

*Journal of*  
***Mechanics of***  
***Materials and Structures***

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***Volume 3, N° 5***

***May 2008***



mathematical sciences publishers

# ANISOTROPY IN HYPOELASTIC SOFT-TISSUE MECHANICS, I: THEORY

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A hypoelastic constitutive theory is derived with an eye towards modeling the passive response of soft anisotropic biological tissues. Anisotropy is handled through a material tensor whose construction is independent of the constitutive formulation. Anisotropy tensors are provided for tissues that have a single dominant fiber family with an elliptic fiber projection onto the transverse plane, and their limiting cases. The tissue is comprised of two constituents: a matrix phase and a fiber phase. The theory is derived in the polar configuration, for ease of handling the derivatives, and then mapped into the commonly used Eulerian configuration. Kirchhoff stress and its conjugate strain-rate are the state variables.

## 1. Introduction

In a landmark paper for the field of biomechanics, [Fung \[1967\]](#) empirically fit the tensile load-deflection curve of rabbit mesentery with the linear, first-order, differential equation

$$\frac{d\sigma}{d\lambda} = \alpha + \beta\sigma \quad \text{or equivalently} \quad \frac{d\sigma}{dt} = (\alpha + \beta\sigma) \frac{d\lambda}{dt}, \quad (1)$$

where  $\sigma$  denotes stress,  $\lambda$  denotes stretch, and  $\alpha$  and  $\beta$  represent material constants. This is Fung's law. The second formula, which follows from the first via the chain rule, is in the format of a hypoelastic constitutive equation. Although originally expressed as a differential equation, Fung's law is almost always referred to in its integrated form

$$\sigma = \frac{\alpha}{\beta}(e^{\beta\epsilon} - 1), \quad (2)$$

where  $\epsilon = \lambda - 1$  denotes strain.

Since then, numerous constitutive formulae have been put forward by various researchers in the field of biomechanics in an effort to extend Fung's one-dimensional law to three dimensions; see Fung's textbook [\[1993\]](#) and the literature reviews [\[Humphrey 2002; Sacks 2000; Viidik 1973; Weiss and Gardiner 2001\]](#). These three-dimensional theories have been derived as hyperelastic constitutive expressions, in spite of the fact that Fung's original one-dimensional formulation was hypoelastic in construction [\[Fung 1967\]](#). Even Fung's own three-dimensional extension of [Equation \(1\)](#) was a hyperelastic theory [\[Fung 1973\]](#), that is,

$$\sigma_{ij} + pI_{ij} = C'_{ijkl}\epsilon_{kl} + \exp(aJ_1 + bJ_2)C_{ijkl}\epsilon_{kl}, \quad (3)$$

where  $\sigma_{ij}$  are components of stress,  $\epsilon_{ij}$  are components of strain,  $J_1$  and  $J_2$  are the first and second strain invariants,  $a$  and  $b$  are constants,  $C'_{ijkl}$  and  $C_{ijkl}$  are elastic moduli, and  $p$  is a Lagrange multiplier that forces an isochoric response. The only nonhyperelastic theory that has been put forward for soft

*Keywords:* anisotropy, finite deformations, hypoelasticity, polar configuration, polar rates, polar strain, polar stress, soft tissue, spin, swirl, vorticity.

tissues was derived by Criscione et al. [2003a], where hyper- and hypoelastic models were combined to describe glutaraldehyde fixed bovine pericardial sacs [Criscione et al. 2003b]. They demonstrated that a hyperelastic material model, alone, is not capable of describing this tissue, because the work done *does* depend upon the path taken.

A *hyperelastic material* defines stress through some potential function in strain or deformation [Ogden 1984]. This potential function has its origin in thermodynamics and produces a theory where the value of stress depends only on the current state of strain and not on the path traversed to reach that state. A *hypoelastic material* defines stress-rate as some *linear* function of strain-rate [Truesdell 1955], where various objective rates have been used in the literature. Hypoelastic materials have the potential of reaching states of stress that do depend upon the path of strain taken. Noll (see Truesdell [1955]) proved a theorem that states: “Every isotropic elastic body is also an isotropic hypoelastic body.” Hypoelasticity is therefore a generalization of hyperelasticity. In contrast, Criscione et al. [2003a] state: “Together they (hyperelasticity and hypoelasticity) define all elastic-type relations, yet they are mutually exclusive.” Obviously, these two statements are inconsistent. The word *hypoelastic* is being used differently in these manuscripts. We will use the term *hypoelastic* in the sense of Truesdell.

This paper provides a rational extension of Fung’s law (1) through an application of Truesdell’s hypoelastic theory [1955] to a class of anisotropic materials that potentially includes soft biological tissues. The theory is also novel in the method by which anisotropy is introduced into it and by the choice of strain rate that is used as its independent variable.

**1.1. Preliminaries.** Scalars are typeset in italics, for example,  $y$ . Vectors are typeset in lower-case bold italics, for example,  $\mathbf{y} = y_i \mathbf{e}_i$ . And tensors are typeset in upper-case bold italics, for example,  $\mathbf{Y} = Y_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ , wherein  $\otimes$  signifies a tensor product. Components  $y_i$  and  $Y_{ij}$  are also typeset in italics, and they are quantified in a rectangular Cartesian basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ . Exceptions to this notation are minimal, but they do arise, both from the use of conventional notations and from the need to introduce multiple variables of similar nature. Components map from one set of coordinates  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  into another set of coordinates  $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$  according to the transformation rules  $\tilde{y}_i = Q_{ia} y_a$  and  $\tilde{Y}_{ij} = Q_{ia} Q_{jb} Y_{ab}$  with  $Q_{ik} Q_{jk} = Q_{ki} Q_{kj} = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker delta function. Repeated indices are summed from 1 to 3 in accordance with Einstein’s summation convention.

The identity tensor  $\mathbf{I} = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_i$  maps vectors into themselves, for example,  $\mathbf{y} = \mathbf{I} \cdot \mathbf{y}$  with the dot denoting a contraction over a pair of indices such that  $y_i = \delta_{ij} y_j$ . Dot notation is used whenever two vectors are contracted together, for example, when defining the vector norm  $\|\mathbf{y}\|_2^2 = \mathbf{y} \cdot \mathbf{y} = y_i y_i$ . However, dot notation is suppressed whenever a tensor contracts with a vector, for example,  $\mathbf{y} = \mathbf{I} \mathbf{y}$ , or whenever two tensors are contracted together, for example,  $\mathbf{YZ} = Y_{ik} Z_{kj} \mathbf{e}_i \otimes \mathbf{e}_j$ . Here contractions are implicit.

The tensor  $\mathbf{Y} = Y_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$  has a transpose  $\mathbf{Y}^T = Y_{ji} \mathbf{e}_i \otimes \mathbf{e}_j$  and an inverse  $\mathbf{Y}^{-1} = Y_{ij}^{-1} \mathbf{e}_i \otimes \mathbf{e}_j$ , whenever it exists, such that  $\mathbf{Y}\mathbf{Y}^{-1} = \mathbf{Y}^{-1}\mathbf{Y} = \mathbf{I}$  where  $Y_{ij}^{-1} \neq (Y_{ij})^{-1}$ . The symmetric tensor operator,  $\text{sym } \mathbf{Y} = \frac{1}{2}(Y_{ij} + Y_{ji}) \mathbf{e}_i \otimes \mathbf{e}_j$ , and the antisymmetric tensor operator,  $\text{skew } \mathbf{Y} = \frac{1}{2}(Y_{ij} - Y_{ji}) \mathbf{e}_i \otimes \mathbf{e}_j$ , satisfy  $\mathbf{Y} = \text{sym } \mathbf{Y} + \text{skew } \mathbf{Y}$ . The trace operator  $\text{tr } \mathbf{Y} = Y_{ii}$  is the diagonal sum. And, from of the Cayley-Hamilton theorem, the determinant obeys the identity  $\det \mathbf{Y} = \frac{1}{3}(\text{tr } \mathbf{Y}^3 - \frac{3}{2} \text{tr } \mathbf{Y} \text{tr } \mathbf{Y}^2 + \frac{1}{2}(\text{tr } \mathbf{Y})^3)$ .

Any modern textbook that addresses the mechanics of continuous media can be used as a reference. A particularly good text has been written by Holzapfel [2000].

## 2. Kinematics

Consider a mass point in a Cartesian coordinate frame  $(1, 2, 3)$  with unit vectors  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  whose original location in some reference configuration  $\kappa_0$  affiliated with time  $t = 0$ , say, is given by a position vector  $\mathbf{X} = X_i \mathbf{e}_i$ . Let its current location in configuration  $\kappa$ , which is affiliated with time  $t$ , be given by another position vector  $\mathbf{x} = x_i \mathbf{e}_i$  that moves with a velocity of  $\mathbf{v} = v_i \mathbf{e}_i$  wherein  $v_i = \dot{x}_i = \partial x_i / \partial t$ .

It is supposed that the motion of this mass point through space can be described by a one-parameter family (in time) of locations  $\mathbf{x} = \chi_i(\mathbf{X}, t) \mathbf{e}_i$  that is considered to be continuous and sufficiently differentiable (in space) to allow a deformation gradient to be defined according to  $\mathbf{F} = F_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$  wherein  $F_{ij} = \partial \chi_i(\mathbf{X}, t) / \partial X_j$ . The inverse to the deformation gradient  $\mathbf{F}^{-1}$  exists because  $J = \det \mathbf{F} > 0$  from the conservation of mass.

Because the tensor  $\mathbf{F}\mathbf{F}^T$  is positive-definite,  $\mathbf{F}$  has a polar decomposition of

$$\mathbf{F} = \mathbf{V}\mathbf{R} = \mathbf{R}\mathbf{U}, \quad (4)$$

with properties  $\mathbf{V} = \mathbf{V}^T$ ,  $\mathbf{U} = \mathbf{U}^T$ ,  $\mathbf{R}^{-1} = \mathbf{R}^T$  such that  $\det \mathbf{R} = +1$ , where  $\mathbf{R}$  is a proper orthogonal tensor, and where  $\mathbf{V}$  and  $\mathbf{U}$  are commonly referred to as the left (or Eulerian) and right (or Lagrangian) stretch tensors, respectively, which are symmetric positive-definite. Also associated with the deformation gradient are the left  $\mathbf{B}$  [Finger 1894] and right  $\mathbf{C}$  [Green 1841] deformation tensors quantified by  $\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2$  and  $\mathbf{C} = \mathbf{F}^T \mathbf{F} = \mathbf{U}^2$ , which are symmetric positive-definite with components  $B_{ij} = F_{ik} F_{jk}$  and  $C_{ij} = F_{ki} F_{kj}$ .

An infinitesimal material vector  $d\mathbf{X}$  is pushed forward into an infinitesimal spatial vector  $d\mathbf{x}$  according to the mapping

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \quad \text{so that} \quad dS^2 = d\mathbf{X} \cdot d\mathbf{X} \quad \text{and} \quad ds^2 = d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X} \cdot \mathbf{C}d\mathbf{X}, \quad (5)$$

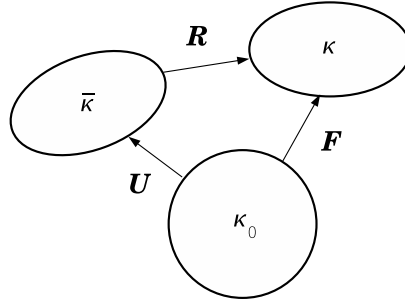
where  $dS$  and  $ds$  are the Euclidean norms of vectors  $d\mathbf{X}$  and  $d\mathbf{x}$ , respectively, both of which are positive. It is apparent from the last formula of the above expression that tensor  $\mathbf{C}$  plays the role of a metric quantifying the deformation that occurs at a mass point, expressed in the Lagrangian viewpoint.

Akin to the deformation gradient tensor  $\mathbf{F}$  is the velocity gradient tensor  $\mathbf{L} = L_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$  with components  $L_{ij} = \partial v_i / \partial x_j$ , so that  $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ . Its symmetric and antisymmetric parts,  $\mathbf{D} = \text{sym } \mathbf{L}$  and  $\mathbf{W} = \text{skew } \mathbf{L}$ , are called the stretching  $\mathbf{D}$  and vorticity  $\mathbf{W}$  tensors, respectively, with  $\mathbf{W}$  quantifying the rate of rotation occurring about the principal axes of  $\mathbf{D}$ . In the literature tensor  $\mathbf{D}$  is also referred to as either the strain-rate tensor or the rate-of-deformation tensor.

**2.1. Polar fields.** By selecting what Dienes [1987; 2003] calls the *polar configuration*  $\bar{\kappa}$ , a material modeler can construct constitutive equations that depend only on stretch and its rate; not on rotation or its rate. This is advantageous when developing mathematical models suitable for finite deformation analysis. All constitutive equations in the PRONTO two-dimensional and three-dimensional finite-element codes [Flanagan and Taylor 1987], for example, are solved in the polar configuration  $\bar{\kappa}$ , and then mapped into the spatial configuration  $\kappa$  for the displaying of results.

Dienes [1987; 2003] refers to any spatial field, for example, vector  $\mathbf{y}$  or tensor  $\mathbf{Y}$ , that is mapped into  $\bar{\kappa}$  (by means of the rotation tensor  $\mathbf{R}$  defined in Equation (4)) as a *polar field*; specifically,

$$\bar{\mathbf{y}} = \mathbf{R}^T \mathbf{y} \quad \text{and} \quad \bar{\mathbf{Y}} = \mathbf{R}^T \mathbf{Y} \mathbf{R}. \quad (6)$$



**Figure 1.** Mappings between the reference  $\kappa_0$ , polar  $\bar{\kappa}$ , and current  $\kappa$  configurations of a body.

Dienes [1979; 1986] distinguishes polar fields, for example,  $\bar{\mathbf{y}}$  and  $\bar{\mathbf{Y}}$ , from their Eulerian counterparts, that is,  $\mathbf{y}$  and  $\mathbf{Y}$ , with a bar—a notation that we adopt. A direct consequence of his definition is that  $\mathbf{e}_1 = \mathbf{R}\bar{\mathbf{e}}_1$ ,  $\mathbf{e}_2 = \mathbf{R}\bar{\mathbf{e}}_2$ , and  $\mathbf{e}_3 = \mathbf{R}\bar{\mathbf{e}}_3$ , where rotation  $\mathbf{R}$  maps *polar axes*  $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$  into *spatial axes*  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ . How the three configurations  $\kappa_0$ ,  $\bar{\kappa}$ , and  $\kappa$  relate to one another is depicted in Figure 1.

Material derivatives of polar fields are described by [Dienes 1979]

$$\dot{\bar{\mathbf{y}}} = \mathbf{R}^T \hat{\mathbf{y}} \text{ wherein } \hat{\mathbf{y}} = \dot{\mathbf{y}} - \boldsymbol{\Omega} \mathbf{y}, \quad \text{and} \quad \dot{\bar{\mathbf{Y}}} = \mathbf{R}^T \hat{\mathbf{Y}} \mathbf{R} \text{ wherein } \hat{\mathbf{Y}} = \dot{\mathbf{Y}} - \boldsymbol{\Omega} \mathbf{Y} + \mathbf{Y} \boldsymbol{\Omega}, \quad (7)$$

where  $\dot{\mathbf{y}} = \partial \mathbf{y} / \partial t + \nabla \mathbf{y} \cdot \mathbf{v}$  defines the material derivative of spatial field  $\mathbf{y}$ , which contains partial derivatives in both time  $\partial \mathbf{y} / \partial t = (\partial y_i / \partial t) \mathbf{e}_i$  and space  $\nabla \mathbf{y} = \frac{\partial y_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j$ , and where  $\boldsymbol{\Omega} = \dot{\mathbf{R}} \mathbf{R}^T$  is the spin tensor, which is the rate of rigid-body rotation about a material point that, like  $\mathbf{W}$ , is antisymmetric. The spatial derivatives  $\hat{\mathbf{y}}$  and  $\hat{\mathbf{Y}}$  are objective rates with  $\hat{\mathbf{Y}}$  being commonly called the Green-Naghdi rate [Green and Naghdi 1965]. Instead, following the naming convention of Dienes [1987; 2003], we shall call  $\dot{\bar{\mathbf{Y}}}$  the *material rate of polar tensor*  $\bar{\mathbf{Y}}$ , and  $\hat{\mathbf{Y}}$  the *polar rate of spatial tensor*  $\mathbf{Y}$ . Freed and Diethelm [2007] have recently extended this class of mappings to include integrals and derivatives of fractional (noninteger) order, which they then used in the construction of a viscoelastic model for isotropic tissues.

There are a variety of objective time derivatives that have been used in constitutive modeling (see Atluri [1984] for a review). Physical reasons for selecting the polar rate over other objective rates and, in particular, over the Jaumann rate  $\dot{\mathbf{Y}} - \mathbf{W} \mathbf{Y} + \mathbf{Y} \mathbf{W}$ , has been the topic of many research papers [Atluri 1984; Dienes 1979; 1979; 1986; 1987; 2003; Flanagan and Taylor 1987; Freed and Diethelm 2007; Johnson and Bammann 1984; Szabó and Balla 1989].

**Remark 1.** In accordance with Equations (4), (6), and (7), Dienes [1979; 1986; 1987; 2003] argues that there is in fact just one *physical* stretch tensor, it being  $\mathbf{V}$ , since  $\bar{\mathbf{V}} = \mathbf{R}^T \mathbf{V} \mathbf{R} \equiv \mathbf{U}$  and  $\dot{\bar{\mathbf{V}}} = \mathbf{R}^T \hat{\mathbf{V}} \mathbf{R} \equiv \dot{\mathbf{U}}$ , wherein

$$\hat{\mathbf{V}} = \dot{\mathbf{V}} - \boldsymbol{\Omega} \mathbf{V} + \mathbf{V} \boldsymbol{\Omega} \quad \text{with evolution} \quad \dot{\mathbf{V}} = \mathbf{L} \mathbf{V} - \mathbf{V} \boldsymbol{\Omega} \quad \text{so that} \quad \hat{\mathbf{V}} = (\mathbf{D} + \mathbf{Z}) \mathbf{V}. \quad (8)$$

In other words, the right-stretch tensor  $\mathbf{U}$  and the polar stretch tensor  $\bar{\mathbf{V}}$  are synonyms for the same tensor field. Similarly, Dienes argues that there is just one *physical* deformation tensor, it being  $\mathbf{B}$ , since  $\bar{\mathbf{B}} = \mathbf{R}^T \mathbf{B} \mathbf{R} \equiv \mathbf{C}$  and  $\dot{\bar{\mathbf{B}}} = \mathbf{R}^T \hat{\mathbf{B}} \mathbf{R} \equiv \dot{\mathbf{C}}$ , wherein

$$\hat{\mathbf{B}} = \dot{\mathbf{B}} - \boldsymbol{\Omega} \mathbf{B} + \mathbf{B} \boldsymbol{\Omega} \quad \text{with evolution} \quad \dot{\mathbf{B}} = \mathbf{L} \mathbf{B} + \mathbf{B} \mathbf{L}^T \quad \text{so that} \quad \hat{\mathbf{B}} = (\mathbf{D} + \mathbf{Z}) \mathbf{B} + \mathbf{B} (\mathbf{D} - \mathbf{Z}), \quad (9)$$

or equivalently,  $\hat{\mathbf{B}} = 2\mathbf{V}\mathbf{D}\mathbf{V}$ . Consequently the right-deformation tensor  $\mathbf{C}$  and the polar deformation tensor  $\bar{\mathbf{B}}$  are synonyms for the same tensor field.

We have introduced another antisymmetric tensor in Equations (8) and (9), it being  $\mathbf{Z} = \mathbf{W} - \mathbf{\Omega}$ , which we call *swirl*. It is the actual vorticity that occurs within a material element, that is, the rigid-body spin  $\mathbf{\Omega}$  has been removed from the overall rotation of vortex  $\mathbf{W}$ . Its influence can be neglected when analyzing most engineered materials, because they cannot sustain finite deformations, but soft tissues can.

In light of Remark 1, we will hereafter break with traditional notation and terminology, somewhat, and adopt the notation and terminology of Dienes when addressing the polar configuration  $\bar{\kappa}$ . We will, however, retain conventional usage whenever a particular field or equation applies to the Lagrangian configuration  $\kappa_0$ . In this regard, a concerted effort has been made herein to distinguish between tensor equations defined on  $\kappa_0$  from those that are defined on  $\bar{\kappa}$ .

### 3. Conjugate stress-strain pairs

The rate at which work is being done on a material element of unit volume, when expressed in either the Eulerian  $\kappa$  or Lagrangian  $\kappa_0$  configuration, is well-known and given by the respective formulae [Holzapfel 2000; Ogden 1984]

$$\dot{W} = \text{tr}(\mathbf{T}\mathbf{D}) = \frac{1}{2} \text{tr}(\dot{\mathbf{S}}\mathbf{C}), \quad (10)$$

where  $\mathbf{T}$  is the Kirchhoff stress of the Eulerian frame, and  $\mathbf{S}$  is the second Piola-Kirchhoff stress of the Lagrangian frame, which relate to the true stress of Cauchy  $\boldsymbol{\sigma}$  via the well-known identities  $\mathbf{T} = \mathbf{J}\boldsymbol{\sigma}$  and  $\mathbf{S} = \mathbf{J}\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$ , recalling that  $\mathbf{J} = \det \mathbf{F}$ .

Upon inserting the polar decomposition of  $\mathbf{F}$ , that is,  $\mathbf{R}\bar{\mathbf{V}}$ , into the work expression for the  $\kappa_0$  configuration, noting that  $\mathbf{S} = \mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} \equiv \bar{\mathbf{V}}^{-1}\mathbf{R}^T\mathbf{T}\mathbf{R}\bar{\mathbf{V}}^{-1}$  and  $\dot{\mathbf{C}} = \mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U} \equiv \bar{\mathbf{V}}\dot{\bar{\mathbf{V}}} + \dot{\bar{\mathbf{V}}}\bar{\mathbf{V}}$  because  $\mathbf{C} = \mathbf{U}^2$ , one can reexpress Equation (10) for the  $\bar{\kappa}$  configuration as

$$\begin{aligned} \dot{W} &= \frac{1}{2} \text{tr}(\bar{\mathbf{V}}^{-1}\mathbf{R}^T\mathbf{T}\mathbf{R}\bar{\mathbf{V}}^{-1}(\bar{\mathbf{V}}\dot{\bar{\mathbf{V}}} + \dot{\bar{\mathbf{V}}}\bar{\mathbf{V}})) = \text{tr}(\mathbf{R}^T\mathbf{T}\mathbf{R}\frac{1}{2}(\dot{\bar{\mathbf{V}}}\bar{\mathbf{V}}^{-1} + \bar{\mathbf{V}}^{-1}\dot{\bar{\mathbf{V}}})) \\ &= \text{tr}(\bar{\mathbf{T}}\dot{\bar{\mathbf{E}}}) = \text{tr}(\bar{\mathbf{T}} \text{sym } \dot{\bar{\mathbf{E}}}) \equiv \text{tr}(\bar{\mathbf{T}}\dot{\bar{\mathbf{E}}}), \end{aligned} \quad (11)$$

where the symmetric *polar stress*  $\bar{\mathbf{T}}$  and its rate  $\dot{\bar{\mathbf{T}}}$  obey the mappings

$$\bar{\mathbf{T}} = \mathbf{R}^T\mathbf{T}\mathbf{R} \quad \text{and} \quad \dot{\bar{\mathbf{T}}} = \mathbf{R}^T\hat{\mathbf{T}}\mathbf{R},$$

the *nonsymmetric polar strain*  $\bar{\mathbf{E}}$  and its rate  $\dot{\bar{\mathbf{E}}} = \dot{\bar{\mathbf{V}}}\bar{\mathbf{V}}^{-1}$  obey the mappings

$$\bar{\mathbf{E}} = \mathbf{R}^T\boldsymbol{\varepsilon}\mathbf{R} \quad \text{and} \quad \dot{\bar{\mathbf{E}}} = \mathbf{R}^T\hat{\boldsymbol{\varepsilon}}\mathbf{R},$$

with  $\text{sym } \hat{\boldsymbol{\varepsilon}} = \mathbf{D}$  and  $\text{skew } \hat{\boldsymbol{\varepsilon}} = \mathbf{Z}$ , so that

$$\bar{\mathbf{E}} = \int_0^t \mathbf{R}^T(0, \tau)(\mathbf{L} - \mathbf{\Omega})(\tau)\mathbf{R}(0, \tau)d\tau = \int_0^t \mathbf{R}^T(0, \tau)(\mathbf{D} + \mathbf{Z})(\tau)\mathbf{R}(0, \tau)d\tau$$

and

$$\boldsymbol{\varepsilon} = \int_0^t \mathbf{R}(\tau, t)(\mathbf{L} - \mathbf{\Omega})(\tau)\mathbf{R}^T(\tau, t)d\tau = \int_0^t \mathbf{R}(\tau, t)(\mathbf{D} + \mathbf{Z})(\tau)\mathbf{R}^T(\tau, t)d\tau,$$

because  $\mathbf{R}(0, t) = \mathbf{R}(\tau, t)\mathbf{R}(0, \tau)^1$ , while the *symmetric polar strain*  $\bar{\mathbf{E}}$  and its rate  $\dot{\bar{\mathbf{E}}}$  obey the mappings<sup>2</sup>

$$\bar{\mathbf{E}} = \mathbf{R}^T \mathbf{E} \mathbf{R} \quad \text{and} \quad \dot{\bar{\mathbf{E}}} = \mathbf{R}^T \hat{\mathbf{E}} \mathbf{R} \equiv \mathbf{R}^T \mathbf{D} \mathbf{R}, \tag{12}$$

so that

$$\bar{\mathbf{E}} = \int_0^t \mathbf{R}^T(0, \tau) \mathbf{D}(\tau) \mathbf{R}(0, \tau) d\tau \quad \text{and} \quad \mathbf{E} = \int_0^t \mathbf{R}(\tau, t) \mathbf{D}(\tau) \mathbf{R}^T(\tau, t) d\tau. \tag{13}$$

In summary,  $\bar{\mathbf{E}} = \text{sym } \bar{\mathcal{E}}$ ,  $\dot{\bar{\mathbf{E}}} = \text{sym } \dot{\bar{\mathcal{E}}}$ ,  $\mathbf{E} = \text{sym } \mathcal{E}$ , and  $\hat{\mathbf{E}} = \text{sym } \hat{\mathcal{E}}$ . The polar rate of strain  $\hat{\mathcal{E}} = \mathbf{L} - \boldsymbol{\Omega}$  has symmetric and antisymmetric parts that are described by  $\text{sym } \hat{\mathcal{E}} = \mathbf{D}$  and skew  $\hat{\mathcal{E}} = \mathbf{Z}$ .

The above development demonstrates that  $\{\bar{\mathbf{T}}, \bar{\mathbf{E}}\}$ ,  $\{\bar{\mathbf{T}}, \dot{\bar{\mathcal{E}}}\}$ ,  $\{\mathbf{T}, \mathbf{E}\}$ , and  $\{\mathbf{T}, \mathcal{E}\}$  are admissible conjugate pairs for stress and strain in the polar  $\bar{\kappa}$  and spatial  $\kappa$  configurations, and that they are associated with the respective strain rates of  $\dot{\bar{\mathbf{E}}}$ ,  $\dot{\bar{\mathcal{E}}} = \dot{\bar{\mathbf{V}}}\bar{\mathbf{V}}^{-1}$ ,  $\hat{\mathbf{E}} \equiv \mathbf{D}$ , and  $\hat{\mathcal{E}} = \mathbf{L} - \boldsymbol{\Omega} \equiv \mathbf{D} + \mathbf{Z}$ . None of these pairs are found in Farahani and Naghdabadi [2003], who recently reviewed the literature of conjugate stress-strain pairs in  $\kappa_0$ .

**3.1. Updating polar stress.** When constructing numerical algorithms for solving constitutive equations in the polar configuration  $\bar{\kappa}$ , it becomes necessary to establish a means for pushing the polar stress forward from the polar configuration  $\bar{\kappa}_N$  of the  $N$ -th step, say, into the polar configuration  $\bar{\kappa}_{N+1}$  of the  $N + 1$ -st step. This process is most easily understood via the diagram presented in Figure 2. Found in this diagram are three, well-known, polar decompositions:  $\mathbf{F}_N = \mathbf{R}_N \mathbf{U}_N$ ,  $\mathbf{F}_{N+1} = \mathbf{R}_{N+1} \mathbf{U}_{N+1}$ , and  $\mathbf{F}_{N+1} = \mathbf{F}_{N+1}^N \mathbf{F}_N$ , wherein  $\mathbf{F}_{N+1}^N$  represents the deformation gradient between  $\kappa_N$  and  $\kappa_{N+1}$ . Also present is the polar decomposition  $\mathbf{F}_{N+1}^N = \mathbf{R}_{N+1}^N \mathbf{U}_{N+1}^N$  that leads to  $\mathbf{U}_{N+1}^N = \mathbf{R}_N \mathbf{U}_{N+1} \mathbf{U}_N^{-1} \mathbf{R}_N^T$  because  $\mathbf{R}_{N+1} = \mathbf{R}_{N+1}^N \mathbf{R}_N$ .

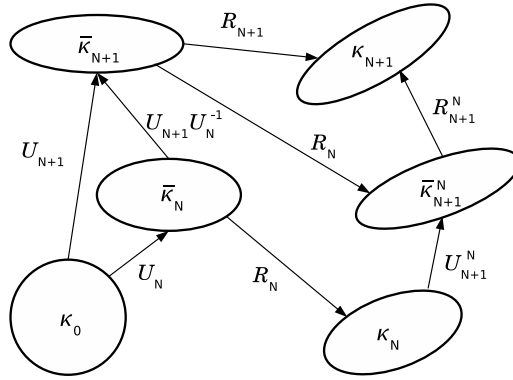
Given a state of polar stress  $\bar{\mathbf{T}}_N$  for the  $\bar{\kappa}_N$  configuration that has been stored for later use by an integrator affixed to the  $\bar{\kappa}_{N+1}$  configuration, for example, the retrieved stress must first be pushed forward from  $\bar{\kappa}_N$  into  $\bar{\kappa}_{N+1}$  before it can be utilized in this new configuration. This is achieved via the mapping

$$\bar{\mathbf{T}}_{N+1}^N = \bar{\mathbf{G}}_{N+1}^N \bar{\mathbf{T}}_N (\bar{\mathbf{G}}_{N+1}^N)^T \quad \text{wherein} \quad \bar{\mathbf{G}}_{N+1}^N = \mathbf{U}_{N+1} \mathbf{U}_N^{-1} \equiv \bar{\mathbf{V}}_{N+1} \bar{\mathbf{V}}_N^{-1} \approx \bar{\mathbf{I}} + \Delta t \dot{\bar{\mathcal{E}}}_N, \tag{14}$$

where  $\bar{\mathbf{T}}_{N+1}^N$  represents the polar stress  $\bar{\mathbf{T}}$  of state  $\bar{\kappa}_N$  pushed forward into state  $\bar{\kappa}_{N+1}$ . This mapping is akin to the well-known mapping that pushes the second Piola-Kirchhoff stress  $\mathbf{S}$  forward into the Kirchhoff stress  $\mathbf{T}$  via  $\mathbf{T} = \mathbf{F} \mathbf{S} \mathbf{F}^T$ . The difference between this scheme and the updated-Lagrangian scheme is that Equation (14) maps the polar stress  $\bar{\mathbf{T}}$  from  $\bar{\kappa}_N$  into  $\bar{\kappa}_{N+1}$ ; whereas the updated-Lagrangian method maps the Kirchhoff stress  $\mathbf{T}$  from  $\kappa_N$  into  $\kappa_{N+1}$  as  $\mathbf{T}_{N+1}^N = \mathbf{F}_{N+1}^N \mathbf{T}_N (\mathbf{F}_{N+1}^N)^T$  [Belytschko et al. 2000]. The approximation for  $\bar{\mathbf{G}}$  in the above formula follows from a Taylor expansion of  $\bar{\mathbf{V}}$  about  $t_N$  with  $\Delta t = t_{N+1} - t_N$ , while recalling that  $\dot{\bar{\mathcal{E}}} = \dot{\bar{\mathbf{V}}}\bar{\mathbf{V}}^{-1}$  and noting that  $\bar{\mathbf{I}} = \mathbf{R}^T \mathbf{I} \mathbf{R} = \mathbf{R}^T \mathbf{R} = \mathbf{I}$ .

<sup>1</sup>Any material rotation  $\mathbf{R}(0, t)$  from  $\kappa_0$  to  $\kappa$  can be decomposed into two sequential rotations: the first rotation  $\mathbf{R}(0, \tau)$  goes from  $\kappa_0$  to  $\kappa_\tau$ , while the second rotation  $\mathbf{R}(\tau, t)$  goes from  $\kappa_\tau$  to  $\kappa$  with  $0 \leq \tau \leq t$  and  $\mathbf{R}(\tau, \tau) = \mathbf{I}$  for all  $\tau$ . The product  $\mathbf{R}(\tau, t)\mathbf{R}(0, \tau)$  of two orthogonal tensors is another orthogonal tensor; however,  $\mathbf{R}(\tau, t)\mathbf{R}(0, \tau) \neq \mathbf{R}(0, \tau)\mathbf{R}(\tau, t)$ , that is, matrix multiplication does not commute between orthogonal tensors.

<sup>2</sup>Definitions for  $\bar{\mathbf{E}}$  and  $\mathbf{E}$  first appeared in Freed and Diethelm [2007]. Earlier, Hill [1968] proved that  $\dot{\bar{\mathbf{E}}}$  approximates Hencky’s definition for strain rate whenever  $\|\mathbf{E}_G \dot{\bar{\mathbf{E}}} \mathbf{E}_G\|$  is sufficiently small with  $\mathbf{E}_G = \frac{1}{2}(\mathbf{C} - \mathbf{I})$  being the Green strain.



**Figure 2.** Mappings between the Lagrangian  $\kappa_0$ , updated Lagrangian  $\kappa_N$ , and Eulerian  $\kappa_{N+1}$  configurations, and their relationships to the polar configurations  $\bar{\kappa}_N$ ,  $\bar{\kappa}_{N+1}$ , and  $\bar{\kappa}_{N+1}^N$  that associate with the updated Lagrangian and Eulerian configurations, and the incremental advancement between them, respectively.

### 4. Simple shear

To demonstrate that the symmetric polar strain  $\bar{E}$  and its spatial counterpart  $E$  are tractable strain measures, let us consider a simple shear flow whose deformation history is described by

$$\chi_1 = X_1 + \gamma X_2, \quad \chi_2 = X_2, \quad \text{and} \quad \chi_3 = X_3,$$

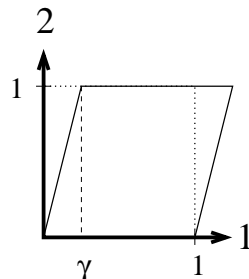
leading to a deformation gradient with components

$$\mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{so that} \quad \mathbf{F}^{-1} = \begin{bmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

wherein  $\gamma$  quantifies the extent of shear, as shown in [Figure 3](#). Simple shear is isochoric, that is,  $\det \mathbf{F} = 1$  independent of material constitution.

This deformation gradient is affiliated with a material rotation of

$$\mathbf{R} = \frac{1}{\sqrt{4 + \gamma^2}} \begin{bmatrix} 2 & \gamma & 0 \\ -\gamma & 2 & 0 \\ 0 & 0 & \sqrt{4 + \gamma^2} \end{bmatrix},$$



**Figure 3.** Simple shear of a unit (dotted) square sheared by  $\gamma$  into a (solid) parallelogram.



which satisfies the required identity that  $\det \mathbf{R} = 1$ . Utilizing this description for  $\mathbf{R}$ , derived by Ogden [1984], leads to simpler expressions for fields described in terms of  $\mathbf{R}$  than one would otherwise arrive at if sines and cosines were used, as appear in classic rotation matrices [Dienes 1979; Flanagan and Taylor 1987].

The velocity gradient  $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$  and stretching  $\mathbf{D} = \text{sym } \mathbf{L}$  tensors have components

$$\mathbf{L} = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{D} \equiv \hat{\mathbf{E}} = \frac{\dot{\gamma}}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

while the vorticity  $\mathbf{W} = \text{skew } \mathbf{L}$ , spin  $\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$ , and swirl  $\mathbf{Z} = \mathbf{W} - \mathbf{\Omega}$  tensors have components

$$\mathbf{W} = \frac{\dot{\gamma}}{2} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{\Omega} = \frac{2\dot{\gamma}}{4 + \gamma^2} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{Z} = \frac{\gamma^2 \dot{\gamma}}{2(4 + \gamma^2)} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where  $\mathbf{W}$  and  $\mathbf{\Omega}$  are seen to be equal only when  $\gamma = 0$ , while  $\mathbf{Z}$  approaches  $\mathbf{W}$  and  $\mathbf{\Omega}$  approaches  $\mathbf{0}$  with increasing values of  $|\gamma|$ .

The symmetric, polar, strain rate  $\dot{\bar{\mathbf{E}}}$  defined in Equation (12) is therefore given by

$$\dot{\bar{\mathbf{E}}} = \frac{\dot{\gamma}}{2(4 + \gamma^2)} \begin{bmatrix} -4\gamma & 4 - \gamma^2 & 0 \\ 4 - \gamma^2 & 4\gamma & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} \dot{\bar{\varepsilon}} & \dot{\bar{\theta}} & 0 \\ \dot{\bar{\theta}} & -\dot{\bar{\varepsilon}} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where, to simplify notation, we have introduced terms

$$\dot{\bar{\theta}} = \frac{4 - \gamma^2}{2(4 + \gamma^2)} \dot{\gamma} \quad \text{and} \quad \dot{\bar{\varepsilon}} = \frac{-2\gamma}{4 + \gamma^2} \dot{\gamma}.$$

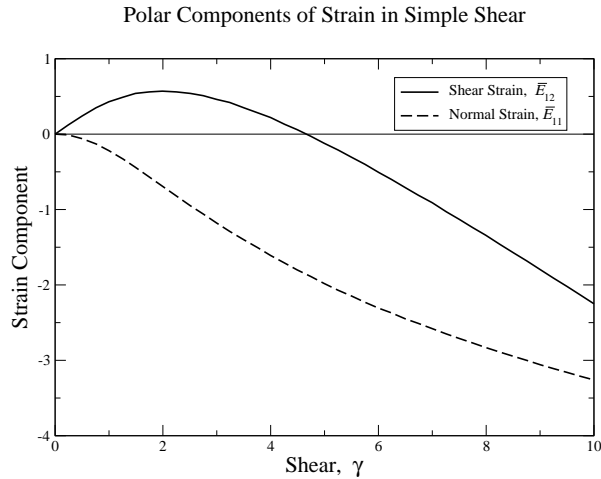
Integrating from 0 to  $\gamma$  produces

$$\bar{\mathbf{E}} = \begin{bmatrix} \bar{\varepsilon} & \bar{\theta} & 0 \\ \bar{\theta} & -\bar{\varepsilon} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{wherein} \quad \bar{\varepsilon} = \ln\left(\frac{4}{4 + \gamma^2}\right) \quad \text{and} \quad \bar{\theta} = 2 \tan^{-1}\left(\frac{\gamma}{2}\right) - \frac{\gamma}{2}, \quad (15)$$

which quantifies the polar strain  $\bar{\mathbf{E}}$  for simple shear. By rotating the polar strain tensor of Equation (15) into the Eulerian frame, according to Equation (12), one gets the physical strain measure

$$\mathbf{E} = \begin{bmatrix} \varepsilon & \theta & 0 \\ \theta & -\varepsilon & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{wherein} \quad \varepsilon = \frac{1}{4 + \gamma^2} ((4 - \gamma^2)\bar{\varepsilon} + 4\gamma\bar{\theta}) \quad \text{and} \quad \theta = \frac{1}{4 + \gamma^2} ((4 - \gamma^2)\bar{\theta} - 4\gamma\bar{\varepsilon}). \quad (16)$$

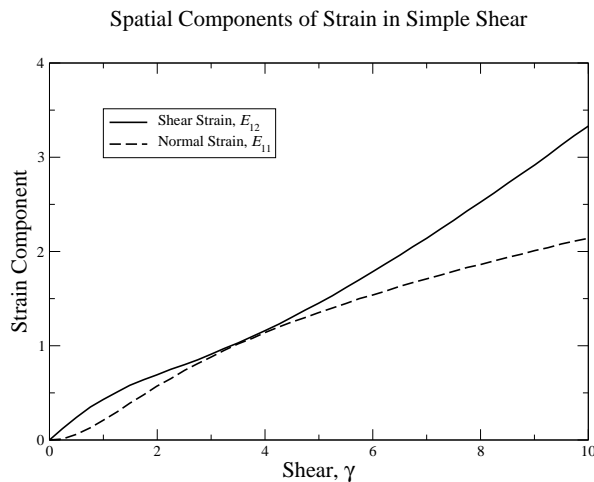
No other strain field, that we are aware of, has these components for simple shear. Having said this, the isotropic hypoelastic material considered by Dienes [1979] has a stress response in simple shear that is proportional to the strain components listed in Equation (16), with the shear modulus being the constant of proportionality. However, Dienes [1986] does not define polar strain as we have done in Equation (13); rather, he advocates using Signorini-strain, that is,  $\mathbf{E}_S = \frac{1}{2}(\mathbf{B} - \mathbf{I})$  [Signorini 1930]. Other differences



**Figure 4.** Polar components of strain in simple shear, where  $\bar{E}_{12} = \bar{\theta}$  and  $\bar{E}_{11} = \bar{\varepsilon}$ .

between Dienes’ formulae and ours for simple shear originate with the different, but equivalent, ways that we assign components to the rotation tensor  $\mathbf{R}$ .

The polar strain components for simple shear are plotted in Figure 4, while their associated spatial components are plotted in Figure 5. Of these, the curves for  $\varepsilon$  and  $\theta$  presented in Figure 5 represent physical components, while those for  $\bar{\varepsilon}$  and  $\bar{\theta}$  presented in Figure 4 are nonphysical in nature. The two normal strains  $\bar{\varepsilon}$  and  $\varepsilon$  are even functions of shear  $\gamma$ , while the two shear strains  $\bar{\theta}$  and  $\theta$  are odd functions of  $\gamma$ . Accordingly, only the right half of the response plane needs to be shown. The shear strain  $E_{12}$  is seen to dip below  $\gamma/2$  over much of the range plotted in Figure 5, yet it asymptotes with  $\gamma/2$  at both small and large values of shear strain; essentially whenever  $|\gamma| < 0.5$  or  $|\gamma| > 100$ . In contrast, the normal strain  $E_{11}$  grows very slow at large values of  $\gamma$ , for example, whenever  $|\gamma| > 100$ , giving it a local appearance of having reached an asymptote.



**Figure 5.** Spatial components of strain in simple shear, where  $E_{12} = \theta$  and  $E_{11} = \varepsilon$ .

## 5. Material anisotropy

The fibrous structures of biological tissues are not strongly aligned; rather, they are dispersed over angles of splay [Finlay et al. 1995]. The use of a distribution density function for the handling of fiber splay in connective tissues was introduced by Lanir [1983]. Since then single-family fiber splays have been adopted by Hurschler et al. [1997] for tendon and ligament, Billiar and Sacks [2000] and Freed et al. [2005] for heart valves, and Usyk et al. [2000] for ventricular myofibers; double-family fiber splays have been adopted by Gasser et al. [2006] for arteries; and triple-family fiber splays have been adopted by Nash and Hunter [2000] for myocardial laminae. In the papers by Freed et al. [2005] and Gasser et al. [2006], the distribution function describing anisotropy was extracted from the overall constitutive formulation yielding a material or structural tensor. In what follows, this material tensor is given additional physical insight through a derivation based on Riemannian Geometry.

In Equation (5), the symmetric positive-definite Lagrangian tensor  $\mathbf{C}$  is shown to be a metric tensor that quantifies deformation. In what follows, a new symmetric positive-semidefinite tensor, call it  $\mathbf{C}^f$ , is introduced that can be thought of as being a metric-like tensor that quantifies just the anisotropic contributions of deformation caused by fiber reinforcement, that is, the isotropic contributions to deformation are removed.

Let us begin by postulating the existence of a masking tensor  $\mathbf{M} = M_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$  defined in  $\kappa_0$  (it is a material tensor; hence it does not change over time) that strips away all isotropic contributions otherwise present in  $d\mathbf{X}$  by scaling only those components of a displacement that align with the material directions of anisotropy in  $\kappa_0$ , as defined by the transformation law

$$d\mathbf{\Xi} = \mathbf{M}d\mathbf{X} \quad \text{with} \quad d\Sigma^2 = d\mathbf{\Xi} \cdot d\mathbf{\Xi} = d\mathbf{X} \cdot \mathbf{A}_0 d\mathbf{X} \quad \text{wherein} \quad \mathbf{A}_0 = \mathbf{M}^T \mathbf{M}, \quad (17)$$

where  $d\Sigma$  is some nonnegative fraction of  $dS$  that quantifies the relative anisotropic stiffening in material direction  $d\mathbf{X}$ , that is,  $0 \leq d\Sigma \leq dS$ . Consequently, the *anisotropy tensor*  $\mathbf{A}_0 = A_{ij}^0 \mathbf{e}_i \otimes \mathbf{e}_j$  is symmetric positive-semidefinite with components  $A_{ij}^0 = M_{ki} M_{kj}$ . The lower bound of  $d\Sigma = 0$  occurs in those directions  $d\mathbf{X}$  where a material has no anisotropic stiffening. Here its stiffness is considered to be due solely to its isotropic constituents. The upper bound of  $d\Sigma = dS$  occurs in those directions  $d\mathbf{X}$  where the material has its greatest anisotropic stiffening. Here its stiffness is considered to be due solely to its anisotropic constituents. All other directions, that is, where  $0 < d\Sigma < dS$ , are considered to have a mixture of isotropic and anisotropic stiffening effects with the anisotropic contribution scaling according to ratio  $d\Sigma/dS \in (0, 1)$ .

Because the masking tensor  $\mathbf{M}$  is a material tensor defined in the reference configuration  $\kappa_0$ , it maps into the current configuration  $\kappa$  through a simple rotation such that

$$d\boldsymbol{\xi} = \mathbf{R}\mathbf{M}\mathbf{R}^T d\mathbf{x} \quad \text{with} \quad d\sigma^2 = d\boldsymbol{\xi} \cdot d\boldsymbol{\xi} = d\mathbf{X} \cdot \mathbf{C}^f d\mathbf{X} \quad \text{wherein} \quad \mathbf{C}^f = \mathbf{U}\mathbf{A}_0\mathbf{U},$$

and, in doing so, analogous to  $d\Sigma$ ,  $d\sigma$  is bound to the interval  $0 \leq d\sigma \leq dS$ . Here we see that the anisotropy tensor  $\mathbf{A}_0$  now takes on the role of a geometric metric within  $\mathbf{C}^f$ , making  $\mathbf{C}^f$  a metric of the anisotropic deformation. Tensor  $\mathbf{C}^f$  possesses the property that  $\mathbf{C}^f \rightarrow \mathbf{C}$  whenever  $\mathbf{A}_0 \rightarrow \mathbf{I}$ , which occurs at all isotropic barriers.

**Remark 2.** The fact that  $d\Sigma/dS \in [0, 1]$  requires that the maximum eigenvalue of tensor  $\mathbf{A}_0$  be 1. This physical constraint is equivalent to the mathematical constraint that

$$\|\mathbf{A}_0\|_2 = 1, \quad (18)$$

where  $\|\mathbf{A}_0\|_2$  is the  $L_2$  norm of  $\mathbf{A}_0$ .

**5.1. Transverse isotropy.** Materials described by a unit vector  $\mathbf{a}_0$  that aligns with a single fiber orientation in some reference configuration  $\kappa_0$  have an anisotropy tensor that is given by<sup>3</sup> Spencer [1972]

$$\mathbf{A}_0 = \mathbf{a}_0 \otimes \mathbf{a}_0, \quad (19)$$

which satisfies the constraint listed in Equation (18), because  $\|\mathbf{a}_0\|_2 = 1$  by definition. The eigenvalues for this material tensor  $\mathbf{A}_0$  are  $\{1, 0, 0\}$ . Highly aligned fiber orientations, as implied by transverse isotropy, are intentionally designed into many man-made materials; whereas fiber architectures in biological tissues usually have splayed or distributed fiber orientations.

**5.1.1. Conic splay.** Gasser et al. [2006] derived an anisotropy tensor from a statistical assessment of fiber splay, where a projection of the fibers onto the transverse plane is uniformly distributed by angle (the fiber distribution is a cone of orientation vectors) with unit vector  $\mathbf{a}_0$  denoting their mean orientation. Their anisotropy tensor has the form

$$\mathbf{A}_0 = \delta \mathbf{I} + (1 - \delta) \mathbf{a}_0 \otimes \mathbf{a}_0, \quad (20)$$

where our presentation of their formula has been normalized so that  $\|\mathbf{A}_0\|_2 = 1$ . Our  $\delta$  relates to their  $\kappa = \frac{1}{4} \int_0^\pi \rho(\Theta) \sin^3(\Theta) d\Theta$  via the formula  $\delta = \kappa / (1 - 2\kappa)$ , where  $\rho(\Theta)$  is an orientation density function for fiber splay, with  $\kappa \in [0, 1/3]$  and therefore  $\delta \in [0, 1]$ . The dispersion parameter  $\delta$  is bound from below by transverse isotropy (that is, the limiting case of  $\delta = 0$ , which simplifies Equation (20) to Equation (19)), and it is bound from above by isotropy in 3-space (that is, whenever  $\delta = 1$ ). The eigenvalues for this material tensor  $\mathbf{A}_0$  are  $\{1, \delta, \delta\}$ .

The dispersion parameter  $\delta$  is to be regarded as a phenomenological material constant. It is a material property derived from a statistical assessment of fiber distribution, just like the shear modulus  $\mu$  is a material property derived from a statistical assessment of the spatial geometry of the molecular backbone in elastomers [Treloar 1975]. No specific knowledge pertaining to the statistical underpinnings of either  $\delta$  or  $\mu$  is required for their use in applications.

**5.1.2. Fan splay.** Freed et al. [2005] derived another anisotropy tensor from a different statistical assessment of fiber dispersion. Here splay was considered to occur as a two-dimensional fan contained within 3-space whose planar normal aligns with unit vector  $\mathbf{b}_0$ . Fibers reside within this plane with a mean orientation of  $\mathbf{a}_0$ . Hence, by definition,  $\mathbf{a}_0 \cdot \mathbf{b}_0 = 0$  and  $\|\mathbf{a}_0\|_2 = \|\mathbf{b}_0\|_2 = 1$ . The anisotropy tensor for this geometry, when their formula has been normalized so that  $\|\mathbf{A}_0\|_2 = 1$ , can be rewritten as

$$\mathbf{A}_0 = \delta \mathbf{I} + (1 - \delta) \mathbf{a}_0 \otimes \mathbf{a}_0 - \delta \mathbf{b}_0 \otimes \mathbf{b}_0, \quad (21)$$

<sup>3</sup>Ateshian [2007] has extended transverse isotropy to address the fact that thin filaments can only carry tensile loads. His adaptation into this framework lies beyond the scope of this paper.

where, like Equation (20), the dispersion parameter  $\delta$  is bound from below by transverse isotropy; see Equation (19). However, unlike Equation (20),  $\delta$  is now bound from above by an isotropy in 2-space. The eigenvalues for this material tensor  $\mathbf{A}_0$  are  $\{1, \delta, 0\}$ .

**5.1.3. Orthotropic splay.** As alluded to in Freed et al. [2005], the two cases of conic and fan splay can be unified into an orthotropic anisotropy tensor for a single dispersed fiber family given by

$$\mathbf{A}_0 = \delta \mathbf{I} + (1 - \delta) \mathbf{a}_0 \otimes \mathbf{a}_0 + (\epsilon - \delta) \mathbf{b}_0 \otimes \mathbf{b}_0,$$

where  $0 \leq \epsilon \leq \delta \leq 1$ . Here the projection of fibers onto the transverse plane about  $\mathbf{a}_0$  produces an ellipse whose minor axis is in the direction of  $\mathbf{b}_0$  with a length of  $\epsilon$ , and whose major axis is in the direction of  $\mathbf{a}_0 \times \mathbf{b}_0$  with a length of  $\delta$ . The limiting case of  $\epsilon = 0$  reduces to fan splay (Equation (21)); the limiting case of  $\epsilon = \delta$  reduces to conic splay (Equation (20)); and the limiting case of  $\epsilon = \delta = 0$  reduces to transverse isotropy (Equation (19)). The eigenvalues for this material tensor  $\mathbf{A}_0$  are  $\{1, \delta, \epsilon\}$ .

These are but a few of the possible anisotropy tensors that can be derived and utilized within the framework of the following proposition. The above cases apply whenever a single family of fibers exists whose geometric interpretation of anisotropy can be described with an elliptic projection onto the transverse plane.

### 6. Hypothesis

**Proposition 1.** *The isotropic constituent (for example, the ground substance **matrix**) and the anisotropic constituent (for example, the collagen **fiber**) that comprise soft tissues are considered to be cohabitant, in a continuum sense, at every mass point throughout a tissue. In these materials, the Kirchhoff stress  $\mathbf{T}$  is considered to be a sum of fiber and matrix stresses governed by the rule of mixtures*

$$\mathbf{T} + \wp \mathbf{I} = \phi \mathbf{T}^f + (1 - \phi) \mathbf{T}^m, \tag{22}$$

while the metric for deformation  $\mathbf{C}$  is considered to decompose according to the sum

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{U} \mathbf{A}_0 \mathbf{U} + \mathbf{U} (\mathbf{I} - \mathbf{A}_0) \mathbf{U}, \tag{23}$$

which is given the notation

$$\mathbf{C} = \mathbf{C}^f + \mathbf{C}^m, \quad \text{wherein } \mathbf{C}^f = \mathbf{U} \mathbf{A}_0 \mathbf{U} \quad \text{and} \quad \mathbf{C}^m = \mathbf{U} (\mathbf{I} - \mathbf{A}_0) \mathbf{U}, \tag{24}$$

where  $\mathbf{C}^f$  and  $\mathbf{T}^f$  are the deformation and stress tensors associated with the anisotropic fibers,  $\mathbf{C}^m$  and  $\mathbf{T}^m$  are the deformation and stress tensors associated with the isotropic matrix, and  $\mathbf{A}_0$  is the anisotropy tensor of (17)—all five of these tensors are symmetric—and where  $\phi$  is the volume fraction occupied by the anisotropic constituents, and  $\wp$  is a Lagrange multiplier used to enforce an isochoric constraint, that is,  $\det \mathbf{C} = 1$ .

Regarding stresses, Equation (22) is the well-known rule-of-mixtures formula from composites theory [Spencer 1972]. It is adopted here for the same reason that Humphrey and Rajagopal [2002] adopted it; specifically, to make boundary value problems tractable. Formula (22) enables residual stresses to be assigned to the fiber and matrix phases via initial conditions, for example, as  $\mathbf{T}_0^f$  and  $\mathbf{T}_0^m = \frac{-\phi}{1-\phi} \mathbf{T}_0^f$ , so that  $\mathbf{T}_0 = \mathbf{0}$  at time  $t = 0$ .

However, regarding deformations, the above proposition differs from all existing hypotheses known to the author. There are two commonly used kinematic splits found in the literature. They are the multiplicative decompositions of the deformation gradient tensor (elastic-plastic theories that originate with Lee [1969]) and the mass-weighted sums of constituent velocity vectors (mixture theories that originate with Truesdell [1957]), respectively. In contrast, here the geometric metric, which we take to be  $\mathbf{I}$  because Cartesian tensor fields are used, is additively split into two parts, that is, isotropic and anisotropic, so that the positive-definite metric of deformation  $\mathbf{C}$  becomes a sum of two positive-semidefinite tensors  $\mathbf{C}^f$  and  $\mathbf{C}^m$  defined according to Equation (24).

**Remark 3.** Because the anisotropy tensor  $\mathbf{A}_0$  is a material tensor, its mapping from  $\kappa_0$  into  $\bar{\kappa}$  is not affected by stretch  $\mathbf{U}$ , and one has  $\bar{\mathbf{A}} \equiv \mathbf{A}_0$ . However, it is affected by rotation  $\mathbf{R}$  when mapped from either  $\kappa_0$  or  $\bar{\kappa}$  into  $\kappa$ , wherein the anisotropy tensor becomes  $\mathbf{A} = \mathbf{R}\bar{\mathbf{A}}\mathbf{R}^T \equiv \mathbf{R}\mathbf{A}_0\mathbf{R}^T$ . In other words, unlike metals, for example, anisotropy in soft tissues is not considered to be deformation induced.

**6.1. Polar representation.** In general tensor analysis the deformation tensor is given by the formula  $\mathbf{C} = \mathbf{F}^T \mathbf{g} \mathbf{F}$ , wherein  $\mathbf{g}$  is the geometric metric tensor of  $\kappa$  [Sokolnikoff 1964]. The deformation metric can be rewritten as  $\mathbf{C} \equiv \bar{\mathbf{B}} = \bar{\mathbf{V}} \mathbf{R}^T \mathbf{g} \mathbf{R} \bar{\mathbf{V}} = \bar{\mathbf{V}} \bar{\mathbf{g}} \bar{\mathbf{V}}$  for  $\bar{\kappa}$ , where  $\bar{\mathbf{g}} = \mathbf{R}^T \mathbf{g} \mathbf{R}$  defines a polar metric. Tensors  $\bar{\mathbf{I}} - \bar{\mathbf{A}} \equiv \mathbf{I} - \mathbf{A}_0$  and  $\bar{\mathbf{A}} \equiv \mathbf{A}_0$  in Equation (23) are analogous to metric  $\bar{\mathbf{g}}$ ; however, they are not true metric tensors, like  $\bar{\mathbf{g}}$  is, in the sense that  $\bar{\mathbf{I}} - \bar{\mathbf{A}}$  and  $\bar{\mathbf{A}}$  are not positive definite; they are only positive semidefinite. If the base vectors  $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$  happen to be affiliated with a curvilinear coordinate system, then  $\mathbf{C}^m$  would become  $\bar{\mathbf{V}}(\bar{\mathbf{g}} - \bar{\mathbf{A}})\bar{\mathbf{V}}$  and  $\bar{\mathbf{A}} \rightarrow \bar{\mathbf{g}}$  at isotropic barriers, with the Riemannian metric  $\bar{\mathbf{g}}$  now replacing the Cartesian metric  $\bar{\mathbf{I}}$ . But, in this document, Cartesian tensors are used.

Isotropic and anisotropic contributions to the polar strain-rates  $\dot{\bar{\mathbf{E}}}$  and  $\dot{\bar{\mathbf{E}}}$  can be readily constructed by replacing  $\dot{\mathbf{C}}$  with  $\dot{\mathbf{C}}^m$  and  $\dot{\mathbf{C}}^f$  in Equation (10), recalling that tensor  $\mathbf{A}_0 \equiv \bar{\mathbf{A}}$  is temporally constant, and then recasting these expressions, as was done in the derivation within Equation (11), which leads to

$$\dot{\bar{\mathbf{E}}} = \dot{\bar{\mathbf{E}}}^f + \dot{\bar{\mathbf{E}}}^m, \quad \text{wherein} \quad \dot{\bar{\mathbf{E}}}^f = \bar{\mathbf{A}} \dot{\bar{\mathbf{E}}} \quad \text{and} \quad \dot{\bar{\mathbf{E}}}^m = (\bar{\mathbf{I}} - \bar{\mathbf{A}}) \dot{\bar{\mathbf{E}}}, \quad (25)$$

and therefore

$$\dot{\bar{\mathbf{E}}} = \dot{\bar{\mathbf{E}}}^f + \dot{\bar{\mathbf{E}}}^m, \quad \text{wherein} \quad \dot{\bar{\mathbf{E}}} = \text{sym} \dot{\bar{\mathbf{E}}}, \quad \dot{\bar{\mathbf{E}}}^f = \text{sym} \dot{\bar{\mathbf{E}}}^f, \quad \text{and} \quad \dot{\bar{\mathbf{E}}}^m = \text{sym} \dot{\bar{\mathbf{E}}}^m, \quad (26)$$

while the isotropic and anisotropic contributions to polar stress obey  $\bar{\mathbf{T}} + \wp \bar{\mathbf{I}} = \phi \bar{\mathbf{T}}^f + (1 - \phi) \bar{\mathbf{T}}^m$ , which follows straightaway from Equations (6) and (22).

## 7. Hypoelasticity

We begin our discussion of general hypoelasticity by considering an isotropic material and finish our discussion by extending that construction into a simple anisotropic material model that has potential in the modeling of the passive response of soft biological tissues.

**7.1. Isotropic materials.** An analytic expression for the polar strain  $\bar{\mathbf{E}}(\bar{\mathbf{V}})$ , as suggested by Equation (12), does not currently exist, to the best of our knowledge; nevertheless, this does not preclude one from using its rate  $\dot{\bar{\mathbf{E}}}$  as the primitive kinematic variable in constitutive formulations. Specifically, instead of the more common hyperelastic constitutive constructions [Holzapfel 2000; Ogden 1984], constitutive

equations for elasticity can be cast in rate form, whose formulae belong to the material class<sup>4</sup> [Dienes 1979]

$$\hat{T} = \psi(T, D) = \psi(R\bar{T}R^T, R\dot{E}R^T) = R\psi(\bar{T}, \dot{E})R^T \quad \text{or equivalently} \quad \dot{\bar{T}} = \psi(\bar{T}, \dot{E}). \quad (27)$$

A hypoelastic solid, in the sense of Truesdell [1955], is that subset to this material class which restricts the tensor-valued material function  $\psi$  to be a linear function in strain rate; here, either  $D$  or  $\dot{E}$  depending on one’s choice of reference frame. Flanagan and Taylor [1987] have published an algorithm that can be used to solve constitutive formulae belonging to the material class  $\dot{\bar{T}} = \psi(\bar{T}, \dot{E})$ .

Leonov [2000] derived a theorem that states: Hypoelastic constitutive equations are physically meaningful (that is, they obey the second law of thermodynamics) if, and only if, they admit a potential. Consequently, Equation (27) becomes physically meaningful if, and only if,  $\psi = \partial\Psi/\partial\dot{E}$  so that

$$\dot{\bar{T}} = \frac{\partial\Psi(\bar{T}, \dot{E})}{\partial\dot{E}}. \quad (28)$$

It necessarily follows that the potential function  $\Psi$  must now be quadratic in  $\dot{E}$  in order for the response to be hypoelastic (that is, stress-rate is linear in strain-rate). In this regard,  $\Psi$  is not so different from a conventional hyperelastic strain-energy potential.

Any material that is an isotropic function of two symmetric tensor fields, as in Equation (28), can be expressed as a function of ten scalar invariants [Spencer 1972]; specifically:  $\text{tr}(\bar{T})$ ,  $\text{tr}(\bar{T}\bar{T})$ ,  $\text{tr}(\bar{T}\bar{T}\bar{T})$ ,  $\text{tr}(\dot{E})$ ,  $\text{tr}(\dot{E}\dot{E})$ ,  $\text{tr}(\dot{E}\dot{E}\dot{E})$ ,  $\text{tr}(\bar{T}\dot{E})$ ,  $\text{tr}(\bar{T}\dot{E}\bar{T})$ ,  $\text{tr}(\dot{E}\bar{T}\dot{E})$ , and  $\text{tr}(\bar{T}\dot{E}\bar{T}\dot{E})$ . Imposing the hypoelastic constraint that  $\Psi$  be quadratic in  $\dot{E}$ , while also assuming that  $\Psi$  be at most linear in  $\bar{T}$ , in accordance with Fung’s law (Equation (1)), allows one to write down a general potential function that has merit for modeling biological tissues; it being,

$$\Psi = \frac{1}{2}(\alpha_1 \text{tr}(\dot{E})^2 + \alpha_2 \text{tr}(\dot{E}\dot{E}) + \alpha_3 \text{tr}(\bar{T}) \text{tr}(\dot{E})^2 + \alpha_4 \text{tr}(\bar{T}) \text{tr}(\dot{E}\dot{E}) + 2\alpha_5 \text{tr}(\bar{T}\dot{E}) \text{tr}(\dot{E}) + \alpha_6 \text{tr}(\dot{E}\bar{T}\dot{E})), \quad (29)$$

wherein the  $\alpha_1, \dots, \alpha_6$  are material constants, thereby resulting in the general constitutive equation

$$\dot{\bar{T}} = ((\alpha_1 + \alpha_3 \text{tr} \bar{T}) \text{tr} \dot{E} + \alpha_5 \text{tr}(\bar{T}\dot{E}))\bar{I} + (\alpha_2 + \alpha_4 \text{tr} \bar{T})\dot{E} + \alpha_5 \text{tr}(\dot{E})\bar{T} + \alpha_6 \text{sym}(\bar{T}\dot{E}), \quad (30)$$

whose structure Truesdell [1955] calls a “hypoelastic material of grade one”. For isochoric flows and for materials whose bulk modulus greatly exceeds its shear modulus, like soft tissues [Fung 1973], this potential function can be further simplified to

$$\Psi = \mu \text{tr}(\dot{E}\dot{E}) + \frac{\alpha}{3} \text{tr}(\bar{T}) \text{tr}(\dot{E}\dot{E}) + \beta \text{tr}(\dot{E}\bar{T}\dot{E}) - \wp(\text{tr} \dot{E} - 0), \quad (31)$$

because  $\text{tr} \dot{E} \equiv \text{tr} D = 0$ , which in turn produces the constitutive equation

$$\dot{\bar{T}} + \wp\bar{I} = 2(\mu - \alpha p)\dot{E} + 2\beta \text{sym}(\bar{T}\dot{E}), \quad (32)$$

<sup>4</sup>Truesdell [1955] used the upper-convected derivative of Oldroyd [1950] for establishing stress rate in his original formulation of hypoelasticity, although no mention was made by Truesdell of Oldroyd’s pioneering work in the field.

where  $\mu$  is the shear modulus,  $\alpha$  and  $\beta$  are dimensionless material constants,  $p = -\frac{1}{3} \text{tr } \bar{\mathbf{T}}$  is the hydrostatic pressure, and  $\hat{\phi}$  is a Lagrange multiplier that is distinct from pressure  $p$ . This Lagrange multiplier is a workless constraint, that is,  $\text{tr } \bar{\mathbf{E}} = 0$ , whose value is indeterminate, and can only be fixed after a boundary value problem has been specified.

Equations (30) and (32) rotate into the Eulerian frame  $\kappa$  by premultiplying them with  $\mathbf{R}$ , and postmultiplying them with  $\mathbf{R}^T$ , thereby producing

$$\hat{\mathbf{T}} = ((\alpha_1 + \alpha_3 \text{tr } \mathbf{T}) \text{tr } \mathbf{D} + \alpha_5 \text{tr}(\mathbf{T}\mathbf{D}))\mathbf{I} + (\alpha_2 + \alpha_4 \text{tr } \mathbf{T})\mathbf{D} + \alpha_5 \text{tr}(\mathbf{D})\mathbf{T} + \alpha_6 \text{sym}(\mathbf{T}\mathbf{D}) \quad (33)$$

and

$$\hat{\mathbf{T}} + \hat{\phi}\mathbf{I} = 2(\mu - \alpha p)\mathbf{D} + 2\beta \text{sym}(\mathbf{T}\mathbf{D}), \quad (34)$$

respectively, where now  $p = -\frac{1}{3} \text{tr } \mathbf{T} \equiv -\frac{1}{3} \text{tr } \bar{\mathbf{T}}$ . Equations (32) and (34) are isotropic three-dimensional generalizations of Fung's one-dimensional law (Equation (1)) for the  $\bar{\kappa}$  and  $\kappa$  configurations, respectively.

**Remark 4.** By assigning  $\alpha \equiv \beta$ , the material model listed in Equations (32) and (34) for the  $\bar{\kappa}$  and  $\kappa$  configurations, respectively, predicts the same stiffening ratios for  $\alpha$  ( $\beta \equiv \alpha$ ) and  $\mu$  as the neo-Hookean solid does for its shear modulus  $\mu$  [Treloar 1975], as they pertain to differences between the experiments of uniaxial and equibiaxial extension (that is, the modulus coefficients are 3 and 6, respectively).

**7.2. Soft tissues.** Existing elastic material models for soft tissues are predominantly *hyperelastic* in construction, and typically consider the stress carried by collagen fibers to be modeled as some generalization of an integrated Fung-like solid (see the review articles of Humphrey [2002], Sacks [2000], Viidik [1973], and Weiss and Gardiner [2001]). In particular, stress is taken to be proportional to some deformation tensor, for example,  $\mathbf{B}$ , with its coefficient of proportionality being a scalar-valued measure of strain written as a Fung-like exponential, as in Equation (2), whose argument is expressed in terms of various components or invariants of strain, as in Equation (3). A notable exception to these theories is the theory of Criscione et al. [2003a] who derived a transversely isotropic pseudo-elastic model for describing the passive response of biomembranes.

In this paper, we break with this longstanding tradition by extending Truesdell's [1955] isotropic, *hypoelastic*, constitutive theory into an anisotropic constitutive theory that is capable of modeling the passive response of soft tissues. Specifically, we consider a class of materials whose anisotropic and isotropic constituents are governed by the respective constitutive formulae

$$\dot{\bar{\mathbf{T}}}^f = \frac{\partial \Psi^f(\bar{\mathbf{T}}^f, \dot{\bar{\mathbf{E}}}^f)}{\partial \dot{\bar{\mathbf{E}}}^f} \quad \text{and} \quad \dot{\bar{\mathbf{T}}}^m = \frac{\partial \Psi^m(\bar{\mathbf{T}}^m, \dot{\bar{\mathbf{E}}}^m)}{\partial \dot{\bar{\mathbf{E}}}^m}, \quad (35)$$

with there being no coupling between the arguments of the two material functions  $\Psi^f$  and  $\Psi^m$ . Consequently, results from the prior section addressing isotropic hypoelastic materials can be applied straightaway, because of Prop. 1, to that of an anisotropic solid which belongs to this class of materials.

In accordance with the constitutive assumption in Equation (35), and assuming an isochoric response, one obtains the general potential

$$\Psi^i = \mu_i \text{tr}(\dot{\bar{\mathbf{E}}}^i \dot{\bar{\mathbf{E}}}^i) + \frac{\alpha_i}{3} \text{tr}(\bar{\mathbf{T}}^i) \text{tr}(\dot{\bar{\mathbf{E}}}^i \dot{\bar{\mathbf{E}}}^i) + \beta_i \text{tr}(\dot{\bar{\mathbf{E}}}^i \bar{\mathbf{T}}^i \dot{\bar{\mathbf{E}}}^i), \quad i \in \{f, m\},$$



that leads to the constitutive formulation

$$\dot{\bar{\mathbf{T}}}^i = 2(\mu_i + \frac{\alpha_i}{3} \text{tr} \bar{\mathbf{T}}^i) \dot{\bar{\mathbf{E}}}^i + 2\beta_i \text{sym}(\bar{\mathbf{T}}^i \dot{\bar{\mathbf{E}}}^i), \quad i \in \{f, m\}, \tag{36}$$

each of which has three parameters:  $\mu_i$  has units of stress and is the small-strain shear modulus of phase  $i$ , while  $\alpha_i$  and  $\beta_i$  are dimensionless parameters that govern the rate of exponential growth in stress with increasing strain in phase  $i$ , wherein  $\alpha_i$  is a bulk-like parameter in that it accounts for influences caused by variations in phase pressure. Combining these formulae, in accordance with Prop. 1, implies that the overall polar stress is governed by the differential equation

$$\dot{\bar{\mathbf{T}}} + \dot{\phi} \bar{\mathbf{I}} = \phi \dot{\bar{\mathbf{T}}}^f + (1 - \phi) \dot{\bar{\mathbf{T}}}^m, \tag{37}$$

where  $\dot{\phi}$  is a Lagrange multiplier used to enforce the constraint  $\text{tr} \dot{\bar{\mathbf{E}}} = 0$ , and where  $\phi$  is the fiber volume fraction.

As with isotropic materials, one can rotate this polar equation for hypoelastic, soft, biological tissues, Equation (36), into the spatial frame  $\kappa$ , resulting in

$$\hat{\mathbf{T}}^i = 2(\mu_i + \frac{\alpha_i}{3} \text{tr} \mathbf{T}^i) \hat{\mathbf{E}}^i + 2\beta_i \text{sym}(\mathbf{T}^i \hat{\mathbf{E}}^i), \quad i \in \{f, m\},$$

while Equation (37) maps into  $\kappa$  as

$$\hat{\mathbf{T}} + \dot{\phi} \mathbf{I} = \phi \hat{\mathbf{T}}^f + (1 - \phi) \hat{\mathbf{T}}^m,$$

wherein, from Equations ((6), (25) and (26)) and Remark 3, the various polar rates of strain obey

$$\hat{\mathbf{E}} = \hat{\mathbf{E}}^f + \hat{\mathbf{E}}^m \equiv \mathbf{D} \text{ with } \hat{\mathbf{E}}^f = \frac{1}{2}(\mathbf{A}(\mathbf{L} - \boldsymbol{\Omega}) + (\mathbf{L}^T + \boldsymbol{\Omega})\mathbf{A}) \equiv \frac{1}{2}(\mathbf{A}(\mathbf{D} + \mathbf{Z}) + (\mathbf{D} - \mathbf{Z})\mathbf{A}),$$

and therefore  $\hat{\mathbf{E}}^m = \mathbf{D} - \hat{\mathbf{E}}^f$  recalling that  $\mathbf{A} = \mathbf{R}\bar{\mathbf{A}}\mathbf{R}^T \equiv \mathbf{R}\mathbf{A}_0\mathbf{R}^T$  from Remark 3.

Applications of this theory to specific tissues will follow in future papers.

### 8. Summary

A constitutive theory designed for modeling the passive behavior of soft biological tissues has been derived. It contains several novel features. Foremost, a hypoelastic construction has been adopted, as in Fung’s original one-dimensional model, instead of the hyperelastic constructions that pervade the literature. Anisotropy has been introduced as a geometric property separate from constitutive construction, and a measure for strain rate that arises naturally from the work potential has been used as the independent variable of the theory. The theory was derived in the polar configuration to ensure objectivity, and then mapped into the spatial (Eulerian) configuration.

### 9. Acknowledgment

The author thanks the anonymous reviewers who helped make this document more precise.

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Received 9 Jan 2008. Revised 21 Mar 2008. Accepted 5 Apr 2008.

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