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# INTEGRATION OF MEASURES AND ADMISSIBLE STRESS FIELDS FOR MASONRY BODIES

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We study the compatibility of loads for bodies made of a no-tension (masonry) material. Loads are defined as weakly compatible if they can be equilibrated by an admissible stress field represented by a tensor valued measure, and strongly compatible if they can be equilibrated by a square integrable function. In the present study, we examine situations in which weak compatibility implies strong compatibility. For families of loads that depend on a parameter and the families of measures that equilibrate these loads, we find that, under some conditions, averaging with respect to the parameter leads to a measure with a square integrable density that equilibrates the loads. We illustrate the procedure on two-dimensional rectangular panels free from gravity, clamped at the bottom, and subjected to various loads on the free part of the boundary.

## 1. Introduction

We study the equilibrium problem of a body made of a no-tension (or masonry–like) material [Di Pasquale 1984; Anzellotti 1985; Giaquinta and Giusti 1985; Del Piero 1989; Lucchesi et al. 1994] under given loads $(\boldsymbol{s}, \boldsymbol{b})$ where $\boldsymbol{s}$ is the force applied to the free part of the boundary and $\boldsymbol{b}$ is the body force. The existence of equilibrium states, or at least the weaker property that the total energy functional of the masonry body be bounded from below, is closely related to the existence of a stress field $\boldsymbol{T}$ that is equilibrated with the applied loads and compatible with the incapability of the material to withstand traction (see Proposition 3.1, below). The problem of finding such an *admissible equilibrating stress field* $\boldsymbol{T}$ is a central problem of limit analysis [Temam 1983, Chapter 1, Section 5; Del Piero 1998; Lucchesi et al. 2008] because these stresses can be used to determine lower bounds for the collapse load and sometimes the collapse load itself. The loads admitting such a stress field are called *compatible*.

It has been shown in [Lucchesi et al. 2004; 2005a; 2005b; 2006; 2007] that the solution in concrete cases simplifies considerably if instead of admissible equilibrating stress fields represented by ordinary functions $\boldsymbol{T}$ one admits also stress fields $\mathsf{T}$ represented by tensor valued measures. This amounts to allowing for singularities of the stress field on one or more surfaces or curves of concentrated stress. In this paper, loads that admit an admissible equilibrating stress represented by a measure are called *weakly compatible* to distinguish them from loads that admit admissible equilibrating stresses represented by a square integrable function, which we call *strongly compatible*. These notions are not equivalent, as the examples show.

*Keywords:* masonry bodies, compatibility of loads, stresses represented by measures.
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Nevertheless, the existence of solutions to the equilibrium problem, and in particular the existence of a lower bound for the total energy functional, is strictly related to the existence of the admissible equilibrating stress field represented by a square integrable function by Proposition 3.1.

In the present paper, we describe a procedure that in certain cases allows us to use the information that loads are weakly compatible to show that they are actually strongly compatible. Crucial to the procedure is the fact that, in applications, both the loads $(\boldsymbol{s}^\lambda, \boldsymbol{b}^\lambda)$ and the admissible equilibrating stress measure $\mathsf{T}^\lambda$ depend on a real parameter $\lambda$. The identification of $\lambda$ depends on the nature of the problem. The idea is to take the average of the stress measure over any set $(\mu - \epsilon, \mu + \epsilon)$, where $\epsilon > 0$ is sufficiently small as dictated by the nature of the solution $\mathsf{T}^\lambda$ and $\mu$ is any point in the set of parameters. Averaging gives the measure

$$\mathsf{T} = \frac{1}{2\epsilon} \int_{\mu-\epsilon,}^{\mu+\epsilon} \mathsf{T}^\lambda \, d\lambda,$$

and it may happen that this measure, in contrast to $\mathsf{T}^\mu$, is absolutely continuous (with respect to the Lebesgue measure) with the density $\boldsymbol{T}$, which is square integrable. If the loads $(\boldsymbol{s}^\lambda, \boldsymbol{b}^\lambda)$ depend linearly on the parameter $\lambda$, as is often the case, then it is automatic that $\boldsymbol{T}$ equilibrates the loads $(\boldsymbol{s}^\mu, \boldsymbol{b}^\mu)$. It is intuitively plausible that the averaging procedure smears out the singularities in $\mathsf{T}^\lambda$ if the set of singularities changes its position with changing $\lambda$. Mathematically, the procedure is based on the coarea formula of the geometric measure theory, which also gives the conditions under which it is really the case.

This paper illustrates the general procedure on rectangular two-dimensional panels. We assume that the panel is free from body forces, clamped at its bottom, and subjected to loads prescribed on the boundary. We consider three types of the boundary loads:

(i) vertical top loads and horizontal loads on one side of the panel,

(ii) uniform vertical top loads and oblique side loads on one side of the panel,

(iii) and uniform vertical top loads and vanishing side loads on a panel with a symmetric opening.

In all cases we use the admissible equilibrating stresses represented by measures constructed in [Lucchesi et al. 2006], and combine them with averaging to produce equilibrating stress fields represented by square integrable functions (in fact, they are bounded in these three cases).

In Section 2 we consider families of vector valued measures, called parametric measures, that are mappings from the set of real parameters to the space of vector valued measures. We define an integral with respect to the parameter of such a mapping, which is the abstract counterpart of the averaging procedure mentioned above. The result of integration is again a measure. Section 3 introduces bodies and the loads applied to them. We define weakly and strongly equilibrating stress fields, and in Propositions 3.2 and 3.3 we describe the averaging procedure. The rest of the paper is devoted to the treatment of the loads (i)–(iii) listed above: Sections 4–5 deal with (i), Section 6 with (ii) and Section 7 with (iii). In general, the average of the parametric measure is difficult to calculate explicitly, and for applications it wholly suffices to know that averaging leads to the existence of a square integrable admissible stress field equilibrating the loads. Such is the case of the loads (i)–(iii). However, in a special subcase of case (i), treated in Section 5, we explicitly determine the result of the averaging.

Throughout, we use the conventions for vectors and second order tensors given in [Gurtin 1981]. Thus Lin denotes the set of all second order tensors on $\mathbb{R}^n$, that is, linear transformations from $\mathbb{R}^n$ into itself;

Sym is the subspace of symmetric tensors; $\mathrm{Sym}^+$ the set of all positive semidefinite elements of Sym; $\mathrm{Sym}^-$ is the set of all negative semidefinite elements of Sym. The scalar product of $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Lin}$ is defined by $\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^\mathsf{T})$ and $|\cdot|$ denotes the associated Euclidean norm on Lin.

## 2. Families of measures

If $\Omega \subset \mathbb{R}^n$ is a Borel set and $V$ a finite-dimensional inner product space, we denote by $\mathcal{M}(\Omega, V)$ the set of all $V$ valued measures (of finite total variation) supported by $\Omega$ [Ambrosio et al. 2000, Chapter 1]. If $\mathbf{m} \in \mathcal{M}(\mathbb{R}^n, V)$, we denote by $|\mathbf{m}|$ the total variation measure of $\mathbf{m}$, and by $\mathrm{M}(\mathbf{m})$ the mass of $\mathbf{m}$, defined by $\mathrm{M}(\mathbf{m}) = |\mathbf{m}|(\mathbb{R}^n)$. We call the elements of $\mathcal{M}(\Omega, \mathrm{Sym})$ *tensor measures*. We denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ [Ambrosio et al. 2000, Definition 1.52], and if $k$ is an integer, $0 \le k \le n$, we denote by $\mathcal{H}^k$ the $k$-dimensional Hausdorff measure in $\mathbb{R}^n$ [Ambrosio et al. 2000, Section 2.8]. If $\phi$ is a nonnegative measure or a $V$ valued measure, we denote by $\phi \llcorner A$ the restriction of $\phi$ to a Borel set $A \subset \mathbb{R}^n$; if $\phi$ is a nonnegative measure, we denote by $f\phi$ the product of the measure $\phi$ by a $\phi$ integrable $V$ valued function $f$ on $\mathbb{R}^n$; we refer to [Lucchesi et al. 2006, Section 2] for details. If $\Omega$ is an open subset of $\mathbb{R}^n$, we denote by $C_0(\Omega, V)$ the space of all continuous $V$ valued functions on $\mathbb{R}^n$ with compact support that is contained in $\Omega$, and denote by $|\cdot|_{C_0}$ the maximum norm on $C_0(\mathbb{R}^n, V)$.

An *integrable parametric measure* is a family $\{\mathbf{m}^\lambda : \lambda \in \Lambda\}$ of $V$ valued measures on $\mathbb{R}^n$ where $\Lambda \subset \mathbb{R}$ is a $\mathcal{L}^1$ measurable set of parameters such that

(i) for every $f \in C_0(\mathbb{R}^n, V)$ the function $\lambda \mapsto \int_{\mathbb{R}^n} f \cdot d\mathbf{m}^\lambda$ is $\mathcal{L}^1$ measurable on $\Lambda$;

(ii) we have

$$c := \int_\Lambda \mathrm{M}(\mathbf{m}^\lambda)\, d\lambda < \infty.$$

We note that the function $\lambda \mapsto \mathrm{M}(\mathbf{m}^\lambda)$ is $\mathcal{L}^1$ measurable on $\Lambda$ as a consequence of condition (i): if $K \subset C_0(\mathbb{R}^n, V)$ is a countable dense set then

$$\mathrm{M}(\mathbf{m}^\lambda) = \sup\left\{\int_{\mathbb{R}^n} f \cdot d\mathbf{m}^\lambda : f \in K, |f|_{C_0} \le 1\right\},$$

and thus the function $\lambda \mapsto \mathrm{M}(\mathbf{m}^\lambda)$ is a supremum of a countable family of $\mathcal{L}^1$ measurable functions. Hence, $\mathcal{L}^1$ measurable.

We note that parametric measures similar to those defined above occur in the contexts of disintegration (slicing) of measures [Ambrosio et al. 2000, Section 2.5] and Young's measures [Müller 1999, Chapter 5].

**Proposition 2.1.** *If $\{\mathbf{m}^\lambda : \lambda \in \Lambda\}$ is an integrable parametric measure, then there exists a unique $V$ valued measure $\mathbf{m}$ on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} f \cdot d\mathbf{m} = \int_\Lambda \int_{\mathbb{R}^n} f \cdot d\mathbf{m}^\lambda\, d\lambda, \tag{1}$$

*for each $f \in C_0(\mathbb{R}^n, V)$.*

We write

$$\mathbf{m} = \int_\Lambda \mathbf{m}^\lambda\, d\lambda, \tag{2}$$

and call $\mathbf{m}$ the *integral of the family* $\{\mathbf{m}^\lambda : \lambda \in \Lambda\}$ with respect to $\lambda$.

*Proof.* We note that for each $f \in C_0(\mathbb{R}^n, V)$, the right hand side of Equation (1) is a well defined real number. Indeed,

$$\begin{aligned}\Big| \int_\Lambda \int_{\mathbb{R}^n} f \cdot d\mathbf{m}^\lambda \, d\lambda \Big| &\leq \int_\Lambda \int_{\mathbb{R}^n} |f| \, d|\mathbf{m}^\lambda| \, d\lambda \\ &\leq |f|_{C_0} \int_\Lambda \mathrm{M}(\mathbf{m}^\lambda) \, d\lambda \\ &\leq c|f|_{C_0}.\end{aligned}$$

Thus, by the Riesz representation theorem [Ambrosio et al. 2000, Theorem 1.54], there exists a measure $\mathbf{m}$ such that Equation (1) holds. □

The following two propositions give two important examples of integrable parametric measures. In both cases the corresponding integral, Equation (2), is absolutely continuous with respect to the Lebesgue measure.

**Proposition 2.2.** *Let $\{h^\lambda : \lambda \in \Lambda\}$ be a family of $V$ valued functions on $\Omega \subset \mathbb{R}^n$ defined for all $\lambda$ from a $\mathscr{L}^1$ measurable set $\Lambda \subset \mathbb{R}$ such that the mapping $(\boldsymbol{x}, \lambda) \mapsto h^\lambda(\boldsymbol{x})$ is $\mathscr{L}^{n+1}$ integrable on $\Omega \times \Lambda$, that is,*

$$\int_\Lambda \int_\Omega |h^\lambda(\boldsymbol{x})| \, d\boldsymbol{x} d\lambda < \infty. \tag{3}$$

*If we define a $V$ valued measure $\mathbf{m}^\lambda$ by*

$$\mathbf{m}^\lambda = h^\lambda \mathscr{L}^n \llcorner \Omega,$$

*then $\{\mathbf{m}^\lambda : \lambda \in \Lambda\}$ is an integrable parametric measure, and we have*

$$\int_\Lambda \mathbf{m}^\lambda \, d\lambda = k\mathscr{L}^n \llcorner \Omega,$$

*where $k(\boldsymbol{x}) = \int_\Lambda h^\lambda(\boldsymbol{x}) \, d\lambda$, for $\mathscr{L}^n$ a.e. $\boldsymbol{x} \in \Omega$.*

*Proof.* This follows directly from Fubini's theorem. □

**Proposition 2.3.** *Let $\Omega_0 \subset \mathbb{R}^n$ be open, let $\varphi : \Omega_0 \to \mathbb{R}$ be locally Lipschitz continuous, and let $g : \Omega_0 \to V$ be $\mathscr{L}^n$ measurable on $\Omega_0$, with*

$$\int_{\Omega_0} |g||\nabla\varphi| \, d\mathscr{L}^n < \infty. \tag{4}$$

*Then for $\mathscr{L}^1$ a.e. $\lambda \in \mathbb{R}$, the function $g$ is $\mathscr{H}^{n-1} \llcorner \varphi^{-1}(\lambda)$ integrable. Denoting by $\Lambda$ the set of all such $\lambda$, we define the measure $\mathbf{m}^\lambda$ by*

$$\mathbf{m}^\lambda := g\mathscr{H}^{n-1} \llcorner \varphi^{-1}(\lambda),$$

*for each $\lambda \in \Lambda$. Then $\{\mathbf{m}^\lambda : \lambda \in \Lambda\}$ is an integrable parametric measure, and we have*

$$\int_\Lambda \mathbf{m}^\lambda \, d\lambda = g|\nabla\varphi|\mathscr{L}^n \llcorner \Omega_0. \tag{5}$$

*Proof.* Let $\mathbf{m}$ be given by Equation (2). If $f \in C_0(\mathbb{R}^n, V)$, then by the coarea formula [Ambrosio et al. 2000, Section 2.12] we have

$$\begin{aligned}\int_{\Omega_0} f \cdot g |\nabla\varphi| \, d\mathcal{L}^n &= \int_{\mathbb{R}} \int_{\varphi^{-1}(\lambda)} f \cdot g \, d\mathcal{H}^{n-1} d\lambda \\ &= \int_{\Lambda} \int_{\mathbb{R}^n} f \cdot d\mathbf{m}^{\lambda} d\lambda \\ &= \int_{\mathbb{R}^n} f \cdot d\mathbf{m}. \qquad \square\end{aligned}$$

## 3. Equilibrated loads

We consider a continuous body represented by a Lipschitz domain [Adams and Fournier 2003] $\Omega \subset \mathbb{R}^n$ and assume that $\mathcal{D}$, $\mathcal{S}$ are two disjoint Borel subsets of $\partial\Omega$ such that $\mathcal{D} \cup \mathcal{S} = \partial\Omega$, where $\mathcal{D}$, $\mathcal{S}$ will be identified below as the set of prescribed boundary displacement and prescribed boundary force.

We set

$$V_0 = \{\boldsymbol{v} \in C^1(\mathrm{cl}\,\Omega, \mathbb{R}^n) : \boldsymbol{v} = \mathbf{0} \text{ on } \mathcal{D}\},$$

and

$$V = \{\boldsymbol{v} \in W^{1,2}(\Omega, \mathbb{R}^n) : \boldsymbol{v} = \mathbf{0} \text{ a.e. on } \mathcal{D}\},$$

where $C^1(\mathrm{cl}\,\Omega, \mathbb{R}^n)$ is the set of all continuously differentiable mappings $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ such that $\boldsymbol{v}$ and its derivative $\nabla\boldsymbol{v}$ have a continuous extension to the closure $\mathrm{cl}\,\Omega$ of $\Omega$, and $W^{1,2}(\Omega, \mathbb{R}^n)$ is the Sobolev space of all $\mathbb{R}^n$ valued maps such that $\boldsymbol{v}$ and the distributional derivative $\nabla\boldsymbol{v}$ of $\boldsymbol{v}$ are square integrable on $\Omega$ [Adams and Fournier 2003]. We have $V_0 \subset V$. For any $\boldsymbol{v} \in V$ we define the infinitesimal strain tensor $\hat{\boldsymbol{E}}(\boldsymbol{v})$ of $\boldsymbol{v}$ by

$$\hat{\boldsymbol{E}}(\boldsymbol{v}) = \tfrac{1}{2}(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^{\mathsf{T}}).$$

The loads of the body are a pair $\mathfrak{L} = (\mathbf{s}, \mathbf{b})$ where $\mathbf{s} \in \mathcal{M}(\mathcal{S}, \mathbb{R}^n)$, $\mathbf{b} \in \mathcal{M}(\Omega, \mathbb{R}^n)$. Here $\mathbf{s}$ represents the force applied to the boundary $\mathcal{S}$ and $\mathbf{b}$ the force applied to the bulk $\Omega$ of the body. Since both $\mathbf{s}$ and $\mathbf{b}$ are measures, the definition admits concentrated forces on $\mathcal{S}$ and in $\Omega$ [Podio-Guidugli 2004; Lucchesi et al. 2006]. See Equation (42) for an example. Below we also consider the special case when these two measures are absolutely continuous with respect to the measures $\mathcal{H}^{n-1}$ and $\mathcal{L}^n$.

We interpret the measures $\mathbf{T} \in \mathcal{M}(\Omega, \mathrm{Sym})$ as stresses. Again, concentration effects are possible. We say that $\mathbf{T} \in \mathcal{M}(\Omega, \mathrm{Sym})$ is admissible if $\mathbf{T}$ takes the values in the set $\mathrm{Sym}^-$ of the negative semidefinite symmetric tensors, that is, if $\mathbf{T}(A)\boldsymbol{a} \cdot \boldsymbol{a} \le 0$ for any Borel set $A \subset \Omega$ and for any $\boldsymbol{a} \in \mathbb{R}^n$. We say that $\mathbf{T}$ *weakly equilibrates the loads* $(\mathbf{s}, \mathbf{b})$ if

$$\int_{\Omega} \hat{\boldsymbol{E}}(\boldsymbol{v}) \cdot d\mathbf{T} = \int_{\Omega} \boldsymbol{v} \cdot d\mathbf{b} + \int_{\mathcal{S}} \boldsymbol{v} \cdot d\mathbf{s},$$

for any $\boldsymbol{v} \in V_0$. We say that the loads $\mathfrak{L} = (\mathbf{s}, \mathbf{b})$ are *weakly compatible* if there exists an admissible $\mathbf{T} \in \mathcal{M}(\Omega, \mathrm{Sym})$ which weakly equilibrates them.

One can consider, in particular, the loads $\mathfrak{L} = (\mathbf{s}, \mathbf{b})$ of the form

$$\mathbf{s} = \boldsymbol{s}\mathcal{H}^{n-1} \llcorner \mathcal{S}, \qquad \mathbf{b} = \boldsymbol{b}\mathcal{L}^n \llcorner \Omega, \tag{6}$$

where

$$\boldsymbol{s} \in L^2(\mathcal{S}, \mathbb{R}^n), \qquad \boldsymbol{b} \in L^2(\Omega, \mathbb{R}^n), \tag{7}$$

with the first $L^2$ space taken relative to the measure $\mathcal{H}^{n-1}$ on $\mathcal{S}$ and the second relative to $\mathcal{L}^n$ on $\Omega$. In this case, we often identify the pair $\mathfrak{L} = (\mathsf{s}, \mathsf{b})$ with the pair $\mathfrak{L} = (\boldsymbol{s}, \boldsymbol{b})$.

One can consider, in particular, the measure $\mathsf{T}$ of the form $\mathsf{T} = \boldsymbol{T}\mathcal{L}^n \llcorner \Omega$, where $\boldsymbol{T} \in L^2(\Omega, \mathrm{Sym})$. We say that $\boldsymbol{T}$ is *admissible* if $\boldsymbol{T}(\boldsymbol{x}) \in \mathrm{Sym}^-$ for $\mathcal{L}^n$ a.e. $\boldsymbol{x} \in \Omega$. This is equivalent to saying that the measure $\mathsf{T} = \boldsymbol{T}\mathcal{L}^n \llcorner \Omega$ is admissible in the sense defined above. We say that $\boldsymbol{T}$ *strongly equilibrates the loads* $\mathfrak{L} = (\boldsymbol{s}, \boldsymbol{b})$ if

$$\int_\Omega \hat{\boldsymbol{E}}(\boldsymbol{v}) \cdot \boldsymbol{T}\, d\mathcal{L}^n = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{b}\, d\mathcal{L}^n + \int_{\mathcal{S}} \boldsymbol{v} \cdot \boldsymbol{s}\, d\mathcal{H}^{n-1},$$

for each $\boldsymbol{v} \in V$. Note that this notion applies only to the special loads represented by $\boldsymbol{s}, \boldsymbol{b}$ as in Equation (6). We say that the loads $\mathfrak{L} = (\boldsymbol{s}, \boldsymbol{b})$ satisfying Equation (7) are *strongly compatible* if there exists an admissible stress field $\boldsymbol{T} \in L^2(\Omega, \mathrm{Sym})$ strongly equilibrating them. In [Šilhavý 2008, Example 9.4], an example is given of loads $(\boldsymbol{s}, \boldsymbol{b})$ satisfying Equation (7) (even with $\boldsymbol{s}$ bounded and $\boldsymbol{b} \equiv \boldsymbol{0}$) such that $(\boldsymbol{s}, \boldsymbol{b})$ are weakly compatible but not strongly compatible.

The importance of the strong compatibility arises from the following statement.

**Proposition 3.1** ([Padovani et al. 2007]). *Let $\mathfrak{L} = (\boldsymbol{s}, \boldsymbol{b})$ be the loads satisfying Equation (7). Define the total energy functional $I : V \to \mathbb{R}$ by*

$$I(\boldsymbol{v}) = \int_\Omega \hat{w}(\hat{\boldsymbol{E}}(\boldsymbol{v}))\, d\mathcal{L}^n - \int_\Omega \boldsymbol{v} \cdot \boldsymbol{b}\, d\mathcal{L}^n - \int_{\mathcal{S}} \boldsymbol{v} \cdot \boldsymbol{s}\, d\mathcal{H}^{n-1},$$

*$\boldsymbol{v} \in V$, where $\hat{w} : \mathrm{Sym} \to [0, \infty)$ is the stored energy of a no-tension material* [Del Piero 1989]. *Then the loads are strongly compatible if and only if*

$$I_0 := \inf\{I(\boldsymbol{v}) : \boldsymbol{v} \in V\} > -\infty.$$

The condition $I_0 > -\infty$, in turn, has a dynamical significance [Padovani et al. 2007]: If $I_0 > -\infty$ then any dynamical process of a masonry body with dissipation stabilizes in the sense that the kinetic energy tends to 0, and if the set of equilibrium states is nonempty, the process asymptotically approaches the set of all equilibrium states. If, on the contrary, $I_0 = -\infty$, then any dynamical process exhibits a (dynamical) collapse in the sense that the total energy approaches $-\infty$, and the $W^{1,1}$ norm of the state at large times converges to $\infty$ (at least if $\boldsymbol{s}$ and $\boldsymbol{b}$ are bounded).

One often encounters the situation in which the loads depend on a parameter $\lambda$ from a subset $\Lambda$ of $\mathbb{R}$; that is, one deals with the family of loads $\mathfrak{L}^\lambda = (\mathsf{s}^\lambda, \mathsf{b}^\lambda)$, $\lambda \in \Lambda$, where

$$\{\mathsf{s}^\lambda, \lambda \in \Lambda\}, \qquad \{\mathsf{b}^\lambda, \lambda \in \Lambda\}, \tag{8}$$

are integrable parametric measures with values in $\mathbb{R}^n$, with $\Lambda$ an $\mathcal{L}^1$ measurable subset of $\mathbb{R}$. A more specific situation arises when the loads $\mathfrak{L}^\lambda$ are of the form

$$\mathsf{s}^\lambda = \boldsymbol{s}(\cdot, \lambda)\mathcal{H}^{n-1} \llcorner \mathcal{S}, \qquad \mathsf{b}^\lambda = \boldsymbol{b}(\cdot, \lambda)\mathcal{L}^n \llcorner \Omega, \tag{9}$$

$\lambda \in \Lambda$, where

$$\boldsymbol{s} \in L^2(\mathcal{S} \times \Lambda, \mathbb{R}^n), \qquad \boldsymbol{b} \in L^2(\Omega \times \Lambda, \mathbb{R}^n), \tag{10}$$

with the first $L^2$ space taken with respect to the measure $\mathscr{H}^{n-1} \otimes \mathscr{L}^1$ on $\mathscr{S} \times \Lambda$ and the second relative to the $n+1$ dimensional Lebesgue measure on $\Omega \times \Lambda$. Each load $\mathfrak{L}^\lambda$ is often weakly equilibrated by a stress field $\mathsf{T}^\lambda \in \mathscr{M}(\Omega, \mathrm{Sym})$ in such a way that

$$\{\mathsf{T}^\lambda, \lambda \in \Lambda\} \tag{11}$$

is an integrable parametric measure. In this situation, we have:

**Proposition 3.2.** *Let $\Lambda \subset \mathbb{R}$ be $\mathscr{L}^1$ measurable, and consider the parametric measures as in Equation (8) and Equation (11). Suppose that for $\mathscr{L}^1$ a.e. $\lambda \in \Lambda$ the stress field $\mathsf{T}^\lambda$ weakly equilibrates the loads $\mathfrak{L}^\lambda = (\mathsf{s}^\lambda, \mathsf{b}^\lambda)$. Then*

(i) *the stress field $\bar{\mathsf{T}} := \int_\Lambda \mathsf{T}^\lambda \, d\lambda$ weakly equilibrates the loads $\bar{\mathfrak{L}} := (\bar{\mathsf{s}}, \bar{\mathsf{b}})$, where*

$$\bar{\mathsf{s}} = \int_\Lambda \mathsf{s}^\lambda \, d\lambda, \qquad \bar{\mathsf{b}} = \int_\Lambda \mathsf{b}^\lambda \, d\lambda;$$

(ii) *if $\mathscr{L}^1(\Lambda) < \infty$ and if the loads are of the form Equation (9)–Equation (10), then the loads $\bar{\mathfrak{L}}$ defined in* (i) *are given by $\bar{\mathfrak{L}} = (\bar{\boldsymbol{s}}, \bar{\boldsymbol{b}})$, where*

$$\bar{\boldsymbol{s}} \in L^2(\mathscr{S}, \mathbb{R}^n), \qquad \bar{\boldsymbol{b}} \in L^2(\Omega, \mathbb{R}^n), \tag{12}$$

*are given by*

$$\bar{\boldsymbol{s}}(\boldsymbol{r}) = \int_\Lambda \boldsymbol{s}(\boldsymbol{r}, \lambda) \, d\lambda, \quad \boldsymbol{r} \in \mathscr{S}, \qquad \bar{\boldsymbol{b}}(\boldsymbol{r}) = \int_\Lambda \boldsymbol{b}(\boldsymbol{r}, \lambda) \, d\lambda, \quad \boldsymbol{r} \in \Omega. \tag{13}$$

*If, additionally,*

$$\int_\Lambda \mathsf{T}^\lambda \, d\lambda = \boldsymbol{T} \mathscr{L}^n \llcorner \Omega, \tag{14}$$

*where $\boldsymbol{T} \in L^2(\Omega, \mathrm{Sym})$ and $V_0$ is dense in $V$, then $\boldsymbol{T}$ strongly equilibrates the loads $\bar{\mathfrak{L}} = (\bar{\boldsymbol{s}}, \bar{\boldsymbol{b}})$.*

We note that $V_0$ is dense in $V$ if $\Omega$ has Lipschitz boundary, and if $\mathscr{D}$ is closed in $\partial\Omega$ with Lipschitz boundary.

*Proof.* (i): We have

$$\int_\Omega \hat{\boldsymbol{E}}(\boldsymbol{v}) \cdot d\mathsf{T}^\lambda = \int_\Omega \boldsymbol{v} \cdot d\mathsf{b}^\lambda + \int_{\mathscr{S}} \boldsymbol{v} \cdot d\mathsf{s}^\lambda,$$

for any $\boldsymbol{v} \in V_0$ and $\mathscr{L}^1$ a.e. $\lambda \in \Lambda$. Integrating over $\Lambda$ and invoking the definitions of integrals of measures, we obtain

$$\int_\Omega \hat{\boldsymbol{E}}(\boldsymbol{v}) \cdot d\bar{\mathsf{T}} = \int_\Omega \boldsymbol{v} \cdot d\bar{\mathsf{b}} + \int_{\mathscr{S}} \boldsymbol{v} \cdot d\bar{\mathsf{s}},$$

and thus $\bar{\mathsf{T}}$ weakly equilibrates the loads $\bar{\mathfrak{L}}$.

(ii): The formulas in Equation (13) are obtained by invoking the definitions of integrals of parametric measures, and exchanging the orders of integration with respect to $\boldsymbol{r}$ and $\lambda$. The inclusions Equation (12) follow from the assumption Equation (10) by using Hölder's inequality. If we have Equation (14), then by (i),

$$\int_\Omega \hat{\boldsymbol{E}}(\boldsymbol{v}) \cdot \boldsymbol{T} \, d\mathscr{L}^n = \int_\Omega \boldsymbol{v} \cdot \bar{\boldsymbol{b}} \, d\mathscr{L}^n + \int_{\mathscr{S}} \boldsymbol{v} \cdot \bar{\boldsymbol{s}} \, d\mathscr{L}^{n-1},$$

for each $\boldsymbol{v} \in V_0$; this extends by density to all $\boldsymbol{v} \in V$. □

Consider, finally, the situation in which the loads $\mathfrak{L}^\lambda$ are of the form Equation (9), and the functions $\boldsymbol{s}(\cdot, \lambda)$, $\boldsymbol{b}(\cdot, \lambda)$ depend on $\lambda$ linearly [Del Piero 1998]. Thus $\mathfrak{L}^\lambda := (\boldsymbol{s}^\lambda, \boldsymbol{b}^\lambda)$ where

$$\boldsymbol{s}^\lambda = \boldsymbol{s}_0 + \lambda \boldsymbol{s}_1, \quad \boldsymbol{b}^\lambda = \boldsymbol{b}_0 + \lambda \boldsymbol{b}_1, \qquad \lambda \in \mathbb{R}, \tag{15}$$

where

$$\boldsymbol{s}_0,\ \boldsymbol{s}_1 \in L^2(\mathcal{S}, \mathbb{R}^n), \qquad \boldsymbol{b}_0,\ \boldsymbol{b}_1 \in L^2(\Omega, \mathbb{R}^n).$$

We call $\boldsymbol{s}_0$, $\boldsymbol{b}_0$ the *permanent part* of the loads, $\boldsymbol{s}_1$, $\boldsymbol{b}_1$ the *variable part* of the loads, and $\lambda$ the *loading multiplier.*

If $\Lambda \subset \mathbb{R}$ is an $\mathcal{L}^1$ measurable set with $0 < \mathcal{L}^1(\Lambda) < \infty$, we abbreviate

$$⨍_\Lambda \boldsymbol{O}\, d\lambda := \frac{1}{\mathcal{L}^1(\Lambda)} \int_\Lambda \boldsymbol{O}\, d\lambda,$$

for any $\lambda$ integrable function $\boldsymbol{O}$ on $\Lambda$.

**Proposition 3.3.** *Assume that the loads $\mathfrak{L}^\lambda := (\boldsymbol{s}^\lambda, \boldsymbol{b}^\lambda)$ are given by Equation (15). Let $\Lambda \subset \mathbb{R}$ be a $\mathcal{L}^1$ measurable set such that $0 < \mathcal{L}^1(\Lambda) < \infty$, put*

$$\mu := ⨍_\Lambda \lambda\, d\lambda,$$

*and let $\{\mathsf{T}^\lambda : \lambda \in \Lambda\}$ be an integrable parametric measure such that for $\mathcal{L}^1$ a.e. $\lambda \in \Lambda$ the measure $\mathsf{T}^\lambda$ weakly equilibrates the loads $\mathfrak{L}^\lambda$. Then*

(i) *the measure*

$$\mathsf{T} := ⨍_\Lambda \mathsf{T}^\lambda\, d\lambda$$

*weakly equilibrates the loads $\mathfrak{L}^\mu$;*

(ii) *if $V_0$ is dense in $V$ and $\mathsf{T} = \boldsymbol{T}\mathcal{L}^n \llcorner \Omega$ where $\boldsymbol{T} \in L^2(\Omega, \mathrm{Sym})$ then $\boldsymbol{T}$ strongly equilibrates the loads $\mathfrak{L}^\mu$.*

*Proof.* (i): This follows from Proposition 3.2 (i) if one notes that

$$⨍_\Lambda \boldsymbol{s}^\lambda\, d\lambda = \boldsymbol{s}^\mu, \qquad ⨍_\Lambda \boldsymbol{b}^\lambda\, d\lambda = \boldsymbol{b}^\mu.$$

(ii): This follows from Proposition 3.2 (ii). □

## 4. A panel under vertical top loads and horizontal side loads

We consider the panel

$$\Omega = (0, b) \times (0, h) \subset \mathbb{R}^2$$

and introduce a coordinate system $x$, $y$ in $\mathbb{R}^2$ with the origin in the upper right corner of $\Omega$ and with the orientation of axes as shown in Figure 1. We denote a general point of $\Omega$ by $\boldsymbol{r} = (x, y)$ and let $\boldsymbol{i}$, $\boldsymbol{j}$ be the coordinate vectors along the axes $x$, $y$, respectively. We set

$$\mathcal{D} = (0, b) \times \{h\}, \quad \mathcal{S} = \partial\Omega \setminus \mathcal{D},$$

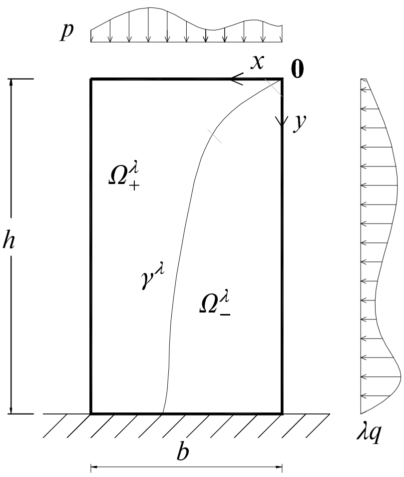


**Figure 1.** The panel under vertical top loads and horizontal side loads.

and consider the loads $\mathfrak{L}^\lambda = (\boldsymbol{s}^\lambda, \boldsymbol{b}^\lambda)$, where $\boldsymbol{b}^\lambda = \boldsymbol{0}$ in $\Omega$, and, for $\boldsymbol{r} = (x, y) \in \mathcal{S}$,

$$\boldsymbol{s}^\lambda(\boldsymbol{r}) = \begin{cases} p(x)\boldsymbol{j}, & \text{on } (0, b) \times \{0\}, \\ \lambda q(y)\boldsymbol{i}, & \text{on } \{0\} \times (0, h), \\ \boldsymbol{0}, & \text{elsewhere,} \end{cases}$$

where $p, q$ are nonnegative continuous functions on $[0, b]$ and $[0, h]$, respectively. We assume that

$$p_0 := p(0) > 0, \qquad q_0 := q(0) > 0.$$

We denote by $P$, $Q$ the primitives of $p, q$, respectively, satisfying $P(0) = Q(0) = 0$, and by $\mathcal{P}$, $\mathfrak{Q}$ the second primitives of $p, q$, respectively, satisfying $\mathcal{P}(0) = \mathcal{P}'(0) = \mathfrak{Q}(0) = \mathfrak{Q}'(0) = 0$. We set

$$\lambda_c = \mathcal{P}(b)/\mathfrak{Q}(h). \tag{16}$$

Since $p, q$ are nonnegative and $p_0 > 0$, $q_0 > 0$, the functions $P$, $Q$ are strictly positive and nondecreasing on the intervals $(0, b)$ and $(0, h)$, respectively. Consequently, $\mathcal{P}$, $\mathfrak{Q}$ are strictly positive and strictly increasing on the intervals $(0, b)$ and $(0, h)$, respectively. If $0 < \lambda \le \lambda_c$, then the range $[0, \lambda\mathfrak{Q}(h)]$ of $\lambda\mathfrak{Q}$ is contained in the range $[0, \mathcal{P}(b)]$ of $\mathcal{P}$. It follows that the set

$$\gamma^\lambda = \{\boldsymbol{r} = (x, y) \in \operatorname{cl}\Omega : \lambda\mathfrak{Q}(y) = \mathcal{P}(x)\},$$

is a graph of an increasing function $\omega^\lambda : [0, t^\lambda] \to [0, h]$, where $t^\lambda$ is determined from the equation $\lambda\mathfrak{Q}(h) = \mathcal{P}(t^\lambda)$. One easily finds that $\omega^\lambda$ is continuously differentiable, and from $\omega^\lambda(0) = 0$, $\omega^\lambda(t^\lambda) = h$ one deduces that $\gamma^\lambda$ is a smooth curve with one endpoint the origin $\boldsymbol{0} \in \mathbb{R}^2$ and the other endpoint $(t^\lambda, h)$. Moreover, except for the endpoints, the curve $\gamma^\lambda$ is contained in $\Omega$. If $\boldsymbol{r} = (x, y) \in \gamma^\lambda$, we denote by $\boldsymbol{t}^\lambda(\boldsymbol{r})$ the unit tangent vector to $\gamma^\lambda$ at $\boldsymbol{r}$, given by

$$\boldsymbol{t}^\lambda(\boldsymbol{r}) = \frac{\lambda Q(y)\boldsymbol{i} + P(x)\boldsymbol{j}}{\sqrt{P^2(x) + \lambda^2 Q^2(y)}}.$$

We note that if $\varphi : \Omega \to \mathbb{R}$ is defined by

$$\varphi(\boldsymbol{r}) = \mathcal{P}(x)/\mathcal{Q}(y), \tag{17}$$

$\boldsymbol{r} = (x, y) \in \Omega$, then for any $\lambda \in (0, \lambda_c)$ the curve $\gamma^\lambda$ is the level set of $\varphi$ corresponding to the value $\lambda$, that is,

$$\gamma^\lambda = \varphi^{-1}(\lambda) := \{\boldsymbol{r} \in \Omega : \varphi(\boldsymbol{r}) = \lambda\}.$$

We note, for future use, that $\varphi$ is continuously differentiable, and

$$|\nabla\varphi(\boldsymbol{r})| = \tau(\boldsymbol{r}), \tag{18}$$

where

$$\tau(\boldsymbol{r}) = \frac{\sqrt{\mathcal{P}'^2(x)\mathcal{Q}^2(y) + \mathcal{P}^2(x)\mathcal{Q}'^2(y)}}{\mathcal{Q}^2(y)}, \tag{19}$$

$\boldsymbol{r} = (x, y) \in \Omega$. The system of curves $\gamma^\lambda$, $\lambda \in (0, \lambda_c)$ forms a nonintersecting family that fully covers the region

$$\Omega_0 = \{\boldsymbol{r} = (x, y) \in \Omega : \mathcal{P}(x)/\mathcal{Q}(y) \in (0, \lambda_c)\} \equiv \varphi^{-1}(0, \lambda_c).$$

For a $\lambda \in (0, \lambda_c]$ the curve $\gamma^\lambda$ divides $\Omega$ into two open sets $\Omega^\lambda_\pm$ defined by

$$\Omega^\lambda_+ = \{\boldsymbol{r} = (x, y) \in \Omega : \text{either } t^\lambda \le x < b \text{ or } 0 < x < t^\lambda \text{ and } y < \omega^\lambda(x)\},$$

$$\Omega^\lambda_- = \{\boldsymbol{r} = (x, y) \in \Omega : 0 < x < t^\lambda \text{ and } y > \omega^\lambda(x)\}, .$$

**Proposition 4.1.** *Let* $0 < \lambda \le \lambda_c$ *and let* $\boldsymbol{T}^\lambda_r : \Omega \to \mathrm{Sym}$ *and* $\boldsymbol{T}^\lambda_s : \gamma^\lambda \to \mathrm{Sym}$ *be defined by*

$$\boldsymbol{T}^\lambda_r(\boldsymbol{r}) = \begin{cases} -p(x)\boldsymbol{j} \otimes \boldsymbol{j}, & \text{if } \boldsymbol{r} \in \Omega^\lambda_+, \\ -\lambda q(y)\boldsymbol{i} \otimes \boldsymbol{i}, & \text{if } \boldsymbol{r} \in \Omega^\lambda_-, \end{cases} \tag{20}$$

*for* $\boldsymbol{r} = (x, y) \in \Omega$ *and by*

$$\boldsymbol{T}^\lambda_s(\boldsymbol{r}) = \sigma^\lambda(\boldsymbol{r})\, \boldsymbol{t}^\lambda(\boldsymbol{r}) \otimes \boldsymbol{t}^\lambda(\boldsymbol{r}), \tag{21}$$

*for* $\boldsymbol{r} \in \gamma^\lambda$, *where* $\sigma^\lambda : \gamma^\lambda \to \mathbb{R}$ *is the unique continuously differentiable function satisfying*

$$\frac{d\sigma^\lambda(\boldsymbol{r})}{ds} = \rho^\lambda(\boldsymbol{r}), \quad \boldsymbol{r} \in \gamma^\lambda, \tag{22}$$

$$\sigma^\lambda(\boldsymbol{0}) = 0, \tag{23}$$

*where* $\rho^\lambda : \gamma^\lambda \to \mathbb{R}$ *is defined by*

$$\rho^\lambda(\boldsymbol{r}) = -\frac{\lambda P(x)Q(y)\big(p(x) + \lambda q(y)\big)}{P^2(x) + \lambda^2 Q^2(y)}, \tag{24}$$

$\boldsymbol{r} = (x, y) \in \gamma^\lambda$, *and where* $d/ds$ *denotes the derivative with respect to the arc length parameter* $s$ *on* $\gamma^\lambda$, *measured from the origin* $\boldsymbol{0}$. *Then* $\boldsymbol{T}^\lambda_r$ *and* $\boldsymbol{T}^\lambda_s$ *are bounded functions on* $\Omega$ *and* $\gamma^\lambda$, *respectively, and the measure*

$$\mathsf{T}^\lambda = \boldsymbol{T}^\lambda_r \mathcal{L}^2 \llcorner \Omega + \boldsymbol{T}^\lambda_s \mathcal{H}^1 \llcorner \gamma^\lambda, \tag{25}$$

*is an admissible stress field weakly equilibrating the loads* $\mathfrak{L}^\lambda$.

*Proof.* We note that the continuity of $p, q$ on the closed intervals $[0, b]$ and $[0, h]$ implies that $\boldsymbol{T}_r^\lambda$ is a bounded function, hence the first term in the right hand side of Equation (25) is a well defined measure. We note that $\rho^\lambda$ is a continuous function on $\gamma^\lambda$. Using the fact that for $x \to 0$, $y \to 0$ we have

$$\mathcal{P}(x) \sim \tfrac{1}{2} p_0 x^2, \quad \mathcal{Q}(y) \sim \tfrac{1}{2} q_0 y^2, \tag{26}$$

to within the errors $o(x^2)$, $o(y^2)$, respectively, and that

$$\lim_{\substack{\boldsymbol{r}=(x,y)\to\boldsymbol{0}\\ \boldsymbol{r}\in\gamma^\lambda}} y/x = \sqrt{p_0/\lambda q_0}, \tag{27}$$

one finds that

$$\lim_{\substack{\boldsymbol{r}\to\boldsymbol{0}\\ \boldsymbol{r}\in\gamma^\lambda}} \rho^\lambda(\boldsymbol{r}) = -\sqrt{\lambda p_0 q_0}. \tag{28}$$

Furthermore, trivially,

$$\lim_{\substack{\boldsymbol{r}\to(t^\lambda,h)\\ \boldsymbol{r}\in\gamma^\lambda}} \rho^\lambda(\boldsymbol{r}) = -\frac{\lambda P(t^\lambda) Q(h)\big(p(t^\lambda) + \lambda q(h)\big)}{P^2(t^\lambda) + \lambda^2 Q^2(h)}.$$

Hence $\sigma^\lambda$ is well defined, bounded, and continuous on $\gamma^\lambda$. This shows that $\boldsymbol{T}_s^\lambda$ is a bounded function on $\gamma^\lambda$, and the second term in the right hand side of Equation (25) is a well defined measure. We further note that $\mathsf{T}_r^\lambda$ is admissible since its density $\boldsymbol{T}_r^\lambda$ is a negative semidefinite tensor for $\mathcal{L}^2$ a.e. $\boldsymbol{r} \in \Omega$. The measure $\mathsf{T}_s^\lambda$ is admissible as well: clearly, $\rho^\lambda$ is nonpositive everywhere on $\gamma^\lambda$, and hence the integration of Equation (22)–Equation (23) shows that $\sigma^\lambda$ is a nonincreasing nonpositive function. Thus Equation (21) shows that the density $\boldsymbol{T}_s^\lambda$ is a negative semidefinite tensor. Consequently, $\mathsf{T}^\lambda$ is also admissible. Finally, one has to show that $\mathsf{T}^\lambda$ weakly equilibrates the loads $\mathfrak{L}^\lambda$. Referring for the details to [Lucchesi et al. 2006, Section 6], we note that this amounts to showing that the normal trace of $\mathsf{T}^\lambda$ equals $\boldsymbol{s}^\lambda$ on $\mathcal{S}$, and that the weak divergence of $\mathsf{T}^\lambda$ in $\Omega$ vanishes. The last is equivalent to proving that the classical divergence of $\boldsymbol{T}_r$ vanishes on $\Omega \setminus \gamma^\lambda$ (which is immediate), and that along $\gamma^\lambda$ the jump condition

$$[\boldsymbol{T}_r]\boldsymbol{n} - \operatorname{div} \boldsymbol{T}_s = \boldsymbol{0}, \tag{29}$$

holds where $[\boldsymbol{T}_r]\boldsymbol{n}$ is the jump of the normal component of $\boldsymbol{T}_r$ across $\gamma^\lambda$ and $\operatorname{div} \boldsymbol{T}_s$ is the linear divergence of $\boldsymbol{T}_s$ along $\gamma^\lambda$. Equation (29) leads to the above described shape of $\gamma^\lambda$ and to the differential equation, Equation (22)–Equation (23). We omit the details. □

Proposition 4.1 is now used to establish the following:

**Proposition 4.2.** *If* $0 < \mu < \lambda_c$, *then the loads* $\mathfrak{L}^\mu$ *are strongly compatible. In fact if* $\Lambda \subset (0, \lambda_c)$ *is any* $\mathcal{L}^1$ *measurable set with* $\mathcal{L}^1(\Lambda) > 0$ *such that*

$$\mu = \fint_\Lambda \lambda \, d\lambda,$$

*then* $\{\mathsf{T}^\lambda : \lambda \in \Lambda\}$ *is an integrable parametric measure, and the measure* $\mathsf{T} = \fint_\Lambda \mathsf{T}^\lambda \, d\lambda$ *is of the form*

$$\mathsf{T} = \boldsymbol{T}\mathcal{L}^2 \llcorner \Omega,$$

*where $\boldsymbol{T}$ is a bounded admissible stress field on $\Omega$ that strongly equilibrates the loads $\mathfrak{L}^{\mu}$. We have $\boldsymbol{T} = \boldsymbol{T}_r + \boldsymbol{T}_s$, where for $\boldsymbol{r} \in \Omega$,*

$$\boldsymbol{T}_r(\boldsymbol{r}) = \fint_{\Lambda} \boldsymbol{T}_r^{\lambda}(\boldsymbol{r})\, d\lambda, \tag{30}$$

$$\boldsymbol{T}_s(\boldsymbol{r}) = \begin{cases} \dfrac{\sigma^{\lambda}(\boldsymbol{r})\tau(\boldsymbol{r})}{\mathscr{L}^1(\Lambda)}\, \boldsymbol{t}^{\lambda}(\boldsymbol{r}) \otimes \boldsymbol{t}^{\lambda}(\boldsymbol{r}), & \text{if } \varphi(\boldsymbol{r}) \in \Lambda, \text{ where } \lambda := \varphi(\boldsymbol{r}), \\ \boldsymbol{0}, & \text{otherwise,} \end{cases} \tag{31}$$

*where $\varphi$ and $\tau$ are defined by Equation (17) and Equation (19).*

For $\mu = \lambda_c$ we have the weak compatibility of the loads $\mathfrak{L}^{\mu}$ by Proposition 4.1, but the above proposition says nothing about the strong compatibility for this limiting value.

*Proof.* We write

$$\mathsf{T}^{\lambda} = \mathsf{T}_r^{\lambda} + \mathsf{T}_s^{\lambda}, \tag{32}$$

where

$$\mathsf{T}_r^{\lambda} = \boldsymbol{T}_r^{\lambda}\mathscr{L}^2 \llcorner \Omega, \qquad \mathsf{T}_s^{\lambda} = \boldsymbol{T}_s^{\lambda}\mathscr{H}^1 \llcorner \gamma^{\lambda}. \tag{33}$$

We note that $\mathsf{T}_r^{\lambda}$ is of the form considered in Proposition 2.2, where $h^{\lambda}$ is to be identified with $\boldsymbol{T}_r^{\lambda}$. One sees that the integrability condition of Equation (3) is satisfied, and hence for any $\mathscr{L}^1$ measurable set $\Lambda \subset [0, \lambda_c]$, the measure

$$\mathsf{T}_r := \fint_{\Lambda} \mathsf{T}_r^{\lambda}\, d\lambda$$

is a measure absolutely continuous with respect to $\mathscr{L}^2 \llcorner \Omega$. Moreover, since the density $h^{\lambda}$ is a bounded function on $\Omega \times \Lambda$, we see that the density of $\mathsf{T}_r$ with respect to $\mathscr{L}^2$ is a bounded function. Thus

$$\mathsf{T}_r = \boldsymbol{T}_r\mathscr{L}^2 \llcorner \Omega,$$

where $\boldsymbol{T}_r$ is a bounded function on $\Omega$ given by Equation (30).

The measure $\mathsf{T}_s^{\lambda}$ is of the form

$$\mathsf{T}_s^{\lambda} = \boldsymbol{G}\mathscr{H}^1 \llcorner \varphi^{-1}(\lambda),$$

where $\boldsymbol{G} : \Omega_0 \to \mathrm{Sym}$ is defined by

$$\boldsymbol{G}(\boldsymbol{r}) = \boldsymbol{T}_s^{\lambda}(\boldsymbol{r})$$

for any $\boldsymbol{r} \in \Omega_0$, and where in the last formula $\lambda$ is an abbreviation for $\varphi(\boldsymbol{r})$.

We now wish to verify that the function $g := \boldsymbol{G}$ satisfies the integrability condition of Equation (4). We shall actually prove that the product $|\nabla\varphi||\boldsymbol{G}|$ is bounded on $\Omega_0$. For this it suffices to prove that for each $\lambda \in (0, \lambda_c)$, the limit

$$L(\lambda) := \lim_{\substack{\boldsymbol{r}\to\boldsymbol{0} \\ \boldsymbol{r}\in\gamma^{\lambda}}} |\nabla\varphi(\boldsymbol{r})||\boldsymbol{G}(\boldsymbol{r})|$$

exists, and the function $L$ is bounded on $(0, \lambda_c)$.

Recalling Equation (26) and Equation (27), we infer from Equation (18) and Equation (19) that

$$\lim_{\substack{\boldsymbol{r}=(x,y)\to\boldsymbol{0} \\ \boldsymbol{r}\in\gamma^{\lambda}}} x|\nabla\varphi(\boldsymbol{r})| = 2\lambda\frac{\sqrt{p_0 + \lambda q_0}}{\sqrt{p_0}}.$$

Furthermore, combining Equation (22), Equation (23), Equation (28) and $ds/dx = \sqrt{1 + \mathcal{P}'^2(x)/\lambda^2 \mathcal{Q}'^2(y)}$ one finds that

$$\lim_{\substack{\boldsymbol{r}=(x,y)\to\boldsymbol{0}\\ \boldsymbol{r}\in\gamma^\lambda}} |\boldsymbol{G}(\boldsymbol{r})|/x = \sqrt{p_0^2 + \lambda p_0 q_0},$$

and hence

$$\lim_{\substack{\boldsymbol{r}\to\boldsymbol{0}\\ \boldsymbol{r}\in\gamma^\lambda}} |\nabla\varphi(\boldsymbol{r})||\boldsymbol{G}(\boldsymbol{r})| = 2\lambda(p_0 + \lambda q_0).$$

This shows that the function $L$ is bounded on $(0, \lambda_c)$, and consequently that $|\nabla\varphi||\boldsymbol{G}|$ is bounded on $\Omega_0$.

In particular, the integrability condition of Equation (4) and Proposition 2.3 say that for any $\mathcal{L}^1$ measurable set $\Lambda \subset \mathbb{R}$ the measure

$$\mathsf{T}_s := \int_\Lambda \mathsf{T}_s^\lambda \, d\lambda$$

is $\mathcal{L}^2$ absolutely continuous over $\Omega$, with the density given by Equation (5). In the present case, this gives $\mathsf{T}_s = \boldsymbol{T}_s \mathcal{L}^2 \llcorner \Omega$, where $\boldsymbol{T}_s$ is given by Equation (31). Noting that $V_0$ is dense in $V$, we see that a combination of Propositions 4.1 and 3.3 completes the proof. $\square$

## 5. Example: Explicit determination of the averaged stress field

The goal of this section is to determine explicitly the density $\boldsymbol{T} = \boldsymbol{T}_r + \boldsymbol{T}_s$ of the measure $\mathsf{T}$ from Proposition 4.2 in a special case. The formula is in Equation (39), below.

We consider the situation of Section 4 and take in particular

$$p = \text{const} \;\text{ on }\; [0, b], \quad q \equiv 1 \text{ on } [0, h].$$

Hence

$$\boldsymbol{s}^\lambda(\boldsymbol{r}) = \begin{cases} p\boldsymbol{j}, & \text{on } (0, b) \times \{0\}, \\ \lambda\boldsymbol{i}, & \text{on } \{0\} \times (0, h) \\ \boldsymbol{0}, & \text{elsewhere on } \mathcal{S}; \end{cases}$$

see Figure 2. The results of Section 4 apply directly.

We find

$$\mathcal{P}(x) = \tfrac{1}{2}px^2, \quad \mathcal{Q}(y) = \tfrac{1}{2}y^2, \qquad 0 \le x \le b, 0 \le y \le h,$$

and Equation (16) gives $\lambda_c = pb^2/h^2$. Furthermore, if $0 \le \lambda \le \lambda_c$, then $\gamma^\lambda$ is the line segment

$$\gamma^\lambda = \{(x, y) \in \Omega : y = \sqrt{p/\lambda}x\}. \tag{34}$$

The regions $\Omega_\pm^\lambda$ are given by

$$\Omega_\pm^\lambda = \{\boldsymbol{r} = (x, y) \in \Omega : \pm(\sqrt{p/\lambda}x - y) > 0\}.$$

The region $\Omega_0$ covered by the segments $\gamma^\lambda$, $\lambda \in (0, \lambda_c)$ is delimited by the main diagonal of $\Omega$; in fact

$$\Omega_0 = \{\boldsymbol{r} = (x, y) \in \Omega : y/x > h/b\}.$$

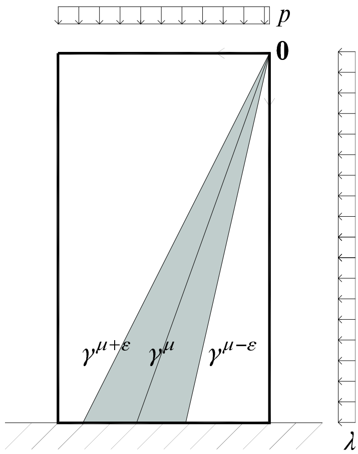


**Figure 2.** The panel under special load conditions.

We consider the measure $\mathsf{T}^\lambda$ given by Equation (25). In the present special case we find from Equation (20) that for $\boldsymbol{r} = (x, y) \in \Omega$,

$$\boldsymbol{T}_r^\lambda(\boldsymbol{r}) = \begin{cases} -p\boldsymbol{j}\otimes\boldsymbol{j}, & \text{if } \boldsymbol{r}\in\Omega_+^\lambda, \\ -\lambda\boldsymbol{i}\otimes\boldsymbol{i}, & \text{if } \boldsymbol{r}\in\Omega_-^\lambda. \end{cases}$$

Furthermore, Equation (24) and Equation (34) give $\rho^\lambda(\boldsymbol{r}) = -px/y$, $\boldsymbol{r} = (x, y) \in \gamma^\lambda$, and hence

$$\sigma^\lambda(\boldsymbol{r}) = -px|\boldsymbol{r}|/y, \tag{35}$$

by Equation (22) and Equation (23). Consequently,

$$\boldsymbol{T}_s^\lambda(\boldsymbol{r}) = -\sqrt{p\lambda}\boldsymbol{r}\otimes\boldsymbol{r}/|\boldsymbol{r}|,$$

for $\boldsymbol{r}\in\gamma^\lambda$, where we note that $\boldsymbol{t}^\lambda(\boldsymbol{r}) = \boldsymbol{r}/|\boldsymbol{r}|$ is the tangent vector to $\gamma^\lambda$.

We now wish to determine the density $\boldsymbol{T} = \boldsymbol{T}_r + \boldsymbol{T}_s$ of the measure $\mathsf{T}$. Recall that the functions $\boldsymbol{T}_r$, $\boldsymbol{T}_s$ are given by Equations (30)and (31). Let $0 < \mu < \lambda_c$, and let $\epsilon > 0$ be such that

$$\Lambda := (\mu - \epsilon, \mu + \epsilon) \subset (0, \lambda_c),$$

and let

$$A = \{\boldsymbol{r} = (x, y) : px^2/y^2 \in \Lambda\}.$$

We refer to Figure 2, where $A$ is the shaded region delimited by segments $\gamma^{\mu-\epsilon}$, $\gamma^{\mu+\epsilon}$, and where $\gamma^\mu$ is the middle segment.

Let us show that from Equation (30) one obtains

$$\boldsymbol{T}_r(\boldsymbol{r}) = \begin{cases} -p\boldsymbol{j}\otimes\boldsymbol{j}, & \text{if } \boldsymbol{r}\in\Omega_+^\lambda\setminus A, \\ -\mu\boldsymbol{i}\otimes\boldsymbol{i}, & \text{if } \boldsymbol{r}\in\Omega_-^\lambda\setminus A, \\ (2\epsilon)^{-1}\big(\alpha(\boldsymbol{r})\boldsymbol{i}\otimes\boldsymbol{i} + \beta(\boldsymbol{r})\boldsymbol{j}\otimes\boldsymbol{j}\big), & \text{if } \boldsymbol{r}\in A, \end{cases} \tag{36}$$

$\boldsymbol{r} \in \Omega$, where for $\boldsymbol{r} = (x, y) \in A$ we set

$$\alpha(\boldsymbol{r}) = \tfrac{1}{2}(p^2x^4/y^4 - (\mu+\epsilon)^2), \qquad \beta(\boldsymbol{r}) = p(\mu - \epsilon - px^2/y^2).$$

Let us derive the third regime of Equation (36); the derivation of the first two regimes is similar and simpler. Thus let $\boldsymbol{r} = (x, y) \in A$, and set $a = px^2/y^2$. We have

$$\boldsymbol{T}_r(\boldsymbol{r}) = (2\epsilon)^{-1}\int_{\mu-\epsilon}^{\mu+\epsilon} \boldsymbol{T}_r^\lambda(\boldsymbol{r})\,d\lambda = (2\epsilon)^{-1}\Big(\int_{\mu-\epsilon}^{a} \boldsymbol{T}_r^\lambda(\boldsymbol{r})\,d\lambda + \int_{a}^{\mu+\epsilon} \boldsymbol{T}_r^\lambda(\boldsymbol{r})\,d\lambda\Big). \tag{37}$$

If $\mu - \epsilon < \lambda < a$, then $\boldsymbol{T}_r^\lambda(\boldsymbol{r}) = -p\boldsymbol{j} \otimes \boldsymbol{j}$; if $a < \lambda < \mu + \epsilon$ then $\boldsymbol{T}_r^\lambda(\boldsymbol{r}) = -\lambda\boldsymbol{i} \otimes \boldsymbol{i}$. Inserting these values into the integrals in Equation (37), and recalling $a = px^2/y^2$, we obtain the value giving the third regime.

To determine $\boldsymbol{T}_s$, we note that from Equation (19) we obtain

$$\tau(\boldsymbol{r}) = 2px|\boldsymbol{r}|/y^3,$$

$\boldsymbol{r} = (x, y) \in \Omega_0$. Consequently, we deduce from Equation (35) and Equation (31) that for $\boldsymbol{r} = (x, y) \in \Omega$,

$$\boldsymbol{T}_s(\boldsymbol{r}) = \begin{cases} -2p^2x^2\boldsymbol{r} \otimes \boldsymbol{r}/2\epsilon\, y^4, & \text{if } \boldsymbol{r} \in A, \\ \boldsymbol{0}, & \text{otherwise.} \end{cases} \tag{38}$$

From Equation (36) and Equation (38) we obtain finally

$$\boldsymbol{T}(\boldsymbol{r}) = \begin{cases} -p\boldsymbol{j} \otimes \boldsymbol{j}, & \text{if } \boldsymbol{r} \in \Omega_+^\lambda \setminus A, \\ -\mu\boldsymbol{i} \otimes \boldsymbol{i}, & \text{if } \boldsymbol{r} \in \Omega_-^\lambda \setminus A, \\ \boldsymbol{S}(\boldsymbol{r}), & \text{if } \boldsymbol{r} \in A, \end{cases} \tag{39}$$

$\boldsymbol{r} \in \Omega$, where

$$\boldsymbol{S}(\boldsymbol{r}) = (2\epsilon)^{-1}\big((p^2x^4/y^4 - (\mu+\epsilon)^2)\boldsymbol{i} \otimes \boldsymbol{i}/2 + p(\mu - \epsilon - px^2/y^2)\boldsymbol{j} \otimes \boldsymbol{j} - 2p^2x^2\boldsymbol{r} \otimes \boldsymbol{r}/y^4\big).$$

Thus, by Proposition 3.3, the function $\boldsymbol{T}$ satisfies

$$\boldsymbol{T}\boldsymbol{n} = \boldsymbol{s}^\mu \quad \text{on } \mathcal{S}, \qquad \operatorname{div} \boldsymbol{T} = \boldsymbol{0} \quad \text{in } \Omega,$$

which can be also verified directly.

## 6. A panel with vertical top loads and oblique side loads

We again consider the panel

$$\Omega = (0, b) \times (0, h),$$

and assume that the top of the panel is subjected to a uniform pressure $p_0$ while the right side of the panel is subjected to oblique loads to be described below. We set

$$\mathcal{D} = (0, b) \times \{h\}, \qquad \mathcal{S} = \partial\Omega \setminus \mathcal{D},$$

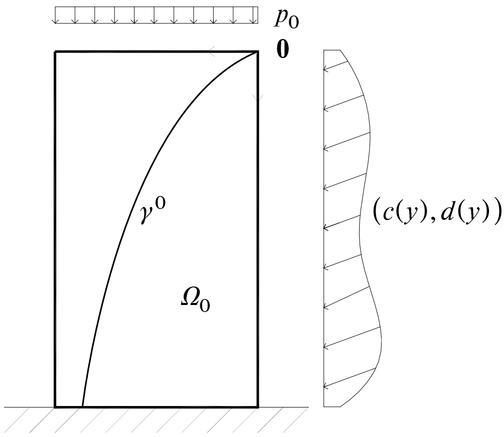


**Figure 3.** Oblique side loads.

$\boldsymbol{b} = \mathbf{0}$ in $\Omega$, and

$$\boldsymbol{s}(\boldsymbol{r}) = \begin{cases} p_0 \boldsymbol{j}, & \text{if } \boldsymbol{r} \in (0, b) \times \{0\}, \\ c(y)\boldsymbol{i} + d(y)\boldsymbol{j}, & \text{if } \boldsymbol{r} = (x, y) \in \{0\} \times [0, h], \\ \mathbf{0}, & \text{if } \boldsymbol{r} \in \{b\} \times (0, h), \end{cases} \tag{40}$$

$\boldsymbol{r} \in \mathcal{S}$, where $p_0 > 0$ and

$$c : [0, h] \to (0, \infty), \qquad d : [0, h] \to (0, \infty)$$

are continuously differentiable functions, see Figure 3. We make a permanent assumption that the functions

$$y \mapsto 1/c(y), \quad y \mapsto d(y)/c(y) \text{ are nondecreasing on } [0, h]. \tag{41}$$

If $0 \le \lambda \le h$, let $\omega^\lambda : \mathbb{R} \to \mathbb{R}$ be given by

$$\omega^\lambda(x) = \alpha(\lambda)x^2 + \beta(\lambda)x + \lambda,$$

$x \in \mathbb{R}$, where

$$\alpha(\lambda) = p_0/2hc(\lambda), \qquad \beta(\lambda) = d(\lambda)/c(\lambda),$$

and let $\gamma^\lambda$ be given by

$$\gamma^\lambda = \{\boldsymbol{r} = (x, \omega^\lambda(x)) \in \Omega : 0 < x < b\}.$$

In the following proposition we consider an auxiliary problem in which $\lambda \in [0, h]$ is fixed and the body is subjected to the loads $(\mathsf{s}^\lambda, \mathbf{0})$ with $\mathsf{s}^\lambda$ given by the measure

$$\mathsf{s}^\lambda = \boldsymbol{s}_0 \mathcal{H}^1 \llcorner \mathcal{S} + \big(c(\lambda)\boldsymbol{i} + d(\lambda)\boldsymbol{j}\big)\delta_{(0,\lambda)}, \tag{42}$$

where $\delta_{(0,\lambda)}$ is the Dirac measure at the point $(0, \lambda)$ and where

$$s_0(\boldsymbol{r}) = \begin{cases} p_0 \boldsymbol{j}/h, & \text{if } \boldsymbol{r} \in (0, b) \times \{0\}, \\ \boldsymbol{0}, & \text{if } \boldsymbol{r} \in \mathcal{S} \setminus (0, b) \times \{0\}, \end{cases}$$

$\boldsymbol{r} \in \mathcal{S}$.

**Proposition 6.1.** *Let* $0 \le \lambda \le h$, *and let* $\mathsf{T}^\lambda$ *be the measure defined by*

$$\mathsf{T}^\lambda = \boldsymbol{T}_r^\lambda \mathcal{L}^2 \llcorner \Omega + \boldsymbol{T}_s^\lambda \mathcal{H}^1 \llcorner \gamma^\lambda,$$

*where* $\boldsymbol{T}_r^\lambda$, $\boldsymbol{T}_s^\lambda$ *are bounded functions on* $\Omega$ *and* $\gamma^\lambda$, *respectively, given by*

$$\boldsymbol{T}_r^\lambda(\boldsymbol{r}) = \begin{cases} -p_0 \boldsymbol{j} \otimes \boldsymbol{j}/h, & \text{if } y < \omega^\lambda(x), \\ \boldsymbol{0}, & \text{if } y > \omega^\lambda(x), \end{cases}$$

$\boldsymbol{r} = (x, y) \in \Omega$, *and*

$$\boldsymbol{T}_s^\lambda(\boldsymbol{r}) = \sigma^\lambda(\boldsymbol{r}) \boldsymbol{t}^\lambda(\boldsymbol{r}) \otimes \boldsymbol{t}^\lambda(\boldsymbol{r}), \tag{43}$$

$\boldsymbol{r} = (x, y) \in \gamma^\lambda$, *where* $\boldsymbol{t}^\lambda(\boldsymbol{r})$ *is the unit tangent vector to* $\gamma^\lambda$ *at* $\boldsymbol{r}$ *and*

$$\sigma^\lambda(\boldsymbol{r}) = -\sqrt{c^2(\lambda) + \big(p_0 x/h + d(\lambda)\big)^2}.$$

*If* $\omega^\lambda(b) \ge h$ *then* $\mathsf{T}^\lambda$ *is an admissible stress field weakly equilibrating the loads* $(\mathsf{s}^\lambda, \boldsymbol{0})$.

Note that one endpoint of $\gamma^\lambda$ is always $(0, \lambda)$; the other endpoint can be either on the side $\{b\} \times (0, h)$ or on the base $[0, b] \times \{h\}$. The condition $\omega^\lambda(b) \ge h$ then says that the latter possibility occurs.

*Proof.* This follows from the considerations in [Lucchesi et al. 2006, Example 2]. □

**Proposition 6.2.** *If* $\omega^0(b) \ge h$ *then the loads* $(\boldsymbol{s}, \boldsymbol{0})$ *are strongly compatible. In fact, there exists a bounded admissible tensor field* $\boldsymbol{T}$ *on* $\Omega$ *strongly equilibrating them.*

The condition $\omega^0(b) \ge h$ says that the initial curve $\gamma^0$ ends on the base $[0, b] \times \{h\}$ of $\Omega$.

*Proof.* One easily finds that

$$\{\mathsf{T}^\lambda : 0 \le \lambda \le h\}, \quad \{\mathsf{s}^\lambda : 0 \le \lambda \le h\},$$

are integrable parametric measures. From conditions Equation (41), one finds that $\omega^\lambda(b) \ge \omega^0(b)$. Thus the hypothesis $\omega^0(b) \ge h$ implies that $\omega^\lambda(b) \ge h$ for all $\lambda \in [0, h]$. Consequently, $\mathsf{T}^\lambda$ weakly equilibrates the loads $(\mathsf{s}^\lambda, \boldsymbol{0})$ whenever $0 \le \lambda \le h$ by Proposition 6.1. By Proposition 3.2 (i), the stress field $\mathsf{T} = \int_0^h \mathsf{T}^\lambda \, d\lambda$ weakly equilibrates the loads $(\bar{\mathsf{s}}, \boldsymbol{0})$, where $\bar{\mathsf{s}} = \int_0^h \mathsf{s}^\lambda \, d\lambda$. If $0 \le \lambda \le h$ and $\boldsymbol{v} \in C_0(\mathbb{R}^2, \mathbb{R}^2)$, then comparing Equation (42) with Equation (40) we obtain

$$\int_{\mathcal{S}} \boldsymbol{v} \cdot d\mathsf{s}^\lambda = \int_{\mathcal{S}} \boldsymbol{v} \cdot \boldsymbol{s}_0 \, d\mathcal{H}^1 + \boldsymbol{v}(0, \lambda) \cdot \boldsymbol{s}(0, \lambda).$$

Hence,

$$\int_0^h \int_{\mathcal{S}} \boldsymbol{v} \cdot \, d\mathsf{s}^\lambda d\lambda = h \int_{\mathcal{S}} \boldsymbol{v} \cdot \boldsymbol{s}_0 \, d\mathcal{H}^1 + \int_0^h \boldsymbol{v}(0, \lambda) \cdot \boldsymbol{s}(0, \lambda) \, d\lambda = \int_{\mathcal{S}} \boldsymbol{v} \cdot \boldsymbol{s} \, d\mathcal{H}^1,$$

which shows that

$$\bar{\mathsf{s}} \equiv \int_0^h \mathsf{s}^\lambda \, d\lambda = \boldsymbol{s}\mathscr{H}^1 \llcorner \mathscr{S}.$$

Thus, we conclude that $\mathsf{T}$ weakly equilibrates the loads $(\boldsymbol{s}, \mathbf{0})$.

Let us now show that $\mathsf{T} = \boldsymbol{T}\mathscr{L}^2 \llcorner \Omega$, where $\boldsymbol{T}$ is a bounded function on $\Omega$. Decompose $\mathsf{T}^\lambda$ into $\mathsf{T}^\lambda_r$, $\mathsf{T}^\lambda_s$ as in Equation (32) and Equation (33). Then $\mathsf{T} = \mathsf{T}_r + \mathsf{T}_s$, where

$$\mathsf{T}_r = \int_0^h \mathsf{T}^\lambda_r \, d\lambda, \qquad \mathsf{T}_s = \int_0^h \mathsf{T}^\lambda_s \, d\lambda.$$

Since $\boldsymbol{T}^\lambda_r$ is bounded independently of $\lambda$, it is found that $\mathsf{T}_r = \boldsymbol{T}_r\mathscr{L}^2 \llcorner \Omega$ where $\boldsymbol{T}_r$ is a bounded function in the same way as in the proof of Proposition 4.2.

Next, we prove that

$$\mathsf{T}_s = \boldsymbol{T}_s\mathscr{L}^2 \llcorner \Omega, \tag{44}$$

where $\boldsymbol{T}_s$ is a bounded function. Let

$$\Omega_0 = \bigcup \{\gamma^\lambda : 0 < \lambda < h\} \equiv \{\boldsymbol{r} = (x, y) \in \Omega : y > \omega^0(x)\}.$$

The assumption Equation (41) and the form of $\omega^\lambda$ imply that for each $\boldsymbol{r} = (x, y) \in \Omega_0$ there exists exactly one $\lambda$ such that

$$y = \omega^\lambda(x).$$

We define $\varphi : \Omega_0 \to \mathbb{R}$ by setting $\varphi(\boldsymbol{r}) = \lambda$, that is, by

$$y = \alpha(\varphi(\boldsymbol{r}))x^2 + \beta(\varphi(\boldsymbol{r}))x + \varphi(\boldsymbol{r}),$$

$\boldsymbol{r} = (x, y) \in \Omega_0$. The implicit function theorem and the differentiability of $\alpha$, $\beta$ imply that $\varphi$ is continuously differentiable and the derivatives of $\varphi$ at $\boldsymbol{r} = (x, y)$ are given by

$$\frac{\partial \varphi}{\partial x} = -\frac{2\alpha x + \beta}{\alpha' x^2 + \beta' x + 1}, \qquad \frac{\partial \varphi}{\partial y} = \frac{1}{\alpha' x^2 + \beta' x + 1}, \tag{45}$$

where $\alpha$, $\beta$, $\alpha'$, $\beta'$ are evaluated at $\varphi(\boldsymbol{r})$. We have $\alpha' \ge 0$, $\beta' \ge 0$ by Equation (41) and hence the denominators in Equation (45) are $\ge 1$. Since the numerators are bounded as $\alpha$, $\beta$ are continuous on $[0, h]$, we see that the partial derivatives Equation (45) are bounded on $\Omega_0$. Hence $|\nabla\varphi|$ is also bounded. We have

$$\mathsf{T}^\lambda_s = \boldsymbol{G}\mathscr{H}^1 \llcorner \varphi^{-1}(\lambda),$$

where $\boldsymbol{G} : \Omega_0 \to \mathrm{Sym}$ is given by

$$\boldsymbol{G}(\boldsymbol{r}) = \boldsymbol{T}^\lambda_s(\boldsymbol{r}),$$

$\boldsymbol{r} \in \Omega_0$, and where $\lambda$ stands for $\varphi(\boldsymbol{r})$. From the expression Equation (43), we find that $\boldsymbol{G}$ is bounded on $\Omega_0$. Proposition 2.3 then says that we have Equation (44), where

$$\boldsymbol{T}_s(\boldsymbol{r}) = \begin{cases} |\nabla\varphi(\boldsymbol{r})|\boldsymbol{G}(\boldsymbol{r}), & \text{if } \boldsymbol{r} \in \Omega_0, \\ \mathbf{0}, & \text{if } \boldsymbol{r} \in \Omega \setminus \Omega_0, \end{cases}$$

$\boldsymbol{r} \in \Omega$. Thus $\boldsymbol{T}_s$ is bounded. Noting that $V_0$ is dense in $V$, we see that a combination of Propositions 6.1 and 3.2 (ii) completes the proof. □

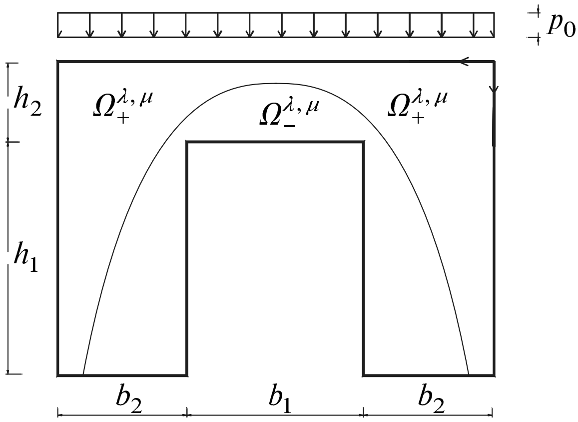


**Figure 4.** Panel with a symmetric opening.

## 7. A panel with a symmetric opening

Let us consider a rectangular panel $\Omega$ with base $b = b_1 + 2b_2$, height $h = h_1 + h_2$, and a symmetric opening of dimensions $b_1$ and $h_1$ (Figure 4), that is clamped at its base and subjected to a vertical load $p_0$, uniformly distributed on its top. We set

$$\mathcal{D} = (0, b_2) \times \{h\} \cup (b_1 + b_2, b) \times \{h\}, \quad \mathcal{S} = \partial\Omega \setminus \mathcal{D},$$

$\boldsymbol{b} = \mathbf{0}$ in $\Omega$,

$$\boldsymbol{s} = \begin{cases} p_0 \boldsymbol{j}, & \text{on } (0, b) \times \{0\}, \\ \mathbf{0}, & \text{on } \mathcal{S} \setminus (0, b) \times \{0\}, \end{cases}$$

$p_0 > 0$.

Let $\lambda > 0$, $\mu > 0$, and consider the parabola

$$\gamma^{\lambda,\mu} = \{(x, \omega^{\lambda,\mu}(x)) \in \mathbb{R}^2 : b/2 - \mu < x < b/2 + \mu\},$$

where $\omega^{\lambda,\mu} : (b/2 - \mu, b/2 + \mu) \to \mathbb{R}$ is defined by

$$\omega^{\lambda,\mu}(x) = \lambda + (h - \lambda)(x - b/2)^2/\mu^2,$$

$b/2 - \mu < x < b/2 + \mu$. Let

$$\mathcal{A} = \{(\lambda, \mu) \in (0, \infty) \times (0, \infty) : \gamma^{\lambda,\mu} \subset \Omega\}$$

be the set of all pairs $(\lambda, \mu)$ for which the parabola $\gamma^{\lambda,\mu}$ is wholly contained in the panel $\Omega$. One has [Lucchesi et al. 2006, Section 6]:

$$\mathcal{A} \text{ is nonempty} \Leftrightarrow \quad \zeta \leq 4\xi(\xi + 1),$$

$$\mathcal{A} \text{ has a nonempty interior} \Leftrightarrow \quad \zeta < 4\xi(\xi + 1),$$

where $\xi := b_2/b_1$, $\zeta := h_1/h_2$. If $(\lambda, \mu) \in \mathscr{A}$, we define the sets $\Omega_{\pm}^{\lambda,\mu}$ by

$$\Omega_{-}^{\lambda,\mu} = \{\boldsymbol{r} = (x, y) \in \Omega : |x - b/2| < \mu,\ y > \omega^{\lambda,\mu}(x)\},$$

$$\Omega_{+}^{\lambda,\mu} = \Omega \setminus (\Omega_{-}^{\lambda,\mu} \cup \gamma^{\lambda,\mu}).$$

**Proposition 7.1.** *Let $(\lambda, \mu) \in \mathscr{A}$ and define the measure $\mathsf{T}^{\lambda,\mu}$ by*

$$\mathsf{T}^{\lambda,\mu} = \boldsymbol{T}_r^{\lambda,\mu} \mathscr{L}^2 \llcorner \Omega + \boldsymbol{T}_s^{\lambda,\mu} \mathscr{H}^1 \llcorner \gamma^{\lambda,\mu}, \tag{46}$$

*where $\boldsymbol{T}_r^{\lambda,\mu}$ and $\boldsymbol{T}_s^{\lambda,\mu}$ are bounded functions on $\Omega$ and $\gamma^{\lambda,\mu}$, respectively, given by*

$$\boldsymbol{T}_r^{\lambda,\mu}(\boldsymbol{r}) = \begin{cases} -p_0 \boldsymbol{j} \otimes \boldsymbol{j}, & \text{if } \boldsymbol{r} \in \Omega_{+}^{\lambda,\mu}, \\ \boldsymbol{0}, & \text{otherwise}, \end{cases}$$

$$\boldsymbol{r} \in \Omega, \quad \text{and} \quad \boldsymbol{T}_s^{\lambda,\mu}(\boldsymbol{r}) = \sigma^{\lambda,\mu}(\boldsymbol{r}) \boldsymbol{T}^{\lambda,\mu}(\boldsymbol{r}) \otimes \boldsymbol{T}^{\lambda,\mu}(\boldsymbol{r}),$$

*where*

$$\sigma^{\lambda,\mu}(\boldsymbol{r}) = -\frac{p_0 \sqrt{\mu^4 + 4(h-\lambda)^2 (x - b/2)^2}}{2(h-\lambda)},$$

*$\boldsymbol{r} = (x, y) \in \gamma^{\lambda,\mu}$. Then $\mathsf{T}^{\lambda,\mu}$ is an admissible stress field weakly equilibrating the loads $(\boldsymbol{s}, \boldsymbol{0})$.*

We emphasize that for all $(\lambda, \mu) \in \mathscr{A}$ the stress field $\mathsf{T}^{\lambda,\mu}$ equilibrates the same loads.

*Proof.* This follows from the considerations in [Lucchesi et al. 2006, Examples 3 and 4]. □

**Proposition 7.2.** *If $\mathscr{A}$ has a nonempty interior, then the loads $(\boldsymbol{s}, \boldsymbol{0})$ are strongly compatible. In fact, there exists a bounded admissible stress field $\boldsymbol{T}$ on $\Omega$ strongly equilibrating them.*

*Proof.* Let $(\lambda_0, \mu_0)$ be an interior point of $\mathscr{A}$, hence $(\lambda, \mu) \in \mathscr{A}$ for all $(\lambda, \mu)$ sufficiently close to $(\lambda_0, \mu_0)$. Therefore, setting

$$\alpha := (h - \lambda_0)/\mu_0^2, \qquad \hat{\lambda}(\mu) = h - \alpha \mu^2,$$

we have $(\hat{\lambda}(\mu), \mu) \in \mathscr{A}$ for all $\mu \in \Lambda := (\mu_0 - \epsilon, \mu_0 + \epsilon)$, where $\epsilon > 0$ is sufficiently small. If $\mathsf{T}^{\lambda,\mu}$ denotes the measure Equation (46), then by Proposition 3.2 (i), the measure

$$\mathsf{T} := \int_{\Lambda} \mathsf{T}^{\hat{\lambda}(\mu),\mu} d\mu$$

weakly equilibrates the loads $(\boldsymbol{s}, \boldsymbol{0})$. We write $\mathsf{T} = \mathsf{T}_r + \mathsf{T}_s$, where

$$\mathsf{T}_r = \int_{\Lambda} \mathsf{T}_r^{\mu} d\mu, \qquad \mathsf{T}_s = \int_{\Lambda} \mathsf{T}_s^{\mu} d\mu,$$

$$\mathsf{T}_r^{\mu} = \boldsymbol{T}_r^{\hat{\lambda}(\mu),\mu} \mathscr{L}^2 \llcorner \Omega, \quad \mathsf{T}_s^{\mu} = \boldsymbol{T}_s^{\hat{\lambda}(\mu),\mu} \mathscr{H}^1 \llcorner \gamma^{\hat{\lambda}(\mu),\mu}.$$

By Proposition 2.2, $\mathsf{T}_r = \boldsymbol{T}_r \mathscr{L}^2 \llcorner \Omega$, where

$$\boldsymbol{T}_r(\boldsymbol{r}) = \int_{\Lambda} \boldsymbol{T}_s^{\hat{\lambda}(\mu),\mu}(\boldsymbol{r}) d\mu,$$

$\boldsymbol{r} \in \Omega$. Since $\boldsymbol{T}_s^{\hat{\lambda}(\mu),\mu}$ is bounded independently of $\mu$ if $\mu_0 - \epsilon < \mu < \mu_0 + \epsilon$, we see that $\boldsymbol{T}_r$ is bounded on $\Omega$.

Furthermore, one finds that $\gamma^{\hat{\lambda}(\mu),\mu} = \varphi^{-1}(\mu)$, where $\varphi : \Omega \to \mathbb{R}$ is defined by

$$\varphi(\boldsymbol{r}) = \sqrt{(x - b/2)^2 + (h - y)/\alpha},$$

$\boldsymbol{r} = (x, y) \in \Omega$. Let

$$\Omega_0 = \bigcup \{\gamma^{\hat{\lambda}(\mu),\mu} : \mu_0 - \epsilon < \mu < \mu_0 + \epsilon\} \equiv \varphi^{-1}((\mu_0 - \epsilon, \mu_0 + \epsilon)).$$

The measure $\mathsf{T}_s^\mu$ can be written as

$$\mathsf{T}_s^\mu = \boldsymbol{G}\mathcal{H}^1 \llcorner \varphi^{-1}(\mu),$$

where $\boldsymbol{G} : \Omega_0 \to \mathrm{Sym}$ is given by

$$\boldsymbol{G}(\boldsymbol{r}) = \boldsymbol{T}_s^{\hat{\lambda}(\mu),\mu}(\boldsymbol{r}),$$

$\boldsymbol{r} \in \Omega_0$, where $\mu$ stands for $\varphi(\boldsymbol{r})$. One easily finds that $\varphi$ is continuously differentiable on $\Omega$ with bounded derivatives on $\Omega_0$; in particular, $|\nabla\varphi|$ is bounded on $\Omega_0$. Furthermore, one has

$$|\sigma^{\hat{\lambda}(\mu),\mu}(\boldsymbol{r})| \le \frac{p_0}{2\alpha}\sqrt{1 + \alpha^2 b^2}, \qquad \boldsymbol{r} \in \gamma^{\hat{\lambda}(\mu),\mu},$$

which implies that $\boldsymbol{G}$ is bounded on $\Omega_0$. Proposition 2.3 then says that $\mathsf{T}_s = \boldsymbol{T}_s \mathcal{L}^2 \llcorner \Omega$, where

$$\boldsymbol{T}_s(\boldsymbol{r}) = \begin{cases} (2\epsilon)^{-1}|\varphi(\boldsymbol{r})|\boldsymbol{G}(\boldsymbol{r}), & \text{if } \boldsymbol{r} \in \Omega_0, \\ \boldsymbol{0}, & \text{otherwise,} \end{cases}$$

$\boldsymbol{r} \in \Omega$, which is a bounded function by the above.

We thus conclude that

$$\mathsf{T} = \boldsymbol{T}\mathcal{L}^2 \llcorner \Omega,$$

where $\boldsymbol{T} = \boldsymbol{T}_r + \boldsymbol{T}_s$ is a bounded function on $\Omega$. A reference to the density of $V_0$ in $V$ and to Propositions 7.1 and 3.2 (ii) then completes the proof. $\square$

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