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THE INFLUENCE OF AN INITIAL SIMPLE SHEAR DEFORMATION ON LONG-WAVE MOTION IN AN ELASTIC LAYER

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Long wave dispersion in an incompressible elastic layer subject to an initial static simple shear deformation is investigated. Long wave approximations of the dispersion relation associated with zero incremental traction on the faces are derived for both low and high-frequency motion. Comparison of approximate and numerical solutions is shown to provide excellent agreement over a surprisingly large wave number range. Within both the low and high-frequency regimes, the approximations are employed to establish the relative asymptotic orders of the displacement components and hydrostatic pressure. In the high-frequency case, the in-plane component of displacement is shown to be asymptotically larger than the normal component; motion is, therefore, essentially that of thickness shear resonance. The influence of this specific form of initial deformation is, therefore, seemingly minor in respect of long-wave high-frequency motion. However, in the long-wave low-frequency case, considerable differences are noted in comparison with both the classical and previously published prestressed cases. Specifically, both the normal and in-plane displacement components are of the same asymptotic order, indicating the absence of any natural analogue of either classical bending or extension.

1. Introduction

The theory of infinitesimal time-dependent motion, superimposed upon a large static primary deformation, has been used by a number of authors to study infinitesimal wave propagation in elastic bodies subject to an initial pure homogeneous strain. Within this present study, our concern is long-wave motion in a layer of finite thickness and infinite lateral extent. The layer is composed of incompressible elastic material and subject to a simple shear primary deformation. Wave propagation in homogeneously prestressed rectangular plates has been discussed in detail by a number of authors — see for example [Ogden and Roxburg 1993; Rogerson 1997; Rogersion and Fu 1995] — in regard to incompressible elastic plates, [Roxburgh and Ogden 1994] and [Nolde et al. 2004] for compressible elastic plates, and [Rogerson 1998] for anisotropic plates. All these studies concern scenarios for which one of the principal axes of primary deformation is normal to the surface of a plate or half-space. In the context of SH waves, Bar and Pal [1985] investigated the possibility of Love waves under initial shear stress. Connor and Ogden [1995] investigated the influence of simple shear on the existence of surface waves and later [1996] discussed its influence on dispersion in a plate. In the latter of these two references, they derived the dispersion relations associated with incrementally traction-free and traction-fixed faces, investigated questions of stability, and provided numerical solutions for the dispersion and bifurcation relations in the traction-free case for plates composed of neo-Hookean material. In the case of a simple shear primary deformation,

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no principal axis of deformation is normal to the plane of the plate or half-space. We remark that a similar phenomenon is also possible with anisotropy and with the specific orientation of the principal axes relative to a free surface; see for example [Fu 2005; Triantafyllidis and Abeyarante 1983].

Shear-type deformations may occur in a number of scenarios and are commonly encountered in geomechanics. As examples, we cite the Earth's crust, for which the influence of hydrostatic pressure is significant; see for example [Gessner et al. 2007]. Natural mechanisms also exist within layered systems whereby movements over long timescales may cause shear-type deformations; see [Ide et al. 2007].

In our present study, an asymptotic long-wave analysis of the dispersion relation associated with infinitesimal wave propagation in a layer with an initial simple shear deformation and incrementally traction-free faces is carried out. An asymptotic methodology has previously been exploited in a number of papers, see for example [Kaplunov et al. 2002] and [Kaplunov et al. 2000]. Our specific aim is to carry out a long-wave asymptotic investigation of the dispersion relation and use it to establish the relative asymptotic orders of the field variables, thereby providing the theoretical framework for long-wave asymptotic models to be derived.

We begin this paper with a review of basic dynamic equations in Section 2. In Section 3, the dispersion relation, derived using traction-free boundary conditions, is established and numerical results concerning a Varga material are presented. In Section 4, long-wave low-frequency approximations for dispersion relations are derived up to, and including, second order. These expansions are derived for the most general appropriate strain energy function. For neo-Hookean materials, a constitutive simplification is exploited, and higher-order approximations are established. The approximations are shown to agree well with a number of numerical solutions.

In Section 5, long-wave high-frequency motion is investigated. In the most general constitutive case, the general form of the cut-off frequencies is established, with second- and third-order corrections established for materials of neo-Hookean type. Again, good agreement between asymptotic and numerical solutions is demonstrated. Finally, in Section 6 the relative asymptotic orders of the displacement components and hydrostatic pressure are established for both long-wave high- and low-frequency motion. Regarding high-frequency motion, the results were very similar to those previously established for prestresses for which one principal axis is normal to the free surface; see [Kaplunov et al. 2002]. However, in respect of low-frequency motion, the results were quite different, with both the in-plane and normal displacement components having the same asymptotic orders. It is then the case that the prestressed analogs of classical bending and extension, as discussed in [Kaplunov et al. 2000], are not possible. This notwithstanding, we show that as the amount of shear tends to zero the relative asymptotic orders of displacement components and incremental pressure do all in fact concur with those of [Kaplunov et al. 2000].

2. Governing equations

In this section, we summarize the basic equations which govern infinitesimal motion superimposed upon a finite static deformation in an incompressible elastic solid. For further details of the basic equations, the reader is referred to [Ogden 1984]. Our ultimate aim is to investigate infinitesimal wave propagation in a layer of finite thickness and infinite lateral extent, composed of incompressible elastic material. The layer is subject to an initial finite simple shear deformation parallel to its incrementally traction free faces. Initially, the layer occupies the unstressed natural isotropic state B_u, upon which the primary simple shear deformation is imposed, resulting in the finitely deformed equilibrium configuration B_e. Finally, a small time-dependent motion is superimposed on B_e, with the ultimate current configuration denoted by B_t. The position vectors of a representative particle, relative to a common Cartesian coordinate system, are denoted by X_A, $x_i(X_A)$, and $\tilde{x}_i(X_A, t)$ in B_u, B_e, and B_t, respectively. These position vectors may be related through

$$\tilde{x}_i(X_A, t) = x_i(X_A) + u_i(x_j, t), \tag{2-1}$$

where $u_i(x_j, t)$ denotes the infinitesimal time-dependent superimposed motion.

In this paper, we specifically consider two-dimensional, two-component infinitesimal motions, with $u_3 = 0$ and u_1 and u_2 independent of x_3. In view of incompressibility, $\lambda_1\lambda_2\lambda_3 = 1$, where $\lambda_i > 0$, $i = 1, 2, 3$, are the principal stretches of the primary deformation. The particular simple shear primary deformation concerned may be defined by

$$x_1 = X_1 + \epsilon X_2, \qquad x_2 = X_2, \qquad x_3 = X_3, \tag{2-2}$$

where ϵ is the finite amount of shear. The equations of motion may be represented in the component form

$$B_{jilk} u_{k,jl} - p_{t,i} = \rho \ddot{u}_i, \tag{2-3}$$

with B_{ijkl} components of the fourth-order elasticity tensor, p_t a time dependent pressure increment, ρ the material density, a comma indicating differentiation with respect to the indicated spatial coordinate component and a dot differentiation with respect to time. As was shown in [Destrade and Ogden 2005], within the so-called Eulerian coordinate system, coincident with the principal axes of the primary deformation there are 6 nonzero components of the elasticity tensor entering our equations of motion. In contrast, within what might be regarded as the plate's natural coordinate system, with one axis normal to the free surfaces, there are 10. An important feature of this simple shear deformation is that no principal axis is normal to the incrementally traction-free faces of the plate, with the Eulerian and natural axes noncoincident. The *natural* (x_1, x_2) and Eulerian (x'_1, x'_2) coordinate systems are connected via the relations

$$\mathbf{x} = \mathbf{R}\mathbf{x}', \qquad \mathbf{x}' = \mathbf{R}^T\mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{x}' = \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix},$$
$$s_\theta = \sin\theta, \qquad c_\theta = \cos\theta, \tag{2-4}$$

where the origin is located on the plate's upper face and occupies $-\infty \le x_1 \le \infty$, $-h \le x_2 \le 0$. With respect to Eulerian axes, with displacement components u'_1 and u'_2, the governing equations are provided by

$$(B_{1111} - B_{1122} + p_0)u'_{1,11} - p'_{t,1} + B_{2121}u'_{1,22} + (B_{2121} - \sigma_2)u'_{2,12} = \rho\ddot{u}'_1,$$
$$(B_{2222} - B_{1122} + p_0)u'_{2,22} - p'_{t,2} + B_{1212}u'_{2,11} + (B_{2121} - \sigma_2)u'_{1,21} = \rho\ddot{u}'_2. \tag{2-5}$$

Within these equations, a comma indicates differentiation with respect to a spatial component of Eulerian coordinates, and p_0 is a static pressure brought into play by incompressibility. Before proceeding, we note the following definitions, symmetries, and a connection between σ_2 and p_0:

$$p_0 = B_{2121} - B_{2112} - \sigma_2, \qquad B_{iijj} = B_{jjii}, \qquad B_{ijji} = B_{jiij},$$
$$\alpha = B_{1212}, \qquad \gamma = B_{2121}, \qquad 2\beta = B_{1111} + B_{2222} - 2B_{1122} - 2B_{1221}. \tag{2-6}$$

We adopt a comma convention whereby the indicated spatial derivatives of $\mathbf{u}'$ are always with respect to $\mathbf{x}'$, and $\mathbf{u}$ always with respect to $\mathbf{x}$. Within this spirit, although the pressure is a scalar, we utilize the notation p_t' and p_t to distinguish between which coordinate system the implied differentiation refers to. In view of the plane strain nature of the problem, we assume also that $\lambda_3 \equiv 1$, and note for future use that

$$\alpha\lambda_2^2 = \gamma\lambda_1^2, \qquad \lambda_1 = \lambda, \qquad \lambda_2 = 1/\lambda, \qquad \alpha = \gamma\lambda^4. \tag{2-7}$$

Recalling that the spatial derivatives in Equation (2-5) are with respect to Eulerian coordinates, we will perform appropriate changes to express these equations within the plate's natural coordinate system and introduce

$$\mathcal{K}_1 = B_{2112} + B_{1122} - B_{1111}, \qquad \mathcal{K}_2 = B_{2112} + B_{1122} - B_{2222}, \qquad \mathcal{K}_1 + \mathcal{K}_2 = -2\beta, \tag{2-8}$$

noting also that the properties of the simple shear deformation dictate that

$$\cot\theta = \lambda, \qquad s_\theta = \frac{1}{\sqrt{\lambda^2+1}}, \qquad c_\theta = \frac{\lambda}{\sqrt{\lambda^2+1}}, \qquad 2\lambda = \epsilon + \sqrt{\epsilon^2+4}. \tag{2-9}$$

The equations of motion, within the natural system, are therefore representable in the form

$$\begin{aligned}
&-\lambda(1+\lambda^2)p_{t,1} - (1+\lambda^2)p_{t,2} - \rho\lambda(1+\lambda^2)\ddot{u}_1 - \rho(1+\lambda^2)\ddot{u}_2\\
&\quad + \lambda\big((\gamma - \mathcal{K}_1 - \sigma_2)\lambda^2 + 2\gamma - \sigma_2\big)u_{1,11} - \lambda\big(\mathcal{K}_1 - \gamma\lambda^2\big)u_{1,22}\\
&\quad - \big((2\mathcal{K}_1 + \sigma_2 + \gamma)\lambda^2 + \sigma_2 - \gamma\big)u_{1,12} - \big(\mathcal{K}_1\lambda^2 - \gamma\big)u_{2,11}\\
&\quad - \big((\sigma_2 - 2\gamma)\lambda^2 - \gamma + \mathcal{K}_1 + \sigma_2\big)u_{2,22} - \lambda\big((\sigma_2 - \gamma)\lambda^2 + 2\mathcal{K}_1 + \sigma_2 + \gamma\big)u_{2,12} = 0,
\end{aligned} \tag{2-10}$$

$$\begin{aligned}
&(1+\lambda^2)p_{t,1} - \lambda(1+\lambda^2)p_{t,2} + \rho(1+\lambda^2)\ddot{u}_1 - \rho\lambda(1+\lambda^2)\ddot{u}_2\\
&\quad - \big((\alpha - \sigma_2 + \gamma)\lambda^2 - \sigma_2 - \mathcal{K}_2 + \gamma\big)u_{1,11} - \big(\alpha - \mathcal{K}_2\lambda^2\big)u_{1,22}\\
&\quad - \lambda\big((\sigma_2 - \gamma)\lambda^2 + 2\mathcal{K}_2 + 2\alpha + \sigma_2 - \gamma\big)u_{1,12} + \lambda\big(\alpha\lambda^2 - \mathcal{K}_2\big)u_{2,11}\\
&\quad - \lambda\big((\sigma_2 + \mathcal{K}_2 - \gamma)\lambda^2 - \alpha + \sigma_2 - \gamma\big)u_{2,22} + \big((2\mathcal{K}_2 + 2\alpha + \sigma_2 - \gamma)\lambda^2 - \gamma + \sigma_2\big)u_{2,12} = 0.
\end{aligned} \tag{2-11}$$

These equations of motion are supplemented by the incremental incompressibility condition

$$u_{1,1} + u_{2,2} = 0. \tag{2-12}$$

Solutions of the three governing equations are sought in the form

$$(u_1, u_2, p_t) = (U_1, U_2, kP)e^{ikqx_2}e^{ik(vt - x_1)}, \tag{2-13}$$

where k is the wave number, v the phase speed and q is a parameter which will be determined from the governing equations. From the incompressibility condition, we first establish that $U_1 = qU_2$, which when used within the two equations of motion provides a system of two homogeneous equations in two unknowns, representable in the form

$$c_{11}U_2 + c_{12}P = 0, \qquad c_{21}U_2 + c_{22}P = 0 \tag{2-14}$$

with

$$c_{11} = k^2(C_1^{(3)}q^3 + C_1^{(2)}q^2 + C_1^{(1)}q + C_1^{(0)}), \quad c_{12} = k^3 v\,(-c_\theta + q s_\theta)\,,$$
$$c_{21} = k^2(C_2^{(3)}q^3 + C_2^{(2)}q^2 + C_2^{(1)}q + C_2^{(0)}), \quad c_{22} = k^3 v\,(q c_\theta + s_\theta)\,,$$

where

$$C_1^{(3)} = -c_\theta\big(\gamma\, {c_\theta}^2 - \mathcal{K}_1 {s_\theta}^2\big), \qquad C_2^{(3)} = s_\theta\big(-K_2 {c_\theta}^2 + \alpha\, {s_\theta}^2\big),$$
$$C_1^{(2)} = -s_\theta\big(-\mathcal{K}_1 {s_\theta}^2 + (3\,\gamma + 2\mathcal{K}_1)\, {c_\theta}^2 + 2\, {c_\theta}^2\mathcal{K}_1\big), C_2^{(2)} = -c_\theta\big(-\mathcal{K}_2 {c_\theta}^2 + (3\,\alpha + 2\mathcal{K}_2)\, {s_\theta}^2\big),$$
$$C_1^{(1)} = c_\theta\big({c_\theta}^2\mathcal{K}_1 - (3\,\gamma + 2\mathcal{K}_1)\, {s_\theta}^2 + \rho v^2\big), \qquad C_2^{(1)} = s_\theta\big(-{s_\theta}^2\mathcal{K}_2 + (3\,\alpha + 2\,\mathcal{K}_2)\, {c_\theta}^2 - \rho v^2\big),$$
$$C_1^{(0)} = s_\theta\big(-\gamma\, {s_\theta}^2 + {c_\theta}^2\mathcal{K}_1 + \rho v^2\big), \qquad C_2^{(0)} = -c_\theta\big(\alpha\, {c_\theta}^2 - {s_\theta}^2\mathcal{K}_2 - \rho v^2\big).$$

This system possesses a nontrivial solution provided the determinant of its coefficients is equal to zero; this leads, after use of (2-7), to the quartic equation

$$q^4 - 2\,\epsilon\, q^3 + \big(4\,\delta + 2 + \epsilon^2 - (1+\delta)\,\hat{v}\big)\, q^2 - 2\,\,(1 + 2\,\delta)\,\epsilon\, q + 1 + (1+\delta)\,\epsilon^2 - (1+\delta)\,\hat{v} = 0, \tag{2-15}$$

in q, where

$$\hat{v} = \frac{\rho v^2}{\sqrt{\alpha\gamma}}, \qquad \delta = \frac{\alpha + \gamma - 2\beta}{2(\beta + \sqrt{\alpha\gamma})} \tag{2-16}$$

are the dimensionless squared wave speed and a material parameter.

We note in passing that strong ellipticity dictates that $\alpha > 0$, $\gamma > 0$, $\beta > -\sqrt{\alpha\gamma}$, thus $\delta + 1 > 0$. The characteristic equation, (2-15), is the same as that previously derived by Connor and Ogden [Connor and Ogden 1996], using a different approach involving employment of a stream function. The motivation for our differing approach is to retain the hydrostatic pressure explicitly within the governing equations. This will later allow comparison of its relative order with the two displacement components, within the long-wave regimes. If we now multiply $(2\text{-}14)_1$ by c_θ and $(2\text{-}14)_2$ by s_θ, then subtract them and make use of (2-9), we obtain

$$P = \mathcal{P}(q)U_2, \qquad \mathcal{P}(q) = \frac{\mathcal{P}^{(3)}q^3 + \mathcal{P}^{(2)}q^2 + \mathcal{P}^{(1)}q + \mathcal{P}^{(0)}}{(\lambda^2+1)^2 k v},$$

with

$$\begin{aligned}
\mathcal{P}^{(3)} &= -\lambda^4\gamma - 2\,\beta\,\lambda^2 - \alpha,\\
\mathcal{P}^{(2)} &= -(2\mathcal{K}_1 + \mathcal{K}_2 + 3\gamma)\lambda^3 + (\mathcal{K}_1 + 3\,\alpha + 2\mathcal{K}_2)\lambda,\\
\mathcal{P}^{(1)} &= (\rho v^2 + \mathcal{K}_1)\lambda^4 - (3\gamma - 4\beta - 2\rho v^2 + 3\alpha)\lambda^2 + \mathcal{K}_2 + \rho v^2,\\
\mathcal{P}^{(0)} &= (\alpha + \mathcal{K}_1)\lambda^3 - (\mathcal{K}_2 + \gamma)\lambda.
\end{aligned} \tag{2-17}$$

The solutions for u_1, u_2, and p_t may now be expressed as the linear combinations

$$\begin{aligned} u_1 &= (q_1 A_1 e^{ikq_1x_2} + q_2 A_2 e^{ikq_2x_2} + q_3 B_1 e^{ikq_3x_2} + q_4 B_2 e^{ikq_4x_2}) e^{ik(vt-x_1)}, \\ u_2 &= (A_1 e^{ikq_1x_2} + A_2 e^{ikq_2x_2} + B_1 e^{ikq_3x_2} + B_2 e^{ikq_4x_2}) e^{ik(vt-x_1)}, \\ p_t &= k(\mathcal{P}(q_1) A_1 e^{ikq_1x_2} + \mathcal{P}(q_2) A_2 e^{ikq_2x_2}) e^{ik(vt-x_1)} + k(\mathcal{P}(q_3) B_1 e^{ikq_3x_2} + \mathcal{P}(q_4) B_2 e^{ikq_4x_2}) e^{ik(vt-x_1)}, \end{aligned} \tag{2-18}$$

where A_1, A_2, B_1, B_2 are constants. The incremental traction components across any plane with outward unit normal along $\mathbf{n}$ in B_e are given by

$$\tau_i = B_{milk} u_{k,l} n_m + p_0 u_{m,i} n_m - p_t n_i. \tag{2-19}$$

Relative to the Eulerian system, the two nonzero traction components associated with the upper and lower faces of the plate are given by

$$\begin{aligned} \tau_1' &= (B_{1111} u_{1,1}' + B_{1122} u_{2,2}') s_\theta + (B_{2112} u_{2,1}' + B_{2121} u_{1,2}') c_\theta + p_0 (u_{1,1}' s_\theta + u_{2,1}' c_\theta) - p_t s_\theta, \\ \tau_2' &= (B_{1212} u_{2,1}' + B_{1221} u_{1,2}') s_\theta + (B_{2211} u_{1,1}' + B_{2222} u_{2,2}') c_\theta + p_0 (u_{1,2}' s_\theta + u_{2,2}' c_\theta) - p_t c_\theta. \end{aligned} \tag{2-20}$$

The spatial derivatives in (2-20) are all referred to the Eulerian system. We now change and recalculate all the derivatives of displacement components and pressure relative to the natural system, obtaining

$$\begin{aligned} \tau_1 (\lambda^2+1)^{5/2} = &-\left(\lambda^2+1\right)^2 p_t \\ &+ \left((-B_{1221} - \gamma + B_{1111})\lambda^4 + (-\mathcal{K}_1 - \gamma + 2B_{1122})\lambda^2 + B_{1122}\right) u_{1,1} \\ &+ \left((2\gamma - \sigma_2 + B_{1122})\lambda^4 + (-\mathcal{K}_1 - 2\sigma_2 + 3\gamma + 2B_{1122})\lambda^2\right) u_{2,2} \\ &+ (-\sigma_2 - \mathcal{K}_1 + \gamma + B_{1122}) u_{2,2} + \left(\lambda^5\gamma - \mathcal{K}_1\lambda + (-K_1 + \gamma)\lambda^3\right) u_{1,2} \\ &\qquad + \left((\gamma - \sigma_2)\lambda^5 + (-\mathcal{K}_1 - \sigma_2)\lambda + (\gamma - 2\sigma_2 - \mathcal{K}_1)\lambda^3\right) u_{2,1}, \end{aligned} \tag{2-21}$$

$$\begin{aligned} \tau_2 (\lambda^2+1)^{5/2} = &-\lambda\left(\lambda^2+1\right)^2 p_t \\ &+ \left(B_{1122}\lambda^5 + (-\mathcal{K}_2 - \alpha + 2B_{1122})\lambda^3 + (-\alpha + B_{1122} - \mathcal{K}_2)\lambda\right) u_{1,1} \\ &+ \left((\gamma + B_{1122} - \sigma_2 - \mathcal{K}_2)\lambda^5 + (-\sigma_2 + \alpha + \gamma + B_{1122})\lambda\right) u_{2,2} \\ &+ (\alpha + 2\gamma - \mathcal{K}_2 - 2\sigma_2 + 2B_{1122})\lambda^3 u_{2,2} + \left(\mathcal{K}_2\lambda^4 - \alpha(\mathcal{K}_2 - \alpha)\lambda^2\right) u_{1,2} \\ &\quad + \left((\mathcal{K}_2 - 2\gamma + \alpha + 2\sigma_2)\lambda^2 - \gamma + (-\gamma + \alpha + \mathcal{K}_2 + \sigma_2)\lambda^4 + \sigma_2\right) u_{2,1}. \end{aligned} \tag{2-22}$$

After using the form of solutions (2-13) for displacements and pressure components, these two traction components may be expressed as

$$\begin{aligned} \tau_1 &= \frac{1}{(\lambda^2+1)^{5/2}} \mathcal{T}_1(q), \qquad \mathcal{T}_1(q) = \mathcal{T}_1^{(3)} q^3 + \mathcal{T}_1^{(2)} q^2 + \mathcal{T}_1^{(1)} q + \mathcal{T}_1^{(0)}, \\ \tau_2 &= \frac{1}{(\lambda^2+1)^{5/2}} \mathcal{T}_2(q), \qquad \mathcal{T}_2(q) = \mathcal{T}_2^{(3)} q^3 + \mathcal{T}_2^{(2)} q^2 + \mathcal{T}_2^{(1)} q + \mathcal{T}_2^{(0)}, \end{aligned} \tag{2-23}$$

where

$$\begin{aligned}
\mathcal{T}_1^{(3)} &= \gamma\lambda^4 + 2\,\beta\,\lambda^2 + \alpha,\\
\mathcal{T}_1^{(2)} &= \lambda^5\gamma - (2\beta - 4\gamma)\lambda^3 + (4\beta - 3\alpha)\lambda,\\
\mathcal{T}_1^{(1)} &= (3\gamma - \sigma_2 - \rho v^2)\lambda^4 + (3\alpha - 2\rho v^2 - 2\sigma_2 + 7\gamma - 4\beta)\lambda^2 + 2\beta - \rho v^2 + \gamma - \sigma_2,\\
\mathcal{T}_1^{(0)} &= (\sigma_2 - \gamma)\lambda^5 + (2\sigma_2 - \alpha - \gamma)\lambda^3 - (2\beta - \gamma - \sigma_2)\lambda
\end{aligned} \tag{2-24}$$

and

$$\begin{aligned}
\mathcal{T}_2^{(3)} &= \gamma\lambda^5 + 2\,\beta\,\lambda^3 + \lambda\,\alpha,\\
\mathcal{T}_2^{(2)} &= (3\gamma - 4\beta)\lambda^4 + (2\beta - 4\alpha)\lambda^2 - \alpha,\\
\mathcal{T}_2^{(1)} &= (2\beta - \sigma_2 + \gamma - \rho v^2)\lambda^5 - (4\beta + 2\sigma_2 - 5\alpha + 2\rho v^2 - 5\gamma)\lambda^3 + (2\alpha - \rho v^2 + \gamma - \sigma_2)\lambda,\\
\mathcal{T}_2^{(0)} &= (2\beta - 2\alpha + \gamma - \sigma_2)\lambda^4 + (3\gamma - \alpha - 2\sigma_2)\lambda^2 - \sigma_2 + \gamma.
\end{aligned} \tag{2-25}$$

The complete general solutions for τ_1 and τ_2, as a sum of partial wave solutions, are provided by

$$\begin{aligned}
\tau_1 &= \mathscr{C}(\mathcal{T}_1(q_1)A_1 e^{ikq_1x_2} + \mathcal{T}_1(q_2)A_2 e^{ikq_2x_2})e^{ik(vt-x_1)} + \mathscr{C}(\mathcal{T}_1(q_3)B_1 e^{ikq_3x_2} + \mathcal{T}_1(q_4)B_2 e^{ikq_4x_2})e^{ik(vt-x_1)},\\
\tau_2 &= \mathscr{C}(\mathcal{T}_2(q_1)A_1 e^{ikq_1x_2} + \mathcal{T}_2(q_2)A_2 e^{ikq_2x_2})e^{ik(vt-x_1)} + \mathscr{C}(\mathcal{T}_2(q_3)B_1 e^{ikq_3x_2} + \mathcal{T}_2(q_4)B_2 e^{ikq_4x_2})e^{ik(vt-x_1)}.
\end{aligned} \tag{2-26}$$

As a numerical illustration, we will employ in turn the neo-Hookean and Varga strain-energy functions

$$W = \tfrac{1}{2}\mu({\lambda_1}^2 + {\lambda_2}^2 + {\lambda_3}^2 - 3), \qquad W = \mu(\lambda_1 + \lambda_2 + \lambda_3 - 3), \tag{2-27}$$

where μ is the infinitesimal shear modulus. For the future reference, we note that the parameter δ from (2-16) is expressible in terms of ϵ:

$$\begin{aligned}
&\text{(neo-Hookean)} \qquad \beta = \alpha + \gamma, \qquad \delta = 0;\\
&\text{(Varga)} \qquad \beta = \sqrt{\alpha\gamma}, \qquad \delta = \frac{\left[\left(\frac{\alpha}{\gamma}\right)^{1/4} - \left(\frac{\gamma}{\alpha}\right)^{-1/4}\right]^2}{4} = \frac{\left(\lambda - \frac{1}{\lambda}\right)^2}{4} = \frac{\epsilon^2}{4}.
\end{aligned} \tag{2-28}$$

In passing, we remark that within the plane strain assumption neo-Hookean and Mooney–Rivlin materials coincide.

3. Analysis of the dispersion relation

The quartic characteristic equation (2-15) has four roots which affect the character of the dispersion relation. In particular, we note that solutions of (2-15) may in general form two complex conjugate pairs, be all real, or there may be two real together with a complex conjugates pair. In order to derive a dispersion relation we specify the boundary conditions:

$$\tau_1 = \tau_2 = 0 \qquad \text{on} \qquad x_2 = 0, -h. \tag{3-1}$$

Before proceeding to the derivation of the dispersion relation, we note that the two traction components (2-26) may be represented as

$$\begin{aligned}\tau_1 &= s_\theta\sqrt{\alpha\gamma}(f(q_1)A_1e^{ikq_1x_2}+f(q_2)A_2e^{ikq_2x_2})e^{ik(vt-x_1)}\\&\qquad\qquad+s_\theta\sqrt{\alpha\gamma}(f(q_3)B_1e^{ikq_3x_2}+f(q_4)B_2e^{ikq_4x_2})e^{ik(vt-x_1)},\\ \tau_2 &= c_\theta\sqrt{\alpha\gamma}(g(q_1)A_1e^{ikq_1x_2}+g(q_2)A_2e^{ikq_2x_2})e^{ik(vt-x_1)}\\&\qquad\qquad+c_\theta\sqrt{\alpha\gamma}(g(q_3)B_1e^{ikq_3x_2}+g(q_4)B_2e^{ikq_4x_2})e^{ik(vt-x_1)};\end{aligned}\tag{3-2}$$

this representation concurs with that of [Connor and Ogden 1996], with $f(q)$ and $g(q)$ given by

$$\begin{aligned}f &= \frac{(q-\lambda)\left(q+\lambda^{-1}\right)^2}{1+\delta}+3\,q-\lambda+2\,\lambda^{-1}+p\,(q-\lambda)-\hat{v}\,q,\quad \text{with } p=\frac{\gamma-\sigma_2}{\sqrt{\alpha\gamma}};\\ g &= \frac{(q-\lambda)^2\left(q+\lambda^{-1}\right)}{1+\delta}+3\,q-2\,\lambda+\lambda^{-1}+p\left(q+\lambda^{-1}\right)-\hat{v}\,q.\end{aligned}\tag{3-3}$$

We now use the solutions for incremental tractions, together with the boundary conditions, to derive a system of four linear homogeneous equations in A_1, A_2, B_1, and B_2. The dispersion relation results from the condition that this system has a nontrivial solution, yielding a 4×4 determinant. After a little algebraic manipulation, this condition is expressible in the form

$$\det\begin{vmatrix} f(q_1)C_1 & f(q_2)C_2 & f(q_3)C_3 & f(q_4)C_4\\ g(q_1)C_1 & g(q_2)C_2 & g(q_3)C_3 & g(q_4)C_4\\ f(q_1)S_1 & f(q_2)S_2 & f(q_3)S_3 & f(q_4)S_4\\ g(q_1)S_1 & g(q_2)S_2 & g(q_3)S_3 & g(q_4)S_4\end{vmatrix}=0,\qquad S_j=\sin\frac{q_jkh}{2},\quad C_j=\cos\frac{q_jkh}{2}.\tag{3-4}$$

We note that the dispersion relation may be shown to always provide a real equation. In the case of a neo-Hookean material [Connor and Ogden 1996] the characteristic equation (2-15) has the roots

$$q_1=i,\qquad q_2=-i,\qquad q_3=\epsilon+i\kappa,\qquad q_4=\epsilon-i\kappa,\qquad \kappa^2=1-\hat{v}.\tag{3-5}$$

When κ is real ($\hat{v}<1$) there are four complex roots exist, and when κ is imaginary ($\hat{v}>1$) two are complex and two real; the case of four real roots is not possible. If we introduce the small dimensionless parameter η, defined as the ratio of a layer thickness h to typical wave length l, so $\eta\equiv h/l\equiv kh$, the dispersion relation for a neo-Hookean material reduces to

$$\begin{aligned}&\big(q_0(p^2-\kappa^2)^2+q_0\kappa^2\,(q_0+2\,p)^2+4\,\kappa^2(p^2-\kappa^2)\,(q_0+2\,p)\big)\sinh(\eta)\sinh(\eta\,\kappa)\\&\qquad\qquad+2\,\kappa\,(pq_0+p^2+\kappa^2)^2\big(\cos(\eta\,\epsilon)-\cosh(\eta)\cosh(\eta\,\kappa)\big)=0,\end{aligned}\tag{3-6}$$

where $q_0=1+\epsilon^2+\kappa^2$.

To illustrate the behavior of the solution for the dispersion relation (3-4), some numerical results are presented for a plate composed of Varga material. Figure 1 shows dispersion curves of dimensionless frequency (vertical axis) against scaled wave number (horizontal axis) for several parameter sets, and Figure 2 has the corresponding plots of dimensionless squared wave speed (vertical scale) against scaled wave number (horizontal scale). From both figures it is possible to note that as ϵ increases a wave-front is observed, most strikingly in the lower right pane. Also, Figure 2 illustrates the fact that depending

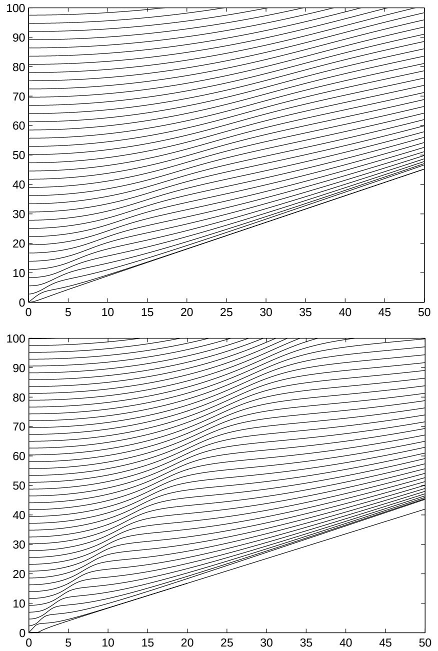

Figure 1. Numerical solution of dispersion relation, showing dimensionless frequency ω (vertical axis) against scaled wave number η (horizontal axis) for a Varga material. Clockwise from top left: $\epsilon = 1, 2, 3$, all with $p = 1$; lower left, $\epsilon = 2,\ p = 1.5$.

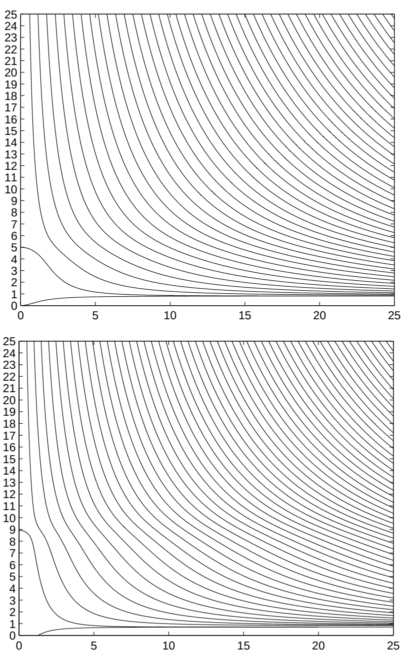

Figure 2. Numerical solution of dispersion relation, showing dimensionless squared wave speed $\hat{v}$ (vertical axis) against scaled wave number η (horizontal axis) for a Varga material. Clockwise from top left: $\epsilon = 1, 2, 3$, all with $p = 1$; lower left, $\epsilon = 2,\ p = 1.5$.

on the numerical values of ϵ and p there may be one (lower left pane) or two fundamental mode long-wave limits (remaining panes). Such phenomena will be investigated analytically through long-wave asymptotic expansion.

4. Long-wave low-frequency motion

4A. ***Leading order.*** Approximations for the phase speed will now be derived in the long-wave region. The term long-wave is introduced to indicate that the wavelength considerably exceeds the plate thickness and therefore $\eta \ll 1$, with $\eta \to 0$ indicating the long-wave limit. Our first consideration is so-called low-frequency motion, for which $\hat{v}$ generally remains finite as $\eta \to 0$. We therefore assume that the roots of the characteristic equation are not large and within the long-wave region

$$C_j = 1 + O(\eta^2), \qquad S_j = \frac{q_j \eta}{2} + O(\eta^3), \qquad j = 1, 2, 3, 4. \tag{4-1}$$

The appropriate leading-order term of the dispersion relation is then provided by

$$D = \det \begin{vmatrix} f(q_1) & f(q_2) & f(q_3) & f(q_4) \\ g(q_1) & g(q_2) & g(q_3) & g(q_4) \\ q_1 f(q_1) & q_2 f(q_2) & q_3 f(q_3) & q_4 f(q_4) \\ q_1 g(q_1) & q_2 g(q_2) & q_3 g(q_3) & q_4 g(q_4) \end{vmatrix} = 0. \tag{4-2}$$

Calculating the 4×4 determinant results in the following quadratic in $\hat{v}$:

$$\mathfrak{D}^{(0)} \equiv \hat{v}^2 + (p^2(\delta+1) - 2p(\delta+1) + \delta - 3 - \epsilon^2)\hat{v} - (p^2-1)(2p(1+\delta) - 2(\delta-1) + 2 + \epsilon^2) = 0, \tag{4-3}$$

with the two roots taking the explicit forms

$$\begin{aligned} \hat{v}_1^{(0)} &= \frac{-(p-1)^2(\delta+1) + \epsilon^2 + 4 + \sqrt{R}}{2}, \\ \hat{v}_2^{(0)} &= \frac{-(p-1)^2(\delta+1) + \epsilon^2 + 4 - \sqrt{R}}{2}, \\ R &= \left((p-1)^2\,\delta + (p+1)^2 - \epsilon^2\right)^2 + 4\,\epsilon^2\,(p+1)^2. \end{aligned} \tag{4-4}$$

In passing, we note that the discriminant of the quadratic in $\hat{v}$ is a perfect square in each of the cases in which δ or ϵ vanish or $p = -1$.

4B. ***Second order.*** To obtain second-order correction terms of dispersion relation, the trigonometrical functions in (3-4) are expanded to the next order, resulting in

$$C_j = 1 - \frac{q_j^2 \eta^2}{8} + O(\eta^4), \qquad S_j = \frac{q_j \eta}{2} - \frac{q_j^3 \eta^3}{48} + O(\eta^4), \qquad j = 1, 2, 3, 4. \tag{4-5}$$

The dispersion relation is then expanded and may be represented in the form

$$\mathfrak{D} = \mathfrak{D}^{(0)} + \eta^2 \mathfrak{D}^{(2)} + O(\eta^4). \tag{4-6}$$

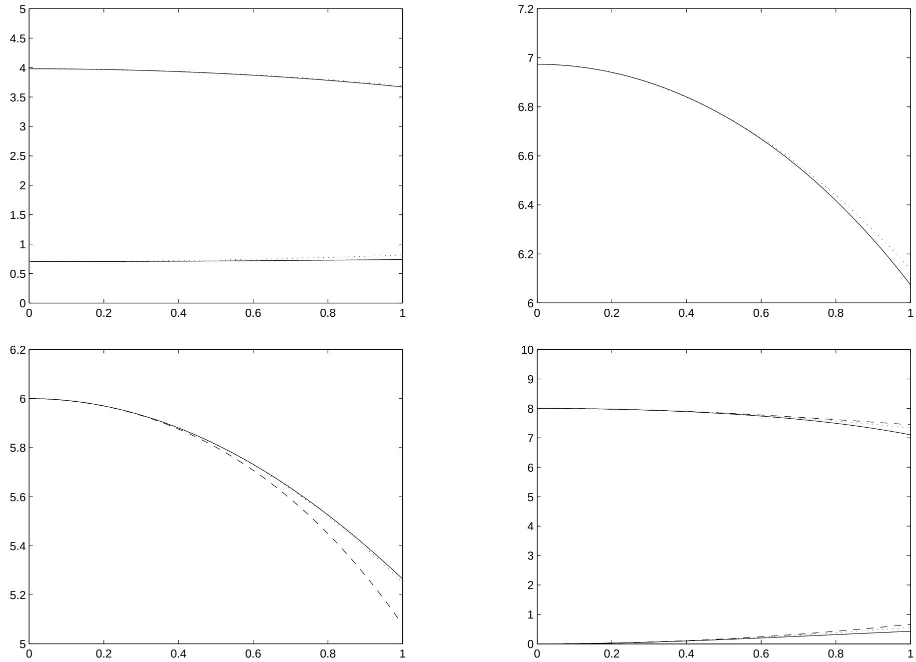

Figure 3. Comparison of numerical (solid lines) and asymptotic results for the fundamental modes, showing $\hat{v}$ (vertical axis) against η (horizontal axis). Top row: Varga material with $\epsilon = 1$, $p = 0.5$ (left, two fundamental modes), and with $\epsilon = 1$, $p = 2$ (right, one fundamental mode); the dotted curve represents the second-order approximation. Bottom row: neo-Hookean material with $\epsilon = 0$, $p = 2$ (left, one fundamental mode), and with $\epsilon = 2$, $p = 1$ (right, two fundamental modes); the dotted curve is the third-order approximation and the dashed curve is the second-order approximation.

We note that both $\mathfrak{D}^{(0)}$ and $\mathfrak{D}^{(2)}$ are quadratic in $\hat{v}$. The form of $\mathfrak{D}^{(2)}$ is sufficiently complicated that it is not of value to write it explicitly here. Notwithstanding this, a corrected expansion may be readily established and written down in the form

$$\hat{v}_i = \hat{v}_i^{(0)} - \eta^2 \frac{\mathfrak{D}^{(2)}(\hat{v}_i^{(0)})}{\mathfrak{D}'^{(0)}(\hat{v}_i^{(0)})} + O(\eta^4). \tag{4-7}$$

Although the explicit representation of $\mathfrak{D}^{(2)}$ is complex, nevertheless approximations afforded by (4-7) can be readily generated and compared with the purely numerical solution. Figure 3 confronts numerical results with second- and third-order approximations for the fundamental modes. The second-order approximation curves in the case of a Varga material (top panes) already show very good agreement over a wide wave number regime; third-order approximation curves are omitted in this case.

4C. ***Long-wave low-frequency approximations in the case $\delta = 0$.*** In this section we consider a class of materials for which $\delta = 0$, implying that $2\beta = \alpha + \gamma$. As first noted in [Connor and Ogden 1996], in such cases the characteristic equation (2-15) factorizes and the dispersion relation takes the particularly simple form given in (3-6). Using (3-6), we are able to derive long-wave low-frequency approximations up to and including $O(\eta^4)$. We begin by noting the following expansions:

$$\begin{aligned}
&\sinh(\eta) = \eta + \tfrac{1}{6}\eta^3 + O(\eta^5), \qquad \cosh(\eta) = 1 + \tfrac{1}{2}\eta^2 + \tfrac{1}{24}\eta^4 + O(\eta^5),\\
&\kappa = \sqrt{1-\hat{v}_0} - \frac{1}{2}\frac{\hat{v}_1\,\eta^2}{\sqrt{1-\hat{v}_0}} - \frac{4\hat{v}_2(1-\hat{v}_0)+\hat{v}_1^2}{8(1-\hat{v}_0)^{3/2}}\eta^4 + O(\eta^5),\\
&\cos(\eta\epsilon) = 1 - \tfrac{1}{2}\eta^2\epsilon^2 + \tfrac{1}{24}\eta^4\epsilon^4 + O(\eta^5),\\
&\sinh(\eta\kappa) = \sqrt{1-\hat{v}_0}\,\eta + \frac{(\hat{v}_0-1)^2-3\hat{v}_1}{6\sqrt{1-\hat{v}_0}}\eta^3 + \frac{(\hat{v}_0-1)^2}{8}\eta^4 + O(\eta^5),\\
&\cosh(\eta\kappa) = 1 + \tfrac{1}{2}\left(1-\hat{v}_0\right)\eta^2 - \tfrac{1}{12}\left(6\hat{v}_1 + (\hat{v}_0-1)^2\right)\eta^4 + O(\eta^5).
\end{aligned} \tag{4-8}$$

Employing expansions (4-8) in (3-6), we obtain a representation of the dispersion relation in a form

$$\mathfrak{D} = \mathfrak{D}^{(0)} + \eta^2\mathfrak{D}^{(2)} + \eta^4\mathfrak{D}^{(4)} + O(\eta^6), \tag{4-9}$$

providing, at leading order,

$$\mathfrak{D}^{(0)} = \sqrt{1-\hat{v}_0}(1-p^2-\hat{v}_0)(\hat{v}_0^2 - 2\epsilon^2\hat{v}_0 + \epsilon^4 + 4\epsilon^2)(2+\epsilon^2-\hat{v}_0+2p) = 0. \tag{4-10}$$

In (4-10) the root $\hat{v}_0 = 1$ is a spurious root associated with a double root of the characteristic equation. Additionally, the third factor cannot provide any real solutions and we therefore conclude that the two long-wave limits are provided by

$$\hat{v}_1^{(0)} = 1 - p^2, \qquad \hat{v}_2^{(0)} = \epsilon^2 + 2p + 2. \tag{4-11}$$

It is now possible to show that the second-order term $\mathfrak{D}^{(2)}$ provides a linear function in $\hat{v}$, from which we deduce the two second-order corrections:

$$\hat{v}_1^{(2)} = \tfrac{1}{12}(\epsilon^2 + (p+1)^2)p^2, \qquad \hat{v}_2^{(2)} = -\tfrac{1}{12}(\epsilon^2 + (p+1)^2). \tag{4-12}$$

Finally, the third dispersion term $\mathfrak{D}^{(4)}$ is again a linear term in $\hat{v}^{(2)}$ and provides the following third-order corrections:

$$\hat{v}_1^{(4)} = \frac{p^2\mathfrak{X}_1}{576(\epsilon^2+1+3p^2-4p)}, \qquad \hat{v}_2^{(4)} = \frac{\mathfrak{X}_2}{576(2p+\epsilon^2+1)(7p+3\epsilon^2+3)}, \tag{4-13}$$

where

$$\begin{aligned}
\mathfrak{X}_1 &= \epsilon^6 + (3p^2-12p-13)\epsilon^4 - (13p^4-24p^3-10p^2+24p+29)\epsilon^2\\
&\qquad - 15p^6 + 4p^5 + 55p^4 + 88p^3 + 23p^2 - 12p - 15,\\
\mathfrak{X}_2 &= -24(p^2+p+1)\epsilon^6 - (136p^3+168p^2-73p-72)\epsilon^4\\
&\qquad -(256p^4+366p^3-28p^2-218p-72)\epsilon^2 - 159p^5 - 252p^4 - 34p^3 + 172p^2 + 121p + 24.
\end{aligned}$$

For the class of materials considered, expansions for the two long-wave low-frequency dimensionless squared wave speeds may be expressed as

$$\hat{v}_m = \hat{v}_m^{(0)} + \eta^2\hat{v}_m^{(2)} + \eta^4\hat{v}_m^{(4)} + O(\eta^6), \tag{4-14}$$

within which $\hat{v}_m^{(n)}$, $m = 1, 2$, $n = 0, 2, 4$, are as given in equations (4-11), (4-12) and (4-13). For certain sets of parameters, these approximations might not be valid because third-order corrections may become large within the vicinity of certain combinations of ϵ and p. These combinations of parameters may be identified as

$$\begin{aligned} \epsilon^2 + 1 + 3\,p^2 - 4\,p &= 0, \\ 2p + \epsilon^2 + 1 &= 0, \\ 7p + 3\epsilon^2 + 3 &= 0, \end{aligned} \tag{4-15}$$

noting that both leading-order approximations should stay positive. Using these conditions, we obtain some regions close to which the approximations are not valid, given by

$$\begin{array}{llll} \text{(a)} & 0 \le \epsilon \le \dfrac{1}{\sqrt{3}} & \text{with} & p = \dfrac{2 \pm \sqrt{1 - 3\epsilon^2}}{3}, \\ \text{(b)} & 0 \le \epsilon \le 1 & \text{with} & p = \dfrac{(\epsilon^2 + 1)}{2}, \\ \text{(c)} & 0 \le \epsilon \le \dfrac{2}{\sqrt{3}} & \text{with} & p = \dfrac{3(\epsilon^2 + 1)}{7}. \end{array} \tag{4-16}$$

The bottom panes in Figure 3 show good agreement between these expansions and numerical solutions for the fundamental modes in the case of a neo-Hookean strain-energy function.

5. Long-wave high-frequency motion

The long-wave high-frequency case can be characterized by the fact that the scaled phase speed $\hat{v} \to \infty$ as $\eta \to 0$. We will also assume that the associated cut-off frequencies are finite. We begin by examining the four roots of the characteristic equation (2-15) when $\hat{v} \gg 1$ and consider expansions of its roots in the form

$$q = s_1 \hat{v} + s_0 + \frac{s_2}{\hat{v}} + \frac{s_3}{\hat{v}^2} + \frac{s_4}{\hat{v}^3}, \qquad \hat{v} = \frac{\Omega}{\eta}. \tag{5-1}$$

After some algebraic manipulation, the following approximations for the roots of (2-15) are obtained:

$$\begin{aligned} q_1 &= i - \frac{i\delta\,(\epsilon - 2\,i)^2\,\eta^2}{2\,(1+\delta)\,\Omega^2} + O(\eta^4), \qquad q_2 = -i + \frac{i\delta\,(\epsilon + 2\,i)^2\,\eta^2}{2\,(1+\delta)\,\Omega^2} + O(\eta^4), \\ q_3 &= \frac{\sqrt{1+\delta}\Omega}{\eta} + \epsilon - \frac{1}{2}\,\frac{(4\,\delta+1)\,\eta}{\sqrt{1+\delta}\Omega} + \frac{2\epsilon\,\delta\,\eta^2}{(1+\delta)\,\Omega^2} - \frac{1}{8}\,\frac{\left(16\,\delta^2 + 20\,\epsilon^2\delta - 8\,\delta + 1\right)\eta^3}{(1+\delta)^{3/2}\,\Omega^3} + O(\eta^4), \\ q_4 &= -\frac{\sqrt{1+\delta}\Omega}{\eta} + \epsilon + \frac{1}{2}\,\frac{(4\,\delta+1)\,\eta}{\sqrt{1+\delta}\Omega} + \frac{2\epsilon\,\delta\,\eta^2}{(1+\delta)\,\Omega^2} + \frac{1}{8}\,\frac{\left(16\,\delta^2 + 20\,\epsilon^2\delta - 8\,\delta + 1\right)\eta^3}{(1+\delta)^{3/2}\,\Omega^3} + O(\eta^4), \end{aligned} \tag{5-2}$$

where Ω is the associated scaled frequency and η the dimensionless wave number.

5A. ***Leading order.*** To derive the leading-order approximations, we represent the dispersion relation in terms of the finite scaled frequency Ω_0 and small scaled wave number η. Before doing this, we establish the roots of the characteristic equation up to and including $O(\eta^2)$, which yields

$$\begin{aligned} q_1 &= i\left(1+O(\eta^2)\right), \qquad q_2 = i\left(1+O(\eta^2)\right), \\ q_3 &= \frac{\Omega_0\sqrt{1+\delta}}{\eta} + \epsilon - \frac{(4\delta+1)\,\eta}{2\Omega_0\sqrt{1+\delta}} + O(\eta^2), \\ q_4 &= -\frac{\Omega_0\sqrt{1+\delta}}{\eta} + \epsilon + \frac{(4\delta+1)\,\eta}{2\Omega_0\sqrt{1+\delta}} + O(\eta^2), \end{aligned} \tag{5-3}$$

allowing the following expansions for the $f(q_i)$ to be obtained:

$$\begin{aligned} f(q_1) &\approx -\frac{((1+\delta)\,p+\delta)\,\lambda^2+1-2\,\delta}{(1+\delta)\,\lambda} - i\left(\frac{\Omega_0^2}{\eta^2} - \frac{1+((1+\delta)\,p+3\,\delta)\,\lambda^2}{(1+\delta)\,\lambda^2}\right), \\ f(q_3) &\approx \left(\frac{2\,\epsilon\,\lambda-\lambda^2+2}{\lambda}\right)\frac{\Omega_0^2}{\eta^2} + \frac{\left(2\,\epsilon^2+2\,\delta+1\right)\lambda+\epsilon}{(1+\delta)\,\lambda^2} \\ &\quad + \left(\frac{1-2\,\epsilon\,\lambda^3+\left(3\,\epsilon^2+1+(1+\delta)\,p+3\,\delta\right)\lambda^2+4\,\epsilon\,\lambda}{\sqrt{1+\delta}\lambda^2}\right)\frac{\Omega_0}{\eta} \\ &\quad + \frac{-(\epsilon^2+(1+\delta)\,(p+1))\lambda^3+\left(\epsilon^3+((1+\delta)\,p+3\,\delta+1)\,\epsilon\right)\lambda^2}{(1+\delta)\,\lambda^2}, \end{aligned} \tag{5-4}$$

and likewise for the $g(q_i)$:

$$\begin{aligned} g(q_1) &\approx \frac{(1-2\,\delta)\,\lambda^2+(1+\delta)\,p+\delta}{(1+\delta)\,\lambda} - i\left(\frac{\Omega_0^2}{\eta^2} - \frac{3\,\delta+\lambda^2+(1+\delta)\,p}{1+\delta}\right), \\ g(q_3) &\approx \left(\frac{2\,\epsilon\,\lambda-2\,\lambda^2+1}{\lambda}\right)\frac{\Omega_0^2}{\eta^2} + \frac{\epsilon^2+\delta+(1+\delta)\,p+1}{(1+\delta)\,\lambda} \\ &\quad + \left(\frac{\lambda^3-4\,\epsilon\,\lambda^2+\left(3\,\epsilon^2+1+(1+\delta)\,p+3\,\delta\right)\lambda+2\,\epsilon}{\sqrt{1+\delta}\lambda}\right)\frac{\Omega_0}{\eta} \\ &\quad + \frac{\epsilon\,\lambda^3-\left(2\,\epsilon^2+2\,\delta+1\right)\lambda^2+\left(\epsilon^3+((1+\delta)\,p+3\,\delta+1)\,\epsilon\right)\lambda}{(1+\delta)\,\lambda}. \end{aligned} \tag{5-5}$$

To derive a leading-order form of the dispersion relation Equation (3-4), expansions for sin and cos are required, these being obtainable as

$$\begin{aligned} S_1 &= \tfrac{1}{2}i\eta + O(\eta^2), \qquad S_2 = -\tfrac{1}{2}i\eta + O(\eta^2), \\ S_3 &= \sin\left(\tfrac{1}{2}\sqrt{1+\delta}\Omega_0\right) + \tfrac{1}{2}\cos\left(\tfrac{1}{2}\sqrt{1+\delta}\Omega_0\right)\epsilon\,\eta + O(\eta^2), \\ S_4 &= -\sin\left(\tfrac{1}{2}\sqrt{1+\delta}\Omega_0\right) + \tfrac{1}{2}\cos\left(\tfrac{1}{2}\sqrt{1+\delta}\Omega_0\right)\epsilon\,\eta + O(\eta^2), \\ C_1 &= 1 + O(\eta^2), \qquad C_2 = 1 + O(\eta^2), \\ C_3 &= \cos\left(\tfrac{1}{2}\sqrt{1+\delta}\Omega_0\right) - \tfrac{1}{2}\sin\left(\tfrac{1}{2}\sqrt{1+\delta}\Omega_0\right)\epsilon\,\eta + O(\eta^2), \\ C_4 &= \cos\left(\tfrac{1}{2}\sqrt{1+\delta}\Omega_0\right) + \tfrac{1}{2}\sin\left(\tfrac{1}{2}\sqrt{1+\delta}\Omega_0\right)\epsilon\,\eta + O(\eta^2). \end{aligned} \tag{5-6}$$

Substituting these expansions, together with (5-4), (5-5) and (5-3), into (3-4) results in a dispersion relation of the form $\mathcal{D}^{(0)} + O(\eta^2)$. At leading order, we obtain $\mathcal{D}^{(0)} = 0$, implying that

$$\sin\left(\sqrt{1+\delta}\Omega_0\right) = 0 \implies \Omega^{(0)} = \frac{n\pi}{\sqrt{1+\delta}}. \tag{5-7}$$

5B. *Long-wave high-frequency approximations in the case $\delta = 0$.* In the case $\delta = 0$, we establish long-wave high-frequency approximations to higher order than in the general case.

5B.1. *Leading-order approximations.* Motivated by the general case, we assume that $\Omega = \Omega_0 + O(\eta^2)$. The following expansions are obtainable for the neo-Hookean material dispersion relation (3-6)

$$\begin{aligned}
&\kappa = \sqrt{1-\frac{\Omega^2}{\eta^2}} = \frac{i\Omega_0}{\eta} - \left(\frac{i}{2\Omega_0}\right)\eta + O(\eta^2),\\
&\cos(\eta\epsilon) = 1 + O(\eta^2), \qquad \sinh(\eta) = \eta + O(\eta^2), \qquad \cosh(\eta) = 1 + O(\eta^2),\\
&\sinh(\eta\kappa) = i\sin(\Omega_0) + O(\eta^2), \qquad \cosh(\eta\kappa) = \cos(\Omega_0) + O(\eta^2),
\end{aligned} \tag{5-8}$$

from which the dispersion relation takes the approximate form

$$\mathcal{D} = \mathcal{D}^{(0)} + O(\eta^2), \qquad \mathcal{D}^{(0)} = i{\Omega_0}^8 \sin(\Omega_0). \tag{5-9}$$

From (5-9) we deduce that the cut-off frequencies are given by $\Omega_0 = n\pi$, a result which could have been obtained directly from (5-7).

5B.2. *Higher-order approximations.* Corrections to the cut-off frequencies are now sought in the form

$$\Omega = n\pi + \Omega_2\eta^2 + \Omega_4\eta^4 + O(\eta^5). \tag{5-10}$$

Keeping in mind that $\cos\Omega_0 = (-1)^n$ and $\sin\Omega_0 = 0$, we obtain the following expansions up to and including $O(\eta^4)$:

$$\begin{aligned}
&\kappa = \frac{i\pi n}{\eta} + \frac{i(-1+2\pi n\Omega_2)}{2\pi n}\eta + \frac{i\left(8\pi^3 n^3\Omega_4 - 1 + 4\pi n\Omega_2\right)}{8\pi^3 n^3}\eta^3 + O(\eta^5),\\
&\sinh(\eta) = \eta + \tfrac{1}{6}\eta^3 + O(\eta^5), \qquad \cosh(\eta) = 1 + \tfrac{1}{2}\eta^2 + \tfrac{1}{24}\eta^4 + O(\eta^5),\\
&\cos(\eta\epsilon) = 1 - \tfrac{1}{2}\eta^2\epsilon^2 + \tfrac{1}{24}\eta^4\epsilon^4 + O(\eta^5),\\
&\sinh(\eta\kappa) = \frac{i(-1)^n(-1+2\pi n\Omega_2)}{2\pi n}\eta^2 + \frac{i(-1)^n\left(8\pi^3 n^3\Omega_4 - 1 + 4\pi n\Omega_2\right)}{8\pi^3 n^3}\eta^4 + O(\eta^5),\\
&\cosh(\eta\kappa) = (-1)^n - -\frac{(-1)^n + 4(-1)^{1+n}\pi n\Omega_2 + 4(-1)^n\pi^2 n^2{\Omega_2}^2}{8\pi^2 n^2}\eta^4 + O(\eta^5).
\end{aligned} \tag{5-11}$$

The dispersion relation correction is now expressible as

$$\mathcal{D} = \mathcal{D}^{(2)}\eta^2 + \mathcal{D}^{(4)}\eta^4 + O(\eta^6), \tag{5-12}$$

leading to the corrections

$$\Omega_2 = \frac{4(p+1)^2\left((-1)^n - 1\right) + (-1)^n\pi^2 n^2}{2\pi^3 n^3(-1)^n}, \qquad \Omega_4 = \frac{\mathcal{H}}{24 n^5\pi^5(-1)^n},$$

where

$$\begin{aligned}\mathcal{H} = &-192\,\pi^4 n^4 (-1)^n \Omega_2{}^2 \\ &+ \Big(240\,\pi\, n((-1)^n-1)(p^2+1) + 36\,\pi^3 n^3 (-1)^n (5+2\epsilon^2) \\ &\qquad + \big(96\,\pi^3 n^3 (-1)^n + 480\,\pi\, n((-1)^n-1)\big)p - 4\,\pi^5 n^5 (-1)^n\Big)\Omega_2 \\ &+ \big((96(1-(-1)^n) + 24\,\pi^2 n^2)p^2 + (96(1-(-1)^n)+48\,\pi^2 n^2)p\big)\epsilon^2 \\ &+ 96\,(1-(-1)^n)p^3 + (24\,(-1)^n \pi^2 n^2 + 312(1-(-1)^n))p^2 \\ &- 336\,((-1)^n-1)p + 12\,\pi^2 n^2 (2-3\,(-1)^n)\epsilon^2 + 120(1-(-1)^n) - 21\,(-1)^n \pi^2 n^2 + 2\,(-1)^n \pi^4 n^4.\end{aligned}$$

Although the corrections, especially Ω_4, are quite complex, they are readily plotted, and provide excellent agreement with numerical solutions (Figure 4).

6. Relative asymptotic orders of displacements and hydrostatic pressure

We now wish to establish the relative orders of displacement and pressure within both the long-wave low and high-frequency regimes. Before moving on to this, we first introduce the notation $\zeta = x_2/h$; after which the solutions for displacement components and hydrostatic pressure may, from (2-18), be represented in the form

$$\begin{aligned} u_1 &= (q_1 A_1 e^{i\eta q_1 \zeta} + q_2 A_2 e^{i\eta q_2 \zeta} + q_3 B_1 e^{i\eta q_3 \zeta} + q_4 B_2 e^{i\eta q_4 \zeta}) e^{ik(vt - x_1)}, \\ u_2 &= (A_1 e^{i\eta q_1 \zeta} + A_2 e^{i\eta q_2 \zeta} + B_1 e^{i\eta q_3 \zeta} + B_2 e^{i\eta q_4 \zeta}) e^{ik(vt - x_1)}, \\ p_t &= k\big(P(q_1) A_1 e^{i\eta q_1 \zeta} + P(q_2) A_2 e^{i\eta q_2 \zeta}\big) e^{ik(vt - x_1)} + k\big(P(q_3) B_1 e^{i\eta q_3 \zeta} + P(q_4) B_2 e^{i\eta q_4 \zeta}\big) e^{ik(vt - x_1)}, \end{aligned} \tag{6-1}$$

where A_1, A_2, B_1 and B_2 may, through the use of (3-1) and (3-2), be related through

$$\begin{aligned} &f(q_1)A_1 + f(q_2)A_2 + f(q_3)B_1 + f(q_4)B_2 = 0, \\ &g(q_1)A_1 + g(q_2)A_2 + g(q_3)B_1 + g(q_4)B_2 = 0, \\ &f(q_1)e^{-iq_1\eta}A_1 + f(q_2)e^{-iq_2\eta}A_2 + f(q_3)e^{-iq_3\eta}B_1 + f(q_4)e^{-iq_4\eta}B_2 = 0, \\ &g(q_1)e^{-iq_1\eta}A_1 + g(q_2)e^{-iq_2\eta}A_2 + g(q_3)e^{-iq_3\eta}B_1 + g(q_4)e^{-iq_4\eta}B_2 = 0. \end{aligned} \tag{6-2}$$

6A. ***Long-wave low-frequency leading-order displacements and hydrostatic pressure.*** In the long-wave low-frequency regime the squared wave speed is generally $O(1)$. Accordingly, there are generally no large or small parameters in the characteristic equation (2-15), and we may assume that to leading order $\exp(-i\eta q_j \zeta) = 1 + O(\eta)$. Thus, in the long-wave low-frequency region

$$\begin{aligned} &u_1 \approx (q_1 A_1 + q_2 A_2 + q_3 B_1 + q_4 B_2), \qquad u_2 \approx A_1 + A_2 + B_1 + B_2, \\ &p_t \approx k(P(q_1)A_1 + P(q_2)A_2 + P(q_3)B_1 + P(q_4)B_2). \end{aligned} \tag{6-3}$$

To obtain the dependence of the coefficients A_1, A_2, B_1, and B_2, and thereby the displacement components and pressure increment, on η, we first use the fact that $\exp(-i\eta q_j) = 1 - i\eta q_j + O(\eta^2)$, for $j \in [1, 4]$, and note the two associated long-wave low-frequency phase speed limits given by (4-4). In complete generality it is possible to establish the relative asymptotic orders of A_1, A_2, B_1, and B_2 and

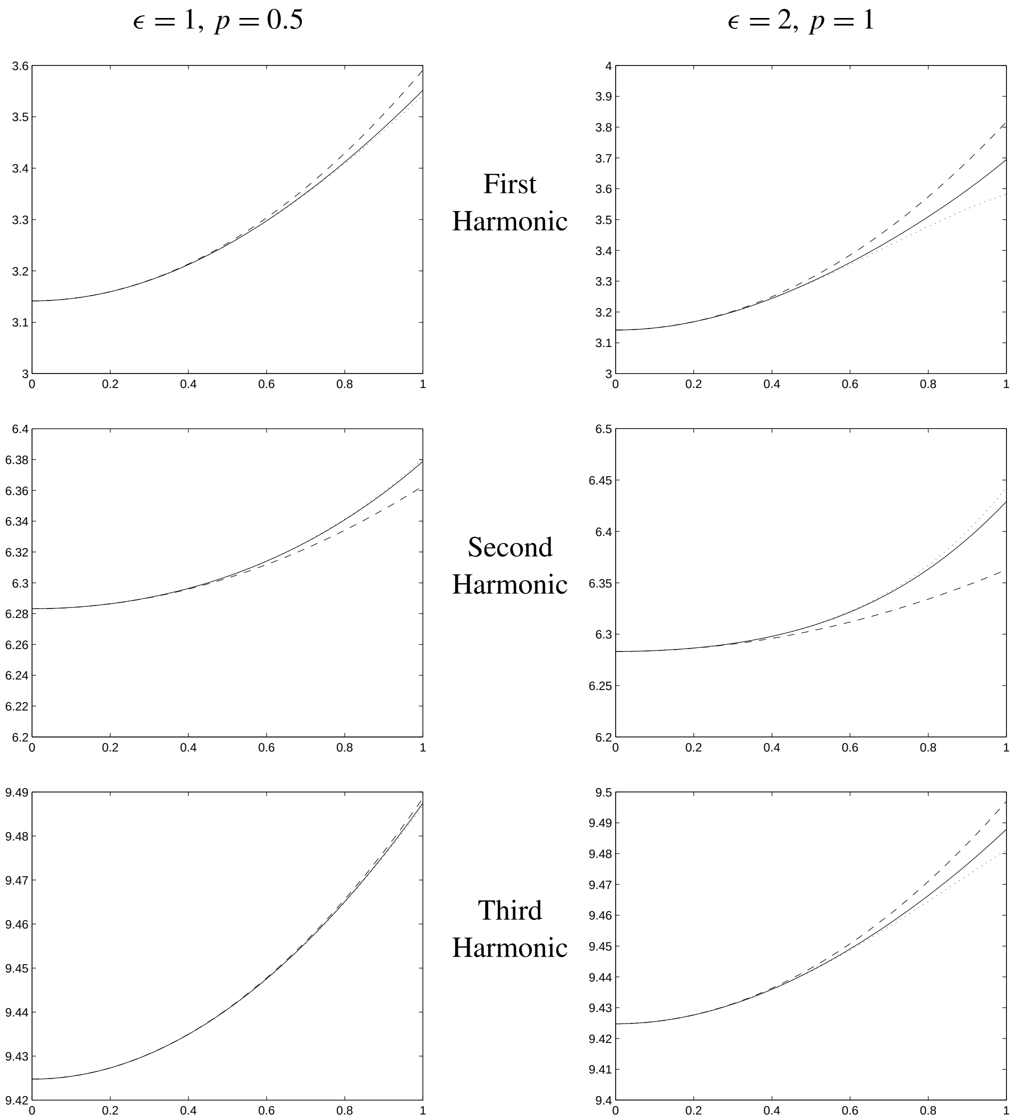

Figure 4. Second- and third-order approximations for first three harmonics compared with numerical results: dimensionless frequency Ω (vertical axis) against scaled wave number η (horizontal axis) for neo-Hookean material. Dotted curves correspond to third order, dashed curves to second order and unbroken curves to numerical results.

hence establish the relative orders of u_1, u_2, and p_t. However, the complexity of the algebra is immense. Accordingly, only explicit results relating to a neo-Hookean material are presented. In this case, A_1, A_2, B_1, and B_2 may be expressed in terms of one parameter, $\tilde{U}$ say, from which the leading order displacements are established as shown in Table 1 at the top of the next page, where we remark that the exponential function $e^{ik(x_1 - vt)}$ has been incorporated into the definition of $\tilde{U}$. It is of interest to note that in Case 1 the in-plane displacement component u_1 has a factor ϵ at leading order, whereas in Case 2, the

Case 1: $\hat{v}^{(0)} = \hat{v}_1^{(0)} = 1 - p^2,$

$$u_1 = \left(\frac{16\mathcal{M}_1^1\left((p-1)^2+\epsilon^2\right)\left((p+1)^2+\epsilon^2\right)p\epsilon}{\left(\epsilon+\sqrt{\epsilon^2+4}\right)^3}\right)\tilde{U}\eta + O(\eta^2),$$

$$u_2 = \left(\frac{32\mathcal{M}_2^1\left((p-1)^2+\epsilon^2\right)\left((p+1)^2+\epsilon^2\right)(p+1)\,p}{\left(\epsilon+\sqrt{\epsilon^2+4}\right)^4}\right)\tilde{U}\eta + O(\eta^2),$$

$$\mathcal{M}_1^1 = \sqrt{\epsilon^2+4}\left(2(p+1)+\epsilon^2(p+4)+\epsilon^4\right)+\epsilon^5+\epsilon^3(p+6)+4\epsilon(p+2)$$

$$\mathcal{M}_2^1 = \left(\epsilon^5+\epsilon^3(p+5)+\epsilon(3p+5)\right)\sqrt{\epsilon^2+4}+\epsilon^6+\epsilon^4(p+7)+\epsilon^2(5p+13)+4(p+1).$$

Case 2: $\hat{v}^{(0)} = \hat{v}_2^{(0)} = \epsilon^2 + 2p + 2,$

$$u_1 = \left(\frac{64\mathcal{M}_1^2\left((p+1)^2+\epsilon^2\right)(p+1)\sqrt{1+\epsilon^2+2p}}{\left(\epsilon+\sqrt{\epsilon^2+4}\right)^3}\right)\tilde{U}\eta + O(\eta^2),$$

$$u_2 = -\left(\frac{128\mathcal{M}_2^2\left((p+1)^2+\epsilon^2\right)\epsilon\sqrt{1+\epsilon^2+2p}}{\left(\epsilon+\sqrt{\epsilon^2+4}\right)^4}\right)\tilde{U}\eta + O(\eta^2),$$

$$\mathcal{M}_1^2 = \left(\epsilon^2 p+3p+1\right)\epsilon\sqrt{\epsilon^2+4}+\epsilon^4 p+\epsilon^2(5p+1)+4$$

$$\mathcal{M}_2^2 = \left(\epsilon^4 p+\epsilon^2(4p+1)+2(p+1)\right)\sqrt{\epsilon^2+4}+\epsilon^5 p+\epsilon^3(6p+1)+4\epsilon(2p+1),$$

Table 1. Leading-order displacements (see previous page).

normal displacement component u_2 has a similar factor. We may therefore conclude that if $\epsilon \sim O(1)$, u_1 and u_2 have the same asymptotic orders. This contrasts with the classical case, discussed for example [Kaplunov et al. 1998], in which $u_2 \gg u_1$ in Case 1 (bending) and $u_1 \gg u_2$ Case 2 (extension). It also contrasts with their prestress counterparts for which one principal axis is normal to the plate — see [Kaplunov et al. 2000] — within which the classical structure is preserved. In the case $\epsilon = 0$, the results above reduce to

Case 1: Bending

$$\hat{v}_1^{(0)} = 1 - p^2, \quad u_1 = O(\eta^2), \quad u_2 = 8\,(p+1)^4\,(p-1)^2\,p\tilde{U}\eta + O(\eta^2), \quad \mathcal{M}_1^2 = \mathcal{M}_2^2 = 4(p+1),$$

Case 2: Extension

$$\hat{v}_2^{(0)} = 2p+2, \quad u_1 = 32\,(p+1)^4\,\tilde{U}\sqrt{1+2p}\eta + O(\eta^2), \quad u_2 = O(\eta^2), \quad \mathcal{M}_1^1 = \mathcal{M}_2^1 = 4(p+1).$$

From this we see that if $\epsilon = 0$, the relative orders agree with previously mentioned classical results. In passing we note that the relative asymptotic orders may need modification within the vicinity of certain critical states of prestress. The order of hydrostatic pressure in case of a neo-hookean material may similarly be established and shown to be of the form

$$p_t = k\mathfrak{Q}\tilde{U}\eta + O(\eta^2), \tag{6-4}$$

with $\mathfrak{Q}$ generally $O(1)$. For both associated fundamental modes, the coefficient $\mathfrak{Q}$ is algebraically complex and is therefore not written explicitly here. However, in the case of $\epsilon = 0$, more explicit results are obtainable, yielding

$$\textbf{Case 1: } \hat{v}_1^{(0)} = 1 - p^2: \qquad \mathfrak{Q}^{(1)} = 0,$$
$$\textbf{Case 2: } \hat{v}_2^{(0)} = 2p + 2: \qquad \mathfrak{Q}^{(2)} = 32\gamma\sqrt{1+2p}(1+p)^5\tilde{U},$$

indicating that in Case 1 (Bending) $p_t \sim \eta^2$ and in Case 2 (extension) $p_t \sim \eta$. These results, together with those for u_1 and u_2, are a particular case of those obtained in [Kaplunov et al. 2000].

6B. ***Long-wave high-frequency leading-order displacements and hydrostatic pressure.*** As $\eta \to 0$, appropriate approximations of the characteristic equation, (2-15), are readily obtainable and the exponential functions in (6-1) are therefore representable in the approximate forms

$$e^{i\eta q_1\zeta} = 1 + O(\eta), \qquad e^{i\eta q_2\zeta} = 1 + O(\eta),$$
$$e^{i\eta q_3\zeta} = e^{i\zeta n\pi} + O(\eta), \qquad e^{i\eta q_4\zeta} = e^{-i\zeta n\pi} + O(\eta).$$

Hence, in the long-wave high-frequency regime, the displacement components and hydrostatic pressure are given approximately by

$$u_1 \approx q_1 A_1 + q_2 A_2 + q_3 B_1 e^{i\zeta n\pi} + q_4 B_2 e^{-i\zeta n\pi},$$
$$u_2 \approx A_1 + A_2 + B_1 e^{i\zeta n\pi} + B_2 e^{-i\zeta n\pi},$$
$$p_t \approx k\big(P(q_1)A_1 + P(q_2)A_2 + P(q_3)B_1 e^{i\zeta n\pi} + P(q_4)B_2 e^{-i\zeta n\pi}\big).$$

In order to estimate the dependence of the coefficients A_1, A_2, B_1, and B_2 on η, we expand the exponential functions in powers of η, up to $O(\eta^2)$, which yields

$$e^{-i\eta q_1} = 1 + \eta + O(\eta^2), \qquad e^{-i\eta q_2} = 1 - \eta + O(\eta^2),$$
$$e^{-i\eta q_3} = (-1)^n(1 - i\eta\epsilon) + O(\eta^2), \qquad e^{-i\eta q_4} = (-1)^n(1 - i\eta\epsilon) + O(\eta^2).$$

The displacement components are now expressible in terms of a single parameter $\tilde{U}$ in the form

$$u_1 = C_6^{(1)}\eta^{-6} + C_5^{(1)}(\eta^{-5}) + O(\eta^{-4}),$$
$$u_2 = C_5^{(2)}(\eta^{-5}) + O(\eta^{-4}),$$
$$C_6^{(1)} = \frac{4n^7\pi^7(-1)^n\left(\lambda^2+1\right)\cosh(\zeta n\pi)\tilde{U}}{\lambda\,(1+\delta)^3},$$
$$C_5^{(1)} \neq 0,$$
$$C_5^{(2)} = \frac{2i\left(\lambda^2+1\right)\pi^5 n^5\tilde{U}}{\lambda^3(1+\delta)^3}\mathscr{C}_u,$$

for

$$\mathscr{C}_u = -2i\pi\,\lambda^2 n(-1)^n\sinh(\zeta n\pi) + 2\left(1-(-1)^n\right)\big(\lambda^4 - 2\lambda^3\epsilon - \lambda^2(p(1+\delta)+\epsilon^2+6+2\delta) + 2\epsilon + 1\big),$$

and the pressure takes the form

$$p_t = kc^{(7)}\eta^{-7} + kc^{(6)}\eta^{-6} + O(\eta^{-5}),$$

where

$$c_7 = c_1^{(7)}\epsilon + c_0^{(7)}, \qquad c_6 = c_2^{(6)}\epsilon^2 + c_1^{(6)}\epsilon + c_0^{(6)}.$$

All the coefficients are quite lengthy both in general and for the neo-Hookean strain energy function. However, in general all coefficients are $O(1)$. We note from the above long-wave representations that the incremental pressure is asymptotically leading, with the in-plane displacement component very much larger than its normal counterpart. We note that in the case $\epsilon = 0$,

$$c_0^{(7)}(\epsilon = 0, \Omega = \Omega^{(0)}) = 0,$$
$$c_0^{(6)}(\epsilon = 0, \Omega = \Omega^{(0)}) = 8\gamma\pi^7 n^7 (-1)^n (p+1)\tilde{U},$$

indicating that now the incremental pressure and in-plane displacement are the same order. This is in agreement with the result in [Kaplunov et al. 2000]. We conclude by remarking that in general the incremental pressure is asymptotically leading and this will have consequences in the derivation of appropriate asymptotic long-wave models.

7. Some concluding remarks

In this paper we have investigated the influence of a simple shear primary deformation on long wave motion in an incompressible elastic layer. For high-frequency motion, the results are broadly similar to those previously published in regard to pure homogeneous strain. However, the results for low-frequency motion show a significant departure from previous investigations. In this case it is not possible to decompose the problem into symmetric and antisymmetric components. It is therefore not possible to establish analogs of classical extension and bending. The reason for this is that whenever the amount of shear is finite, within the long-wave low-frequency regime both the in-plane and normal displacement components have the same asymptotic order. This contrasts with the classical cases of extension and bending, together with their homogeneously strained counterparts, for which the in-plane and normal component are asymptotically leading, respectively. Knowledge of the asymptotic orders of displacement components will provide the necessary basis for deriving an appropriate asymptotic model. Although such a model would be a great simplification, with the displacements having the same order, it would seem unlikely that it will be one-dimensional. Work to clarify this is currently being carried out and the results will be published in due course.

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