$ is analytic for $\bar{z}$ in region S.

The interaction between an interface crack and a dislocation may be solved by superposing the solutions for (i) a dislocation in S_2 near an interface and (ii) an interface crack loaded with the negative of the stresses produced by (i). By simply replacing α with $-\alpha$, β with $-\beta$ and by switching the subscripts 1 and 2 of Φ^D and Ψ^D from Mukai's results [1990], we get the potentials

$$\Phi^D_{1,\text{singular}} = \Psi^D_{1,\text{singular}} = 0, \tag{A.10}$$

$$\Phi^D_{2,\text{singular}} = \frac{B}{z - s_0}, \tag{A.11}$$

$$\Psi^D_{2,\text{singular}} = \bar{B}\frac{1}{z - s_0} + B\frac{\bar{s}_0}{(z - s_0)^2}, \tag{A.12}$$

$$\Phi^D_{1,\text{continuation}} = B\left(\frac{(1-\alpha)}{(1+\beta)(z - s_0)}\right), \tag{A.13}$$

$$\Psi^D_{1,\text{continuation}} = A\left(\frac{(1-\alpha)(\bar{s}_0 - s_0)}{(1-\beta)(z - s_0)^2}\right) + \bar{A}\left(\frac{(1-\alpha)}{(1-\beta)(z - s_0)}\right) - \Phi^D_{1,\text{continuation}} - z\Phi^{D'}_{1,\text{continuation}}, \tag{A.14}$$

$$\Phi^D_{2,\text{continuation}} = B\left(\frac{(\beta-\alpha)}{(1-\beta)(z - \bar{s}_0)}\right) + \bar{B}\left(\frac{(\beta-\alpha)(s_0 - \bar{s}_0)}{(1-\beta)(z - \bar{s}_0)^2}\right), \tag{A.15}$$

$$\Psi^D_{2,\text{continuation}} = \bar{A}\left(\frac{-(\alpha+\beta)}{(1+\beta)(z - \bar{s}_0)}\right) - \Phi^D_{2,\text{continuation}} - z\Phi^{D'}_{2,\text{continuation}}, \tag{A.16}$$

where B is the Burgers vector of Equation (2) and α and β are Dundur's constants defined in Equation (6).

When the main crack is introduced, the stresses due to the dislocation near an interface will be removed from the crack faces by the two potentials Φ^C and Ψ^C. Many investigators [Suo 1989; Mukai et al. 1990; Rice et al. 1990] have presented the solutions for crack problems. For a semiinfinite crack on an interface, the interface boundary conditions at $y=0$ are

$$\begin{aligned} (\sigma_{yy}-i\sigma_{xy})_1-(\sigma_{yy}-i\sigma_{xy})_2 &= 0 && \text{for } |x|<\infty,\\ \left(\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}\right)_1-\left(\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}\right)_2 &= 0 && \text{for } x>0,\\ (\sigma_{yy}-i\sigma_{xy}) &= f(x) && \text{for } x<0. \end{aligned}$$

In terms of the jump potentials the first boundary condition is simply

$$\Omega_{S1}(x)-\Omega_{S2}(x)=0 \quad \text{for } |x|<\infty.$$

Since Ω_S is analytic everywhere and bounded, by Liouville's theorem it must be constant. Moreover, for zero stress at infinity, $\Omega_S^C=0$.

The second boundary condition in terms of the jump potentials is

$$\Omega_{D1}(x)-\Omega_{D2}(x)=0 \quad \text{for } x>0.$$

The third boundary conditions in terms of the jump potentials is

$$\begin{aligned} Q_1\Omega_{D1}(x^+)+Q_2\Omega_{D2}(x^-) &= f(x^+)=f(x) && \text{for } x<0,\\ Q_2\Omega_{D2}(x^-)+Q_1\Omega_{D1}(x^+) &= f(x^-)=f(x) && \text{for } x<0. \end{aligned}$$

or,

$$\Omega_D^+ + m\Omega_D^-(x)=\frac{1}{Q_1}f(x) \quad \text{for } x<0, \text{ where } \quad m=\frac{Q_2}{Q_1}=\frac{1+\beta}{1-\beta}.$$

These two last boundary conditions define a Hilbert problem with the solution

$$\Omega_D(z)=\frac{\chi(z)}{2\pi i}\int_{-\infty}^{0}\frac{1}{Q_1}\frac{f(x)}{\chi^+(x)(x-z)}dx+\chi(z)P(z) \tag{A.17}$$

in which $\chi(z)=z^{\gamma-1}$ is the solution of the homogeneous Hilbert problem defined above,

$$\gamma=\frac{1}{2}-\frac{i}{2\pi}\log|m|=\frac{1}{2}+i\varepsilon, \quad \text{where} \quad \varepsilon=\frac{1}{2\pi}\log\left|\frac{1}{m}\right|,$$

and $P(z)$ is equal to zero [Suo 1989; Mukai et al. 1990]. After Ω_D is obtained, the jump potentials can be inverted back to standard potentials using Equations (A.6)–(A.9). To remove the stresses on the crack caused by the dislocation solution, the integration in Equation (A.17) is carried out with $f(x)$ opposite to the tractions due to a dislocation near the interface. These tractions are obtained by substituting the Equations (A.11), (A.12), (A.15) and (A.16) into Equation (A.1):

$$(\sigma_{yy}-i\sigma_{xy})_c=B\left(\frac{(1-\alpha)}{(1+\beta)(x-s_0)}+\frac{(1-\alpha)}{(1-\beta)(x-\bar{s}_0)}\right)+\bar{B}\left(\frac{(1-\alpha)(s_0-\bar{s}_0)}{(1-\beta)(x-\bar{s}_0)^2}\right).$$

So the jump potential is

$$\Omega_D(z) = -\frac{\chi(z)}{2\pi i Q_1}\int_{-\infty}^{0}\frac{(\sigma_{yy}-i\sigma_{xy})_c}{\chi^+(x)(x-z)}dx,$$

and it can be obtained by following Suo's procedure [1989], giving

$$\Omega_D^C(z) = -B\frac{(1-\alpha)(1-\beta)}{Q_1}\left(\frac{F(z,s_0)}{1+\beta}+\frac{F(z,\bar{s}_0)}{1-\beta}\right)-\overline{B}\frac{(1-\alpha)(1-\beta)}{Q_1}\left(\frac{(s_0-\bar{s}_0)G(z,\bar{s}_0)}{1-\beta}\right),$$

where

$$F(z,a)=\frac{1}{2(z-a)}\left(1-\frac{\chi(z)}{\chi(a)}\right) \quad\text{and}\quad G(z,a)=\frac{\partial F(z,a)}{\partial a}. \tag{A.18}$$

Inverting these to standard potentials gives

$$\Phi_1^C(z) = -B(1-\alpha)(1-\beta)\left[\frac{F(z,s_0)}{1+\beta}+\frac{F(z,\bar{s}_0)}{1-\beta}\right]-\overline{B}(1-\alpha)(1-\beta)\left[\frac{(s_0-\bar{s}_0)G(z,\bar{s}_0)}{1-\beta}\right]$$

$$\Phi_2^C(z) = -B(1-\alpha)(1+\beta)\left[\frac{F(z,s_0)}{1+\beta}+\frac{F(z,\bar{s}_0)}{1-\beta}\right]-\overline{B}(1-\alpha)(1+\beta)\left[\frac{(s_0-\bar{s}_0)G(z,\bar{s}_0)}{1-\beta}\right]$$

$$\Psi_1^C(z) = -B(1-\alpha)(1+\beta)\left[\frac{(\bar{s}_0-s_0)\bar{G}(z,\bar{s}_0)}{1-\beta}\right]-\overline{B}(1-\alpha)(1+\beta)\left[\frac{\bar{F}(z,s_0)}{1+\beta}+\frac{\bar{F}(z,\bar{s}_0)}{1-\beta}\right]-\Phi_1^C(z)-z\Phi_1^{C'}(z)$$

$$\Psi_2^C(z) = -B(1-\alpha)(1-\beta)\left[\frac{(\bar{s}_0-s_0)\bar{G}(z,\bar{s}_0)}{1-\beta}\right]-\overline{B}(1-\alpha)(1-\beta)\left[\frac{\bar{F}(z,s_0)}{1+\beta}+\frac{\bar{F}(z,\bar{s}_0)}{1-\beta}\right]-\Phi_2^C(z)-z\Phi_2^{C'}(z)$$

The final potentials that solve the interaction between a discrete dislocation and an interface crack are

$$\Phi=\Phi^D+\Phi^C=\Phi^D_{\text{singular}}+\Phi^D_{\text{continuation}}+\Phi_C$$
$$\Psi=\Psi^D+\Psi^C=\Psi^D_{\text{singular}}+\Psi^D_{\text{continuation}}+\Psi^C,$$

where the dislocation (D) potentials and the crack (C) potentials are all defined above. Thus, the traction at z on $\theta=-\omega$ can be written as

$$\sigma_{\theta\theta}(t)+i\sigma_{r\theta}(t)=2\overline{B}e^{-i\omega}(t-\eta)^{-1}+BH_1(t,\eta)+\overline{B}H_2(t,\eta),$$

where

$$H_1(t,\eta)=H_{10}(t,\eta)+H_{11}(t,\eta),$$
$$H_2(t,\eta)=H_{20}(t,\eta)+H_{21}(t,\eta),$$

and

$$H_{10} = -\delta\left[\frac{1}{z-\bar{s}_0} + \frac{(\bar{s}_0 - s_0)}{(\bar{z}-s_0)^2} + e^{-2i\omega}\frac{(\bar{s}_0 - s_0)}{(z-\bar{s}_0)^2}\right],$$

$$H_{20} = -\delta\left[\frac{1}{\bar{z}-s_0} + \frac{(s_0 - \bar{s}_0)}{(z-\bar{s}_0)^2} + e^{-2i\omega}\frac{(s_0 - \bar{s}_0)(z+\bar{s}_0-2\bar{z})}{(z-\bar{s}_0)^3}\right] - \frac{\lambda}{z-\bar{s}_0}e^{-2i\omega},$$

$$H_{11} = -(1-\alpha)(1+\beta)L\left[\frac{F(z,s_0)}{1+\beta} + \frac{F(z,\bar{s}_0)}{1-\beta}, \frac{(s_0-\bar{s}_0)G(z,\bar{s}_0)}{1-\beta}\right],$$

$$H_{21} = -(1-\alpha)(1+\beta)L\left[\frac{(s_0-\bar{s}_0)G(z,\bar{s}_0)}{1-\beta}, \frac{F(z,s_0)}{1+\beta} + \frac{F(z,\bar{s}_0)}{1-\beta}\right],$$

where

$$L(\phi(z),\varphi(z)) = \phi(z) + \overline{\varphi(z)} + e^{-2i\omega}\left[(\bar{z}-z)\phi'(z) + \frac{1+\beta}{1-\beta}\bar{\varphi}(z) - \phi(z)\right] \quad \text{and} \quad \begin{aligned}\delta &= \frac{\alpha-\beta}{1-\beta},\\ \lambda &= \frac{\alpha+\beta}{1+\beta}.\end{aligned}$$

The functions H_{10} and H_{20} represent the effects of a dislocation below the interface where the material does not crack. The functions H_{11} and H_{21} are additional terms needed to satisfy the traction-free condition on the semiinfinite crack.

The traction expression is very similar to what had been reported by Hutchinson and Suo [1992]. Our expressions for H_{10} and H_{20} are the same as theirs, but our H_{11} and H_{21} are different, in that theirs are missing all terms related to $(s_0-\bar{s}_0)G(z,\bar{s}_0)/(1-\beta)$.

Appendix B. Formula for asymptotic stresses of an interfacial crack

When a crack branches into material 2, one can formulate $\sigma^0_{\theta\theta}(t) + i\sigma^0_{r\theta}(t)$ in terms of potential functions [Rice 1988; He and Hutchinson 1989] as

$$\sigma^0_{\theta\theta}(t) + i\sigma^0_{r\theta}(t) = \phi_0'(z) + \bar{\phi}_0'(z) + e^{-2i\omega}\left(\bar{z}\phi_0''(z) + \chi_0'(z)\right),$$

where

$$\phi_0'(z) = \frac{1}{2\sqrt{2\pi}\cosh(\pi\varepsilon)}e^{\varepsilon\pi}\bar{K}z^{-(1/2+i\varepsilon)},$$

$$\chi_0'(z) = \frac{1}{2\sqrt{2\pi}\cosh(\pi\varepsilon)}\left(e^{-\varepsilon\pi}Kz^{-1/2+i\varepsilon} - e^{\varepsilon\pi}\left(\frac{1}{2}-i\varepsilon\right)\bar{K}z^{-(1/2+i\varepsilon)}\right).$$

Sun and Qian [1996] demonstrated that this expression of the asymptotic stresses around an interfacial crack tip is appropriate for calculating the stress field in either the traditional linear fracture model or the contact model.

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