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SPATIAL BEHAVIOUR FOR CONSTRAINED MOTION OF A CYLINDER MADE OF A STRONGLY ELLIPTIC ANISOTROPIC MATERIAL

VINCENZO TIBULLO AND MASSIMO VACCARO

This paper studies the spatial behaviour for the motion of a semi-infinite cylinder composed of an anisotropic linear elastic material and subject to zero body force and zero lateral boundary conditions. The elasticity tensor is strongly elliptic, with motion induced by a time-dependent displacement specified pointwise over the base.

1. Introduction

We consider a semi-infinite prismatic cylinder occupied by an anisotropic linear elastic material and subject to zero body force, zero lateral boundary conditions, and zero initial conditions. The motion is induced by a time-dependent displacement specified pointwise over the base. The elasticity tensor is strongly elliptic.

The primary purpose of this paper is to examine how the solutions evolve with respect to the axial variable. To this end, we make an association with the solution of the initial boundary value problem of an appropriate time-weighted cross-section power function, and prove that the strong ellipticity conditions assure this to be an acceptable measure. We then establish a set of differential inequalities that describe the spatial behavior of the measure in concern. This proves that there is a positive constant γ such that the whole activity is zero in that part of the cylinder where the axial distance to the loaded end is greater than γt (that is, a domain influence result holds true), while in the remaining part an exponential decay estimate of Saint Venant type holds true. The results are illustrated for transversely isotropic materials as well as for the rhombic systems.

2. Formulation of problem

Consider a semi-infinite prismatic cylinder $B \subset \mathbb{R}^3$ whose bounded uniform cross section $D \subset \mathbb{R}^2$ has a piecewise continuously differentiable boundary ∂D. The origin of a rectangular Cartesian coordinate system is located in the cylinder's base, and the positive x_3-axis is directed along that of the cylinder. It is convenient to introduce the further abbreviation

$$B_z = \{x \in B : z > x_3\}. \tag{2.1}$$

Moreover, we employ $D(x_3, t)$ to indicate that relevant quantities are to be evaluated at time t over the cross section whose distance from the origin is x_3.

The cylinder is occupied by an anisotropic elastic material and is subject to a deformation in which the displacement field $u(x, t)$ is a smooth vector function satisfying the requirements of the classical

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dynamical theory [Gurtin 1972]. The corresponding stress tensor $\mathbf{S}(x,t)$ has Cartesian components given by

$$S_{ij} = C_{ijkl} u_{k,l}, \tag{2.2}$$

where the constant elasticities C_{ijkl} possess the symmetries

$$C_{ijkl} = C_{klij} = C_{jikl} \tag{2.3}$$

and satisfy the strong ellipticity condition

$$C_{ijkl} m_i m_k n_j n_l > 0 \quad \text{for all nonzero vectors } (m_1, m_2, m_3),\ (n_1, n_2, n_3). \tag{2.4}$$

The cylinder is set in motion with a pointwise prescribed base displacement, zero body force, zero initial conditions, and zero displacement on the lateral surface. Furthermore, the prescribed displacements are such that a classical solution exists on the interval $[0, \infty)$. Consequently, the problem to be considered is specified by

$$(C_{jikl} u_{k,l})_{,j} = \rho \ddot{u}_i, \qquad (x,t) \in B \times [0,\infty), \tag{2.5}$$

$$u_i(x,0) = 0, \qquad \dot{u}_i(x,0) = 0, \quad x \in B, \tag{2.6}$$

$$u_i(x,t) = 0, \qquad (x,t) \in \partial D \times [0,\infty) \times [0,\infty), \tag{2.7}$$

$$u_i(x,t) = f_i(x_1, x_2, t), \qquad (x,t) \in D(0) \times [0,\infty), \tag{2.8}$$

where the dots denote differentiation with respect to time, a subscript comma indicates partial differentiation, ρ is the constant positive mass density, and $f_i\,(x_1, x_2, t)$ is a prescribed differentiable function compatible with the initial and lateral boundary conditions.

3. Transversely isotropic materials

Many natural and man made materials are classified as *transversely isotropic* (or *hexagonal*). Such materials are characterized by the fact that one can find a line that allows a rotation of the material about it without changing its properties. The plane, which is perpendicular to this line (the axis of rotational symmetry) is called a *plane of elastic symmetry* or *plane of isotropy*. A modern example of such a material is a laminate made of randomly oriented, chopped fibers that are in general placed in a particular plane. The effective material properties for a bundled structure have no preferred direction in that plane, which makes it a plane of elastic symmetry. Each plane that contains an axis of rotation is a plane of symmetry. Therefore, transversely isotropic materials admit an infinite number of elastic symmetries.

Necessary and sufficient conditions for strong ellipticity to hold for a transversely isotropic linearly elastic solid are established by Merodio and Ogden [2003] and Chiriţă [2006]. In this connection we recall the standard notation

$$c_{ij} = C_{iijj}, \quad i, j \in \{1,2,3\} \quad \text{(not summed)},$$

$$c_{22} = c_{11}, \quad c_{23} = c_{13}, \quad c_{44} = c_{55} = C_{2323} = C_{1313}, \quad c_{66} = C_{1212} = \tfrac{1}{2}(c_{11} - c_{12}), \tag{3.1}$$

corresponding to the direction of transverse isotropy coinciding with the x_3 coordinate axis. Apart from terms obtained by use of the symmetries (2.3), these are the only nonzero components C_{ijkl}. Then the necessary and sufficient conditions for strong ellipticity to hold are [Merodio and Ogden 2003; Chiriţă 2006; Chiriţă and Ciarletta 1999]

$$c_{11} > 0, \quad c_{33} > 0, \quad c_{55} > 0, \quad c_{11} > c_{12}, \tag{3.2}$$

$$|c_{13} + c_{55}| < c_{55} + \sqrt{c_{11}c_{33}}. \tag{3.3}$$

Moreover, equations (2.5) become

$$\begin{aligned}
\rho\ddot{u}_1 &= c_{11}u_{1,11} + (c_{12}+c_{66})\,u_{2,21} + c_{66}u_{1,22} + (c_{13}+c_{55})\,u_{3,31} + c_{55}u_{1,33},\\
\rho\ddot{u}_2 &= (c_{12}+c_{66})\,u_{1,12} + c_{66}u_{2,11} + c_{11}u_{2,22} + (c_{13}+c_{55})\,u_{3,32} + c_{55}u_{2,33},\\
\rho\ddot{u}_3 &= c_{55}\left(u_{3,11}+u_{3,22}\right) + (c_{13}+c_{55})\left(u_{1,1}+u_{2,2}\right)_{,3} + c_{33}u_{3,33}.
\end{aligned} \tag{3.4}$$

By a straightforward calculation from the basic equations (3.4) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big\{\tfrac{1}{2}e^{-\sigma t}[\rho\dot{u}_i\dot{u}_i + c_{11}(u_{1,1}+u_{2,2})^2 + c_{66}(u_{1,2}-u_{2,1})^2 + c_{33}u_{3,3}^2 + c_{55}(u_{3,1}^2+u_{3,2}^2+u_{1,3}^2+u_{2,3}^2)\\
&\qquad + 2(c_{13}+c_{55})(u_{1,3}u_{3,1}+u_{2,3}u_{3,2}) + 2(c_{66}u_{1,2}u_2)_{,1} - 2(c_{66}u_{1,1}u_2)_{,2}]\Big\}\\
&\quad + \tfrac{1}{2}\sigma e^{-\sigma t}[\rho\dot{u}_i\dot{u}_i + c_{11}(u_{1,1}+u_{2,2})^2 + c_{66}(u_{1,2}-u_{2,1})^2 + c_{33}u_{3,3}^2 + c_{55}(u_{3,1}^2+u_{3,2}^2+u_{1,3}^2+u_{2,3}^2)\\
&\qquad + 2(c_{13}+c_{55})(u_{1,3}u_{3,1}+u_{2,3}u_{3,2}) + 2(c_{66}u_{1,2}u_2)_{,1} - 2(c_{66}u_{1,1}u_2)_{,2}]\\
&= e^{-\sigma t}\left\{\dot{u}_1[(c_{13}+c_{55})u_{3,1}+c_{55}u_{1,3}] + \dot{u}_2[(c_{13}+c_{55})u_{3,2}+c_{55}u_{2,3}] + c_{33}\dot{u}_3u_{3,3}\right\}_{,3}\\
&\quad + e^{-\sigma t}\left\{\dot{u}_1[c_{11}u_{1,1}+(c_{12}+c_{66})u_{2,2}] + c_{66}u_{2,1}\dot{u}_2 + \dot{u}_3[c_{55}u_{3,1}+(c_{13}+c_{55})u_{1,3}]\right\}_{,1}\\
&\quad + e^{-\sigma t}\left\{c_{66}u_{1,2}\dot{u}_1 + \dot{u}_2[(c_{12}+c_{66})u_{1,1}+c_{11}u_{2,2}] + \dot{u}_3[c_{55}u_{3,2}+(c_{13}+c_{55})u_{2,3}]\right\}_{,2},
\end{aligned} \tag{3.5}$$

where σ is a strictly positive parameter at our disposal.

In view of our null initial and lateral boundary conditions (2.6) and (2.7), and by direct integration of (3.5) over $D(x_3, t) \times [0, t]$, we get

$$\begin{aligned}
&\frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}[\rho\dot{u}_i\dot{u}_i + c_{11}(u_{1,1}+u_{2,2})^2 + c_{66}(u_{1,2}-u_{2,1})^2 + c_{33}u_{3,3}^2\\
&\qquad + c_{55}(u_{3,1}^2+u_{3,2}^2+u_{1,3}^2+u_{2,3}^2) + 2(c_{13}+c_{55})(u_{1,3}u_{3,1}+u_{2,3}u_{3,2})]\,\mathrm{d}a\\
&\quad + \frac{\sigma}{2}\int_0^t\int_{D(x_3,s)} e^{-\sigma s}[\rho\dot{u}_i\dot{u}_i + c_{11}(u_{1,1}+u_{2,2})^2 + c_{66}(u_{1,2}-u_{2,1})^2 + c_{33}u_{3,3}^2\\
&\qquad + c_{55}(u_{3,1}^2+u_{3,2}^2+u_{1,3}^2+u_{2,3}^2) + 2(c_{13}+c_{55})(u_{1,3}u_{3,1}+u_{2,3}u_{3,2})]\,\mathrm{d}a\,\mathrm{d}s\\
&= \left(\int_0^t\int_{D(x_3,s)} e^{-\sigma s}\{\dot{u}_\alpha[(c_{13}+c_{55})u_{3,\alpha}+c_{55}u_{\alpha,3}] + c_{33}\dot{u}_3u_{3,3}\}\,\mathrm{d}a\,\mathrm{d}s\right)_{,3}.
\end{aligned} \tag{3.6}$$

Moreover, on the basis of the lateral boundary condition (2.7), we have

$$\int_{D(x_3,t)} e^{-\sigma t}u_{\alpha,\alpha}u_{3,3}\,\mathrm{d}a = \left(\int_{D(x_3,t)} e^{-\sigma t}u_{\alpha,\alpha}u_3\,\mathrm{d}a\right)_{,3} + \int_{D(x_3,t)} e^{-\sigma t}u_{\alpha,3}u_{3,\alpha}\,\mathrm{d}a. \tag{3.7}$$

Employing the initial condition (2.6) gives

$$\int_{D(x_3,t)} e^{-\sigma t} u_{\alpha,\alpha} u_3 \,\mathrm{d}a = \int_0^t \int_{D(x_3,s)} e^{-\sigma s} \left(\dot{u}_{\alpha,\alpha} u_3 + u_{\alpha,\alpha} \dot{u}_3 - \sigma u_3 u_{\alpha,\alpha}\right) \mathrm{d}a \,\mathrm{d}s$$
$$= \int_0^t \int_{D(x_3,s)} e^{-\sigma s} \left(-\dot{u}_{\alpha} u_{3,\alpha} + u_{\alpha,\alpha} \dot{u}_3 - \sigma u_3 u_{\alpha,\alpha}\right) \mathrm{d}a \,\mathrm{d}s. \tag{3.8}$$

Therefore, relations (3.7) and (3.8) imply

$$\int_{D(x_3,t)} e^{-\sigma t} u_{\alpha,\alpha} u_{3,3} \,\mathrm{d}a$$
$$= \left(\int_0^t \int_{D(x_3,s)} e^{-\sigma s} \left(-\dot{u}_{\alpha} u_{3,\alpha} + u_{\alpha,\alpha} \dot{u}_3 - \sigma u_3 u_{\alpha,\alpha}\right) \mathrm{d}a \,\mathrm{d}s\right)_{,3} + \int_{D(x_3,t)} e^{-\sigma t} u_{\alpha,3} u_{3,\alpha} \,\mathrm{d}a. \tag{3.9}$$

By combining relations (3.6) and (3.9) we obtain the identity

$$\frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t} \rho \dot{u}_i \dot{u}_i \,\mathrm{d}a$$
$$+ \frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}[c_{11}(u_{1,1}+u_{2,2})^2 + c_{66}(u_{1,2}-u_{2,1})^2 + c_{33}u_{3,3}^2 + 2(c_{13}+\varkappa)(u_{1,1}+u_{2,2})u_{3,3}]\,\mathrm{d}a$$
$$+ \frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}[c_{55}(u_{3,1}^2 + u_{3,2}^2 + u_{1,3}^2 + u_{2,3}^2) + 2(c_{55}-\varkappa)(u_{1,3}u_{3,1} + u_{2,3}u_{3,2})]\,\mathrm{d}a$$
$$+ \frac{\sigma}{2}\int_0^t \int_{D(x_3,s)} e^{-\sigma s} \rho \dot{u}_i \dot{u}_i \,\mathrm{d}a \,\mathrm{d}s$$
$$+ \frac{\sigma}{2}\int_0^t \int_{D(x_3,s)} e^{-\sigma s}[c_{11}(u_{1,1}+u_{2,2})^2 + c_{66}(u_{1,2}-u_{2,1})^2 + c_{33}u_{3,3}^2 + 2(c_{13}+\varkappa)(u_{1,1}+u_{2,2})u_{3,3}]\,\mathrm{d}a \,\mathrm{d}s$$
$$+ \frac{\sigma}{2}\int_0^t \int_{D(x_3,s)} e^{-\sigma s}[c_{55}(u_{3,1}^2 + u_{3,2}^2 + u_{1,3}^2 + u_{2,3}^2) + 2(c_{55}-\varkappa)(u_{1,3}u_{3,1} + u_{2,3}u_{3,2})]\,\mathrm{d}a \,\mathrm{d}s$$
$$= \left(\int_0^t \int_{D(x_3,s)} e^{-\sigma s}\{\dot{u}_{\alpha}[(c_{55}-\varkappa)u_{3,\alpha} + c_{55}u_{\alpha,3}] + \dot{u}_3[c_{33}u_{3,3} + (c_{13}+\varkappa)u_{\alpha,\alpha}]\}\,\mathrm{d}a \,\mathrm{d}s\right)_{,3}, \tag{3.10}$$

where $\varkappa \in (0, 2c_{55})$ is a positive parameter at our disposal.

This identity suggests that we treat the spatial behaviour of solutions by introducing the following function

$$J_{\varkappa}(x_3, t) = -\int_0^t \int_{D(x_3,s)} e^{-\sigma s} \left\{\dot{u}_{\alpha}\left[(c_{55}-\varkappa)\, u_{3,\alpha} + c_{55}u_{\alpha,3}\right] + \dot{u}_3\left[c_{33}u_{3,3} + (c_{13}+\varkappa)\, u_{\alpha,\alpha}\right]\right\} \mathrm{d}a \,\mathrm{d}s, \tag{3.11}$$

with domain $x_3 \in [0, \infty)$, $t \in [0, \infty)$.

Theorem 1. *Let $u_i(x, t)$ be a solution of the initial boundary value problem defined by (3.4), (2.6), (2.7) and (2.8). Then $J_{\varkappa}(x_3, t)$ as defined by (3.11) represents a measure of the solution, and, for each $t > 0$, evolves as follows. There is a positive constant γ, depending on the elastic constants, such that*

(a) *for* $x_3 \geq \gamma t$ *we have* $J_\varkappa(x_3, t) = 0$ *so that*

$$u_i(x_1, x_2, x_3, t) = 0 \quad \text{for all } x_3 \geq \gamma t; \tag{3.12}$$

(b) *for* $x_3 \leq \gamma t$, *the spatial behavior is described by*

$$0 \leq J_\varkappa(x_3, t) \leq J_\varkappa(0, t) \exp\Big(-\frac{\sigma}{\gamma} x_3\Big). \tag{3.13}$$

Proof. From (3.10) and (3.11), we can deduce

$$\begin{aligned}
-\frac{\partial J_\varkappa}{\partial x_3}&(x_3, t)\\
&= \frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t} \rho \dot{u}_i \dot{u}_i \,\mathrm{d}a\\
&\quad + \frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}[c_{11}(u_{1,1}+u_{2,2})^2 + c_{66}(u_{1,2}-u_{2,1})^2 + c_{33}u_{3,3}^2 + 2(c_{13}+\varkappa)(u_{1,1}+u_{2,2})u_{3,3}]\,\mathrm{d}a\\
&\quad + \frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}[c_{55}(u_{3,1}^2 + u_{3,2}^2 + u_{1,3}^2 + u_{2,3}^2) + 2(c_{55}-\varkappa)(u_{1,3}u_{3,1} + u_{2,3}u_{3,2})]\,\mathrm{d}a\\
&\quad + \frac{\sigma}{2}\int_0^t\int_{D(x_3,t)} e^{-\sigma s} \rho \dot{u}_i \dot{u}_i \,\mathrm{d}a\,\mathrm{d}s\\
&\quad + \frac{\sigma}{2}\int_0^t\int_{D(x_3,s)} e^{-\sigma s}[c_{11}(u_{1,1}+u_{2,2})^2 + c_{66}(u_{1,2}-u_{2,1})^2 + c_{33}u_{3,3}^2 + 2(c_{13}+\varkappa)(u_{1,1}+u_{2,2})u_{3,3}]\,\mathrm{d}a\,\mathrm{d}s\\
&\quad + \frac{\sigma}{2}\int_0^t\int_{D(x_3,s)} e^{-\sigma s}[c_{55}(u_{3,1}^2 + u_{3,2}^2 + u_{1,3}^2 + u_{2,3}^2) + 2(c_{55}-\varkappa)(u_{1,3}u_{3,1} + u_{2,3}u_{3,2})]\,\mathrm{d}a\,\mathrm{d}s.
\end{aligned} \tag{3.14}$$

Now, in view of (3.3), we can choose $\varkappa \in (0, 2c_{55})$ so that

$$\max(-c_{13} - \sqrt{c_{11}c_{33}}, 0) < \varkappa < \min(2c_{55}, -c_{13} + \sqrt{c_{11}c_{33}}) \tag{3.15}$$

and hence to have

$$|c_{55} - \varkappa| < c_{55}. \tag{3.16}$$

Consequently, we obtain

$$c_{55}(u_{3,1}^2 + u_{3,2}^2 + u_{1,3}^2 + u_{2,3}^2) + 2(c_{55}-\varkappa)(u_{1,3}u_{3,1} + u_{2,3}u_{3,2}) \geq \nu_1(u_{3,1}^2 + u_{3,2}^2 + u_{1,3}^2 + u_{2,3}^2) \tag{3.17}$$

and

$$\begin{aligned}
c_{11}(u_{1,1}+u_{2,2})^2 + c_{66}(u_{1,2}-u_{2,1})^2 + c_{33}u_{3,3}^2 + 2(c_{13}+\varkappa)(u_{1,1}+u_{2,2})u_{3,3}&\\
\geq \nu_2[(u_{1,1}+u_{2,2})^2 + u_{3,3}^2],&
\end{aligned} \tag{3.18}$$

where

$$\nu_1 = \min(2c_{55} - \varkappa, \varkappa), \quad \nu_2 = \tfrac{1}{2}\Big(c_{11} + c_{33} - \sqrt{(c_{11}-c_{33})^2 + 4(c_{13}+\varkappa)^2}\Big). \tag{3.19}$$

With these in mind, we can see that

$$
\begin{aligned}
-\frac{\partial J_\varkappa}{\partial x_3}(x_3,t) \geq{} & \frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}\rho\dot{u}_i\dot{u}_i\,\mathrm{d}a + \frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}\nu_2[(u_{1,1}+u_{2,2})^2+u_{3,3}^2]\,\mathrm{d}a \\
& + \frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}\nu_1(u_{3,1}^2+u_{3,2}^2+u_{1,3}^2+u_{2,3}^2)\,\mathrm{d}a + \frac{\sigma}{2}\int_0^t\int_{D(x_3,s)} e^{-\sigma s}\rho\dot{u}_i\dot{u}_i\,\mathrm{d}a\,\mathrm{d}s \\
& + \frac{\sigma}{2}\int_0^t\int_{D(x_3,s)} e^{-\sigma s}\nu 2[(u_{1,1}+u_{2,2})^2+u_{3,3}^2]\,\mathrm{d}a\,\mathrm{d}s \\
& + \frac{\sigma}{2}\int_0^t\int_{D(x_3,s)} e^{-\sigma s}\nu_1(u_{3,1}^2+u_{3,2}^2+u_{1,3}^2+u_{2,3}^2)\,\mathrm{d}a\,\mathrm{d}s \\
\geq{} & 0. \qquad (3.20)
\end{aligned}
$$

On the other hand, by direct differentiation in (3.11) we obtain

$$
\frac{\partial J_\varkappa}{\partial t}(x_3,t) = -\int_{D(x_3,t)} e^{-\sigma t}\left\{\dot{u}_\alpha[(c_{55}-\varkappa)\,u_{3,\alpha}+c_{55}u_{\alpha,3}]+\dot{u}_3[c_{55}u_{3,3}+(c_{13}+\varkappa)\,u_{\alpha,\alpha}]\right\}\mathrm{d}a. \qquad (3.21)
$$

Furthermore, by means of the arithmetic–geometric and Schwarz inequalities, we obtain

$$
\begin{aligned}
\left|\frac{\partial J_\varkappa}{\partial t}(x_3,t)\right| \leq \int_{D(x_3,t)} e^{-\sigma t}\Big\{ & \max(|c_{55}-\varkappa|,c_{55})[\varepsilon_1\dot{u}_\alpha\dot{u}_\alpha+\frac{1}{2\varepsilon_1}(u_{3,\alpha}u_{3,\alpha}+u_{\alpha,3}u_{\alpha,3})] \\
& + \max(|c_{13}+\varkappa|,c_{55})[\varepsilon_2\dot{u}_3\dot{u}_3+\frac{1}{2\varepsilon_2}(u_{\alpha,\alpha}u_{\rho,\rho}+u_{3,3}^2)]\Big\}\,\mathrm{d}a, \qquad (3.22)
\end{aligned}
$$

so that, by setting $\varepsilon_1=\sqrt{\frac{\rho}{2\nu_1}}$ and $\varepsilon_2=\sqrt{\frac{\rho}{2\nu_2}}$, we get

$$
\begin{aligned}
\left|\frac{\partial J_\varkappa}{\partial t}(x_3,t)\right| \leq{} & \frac{c_{55}}{\sqrt{2\rho\nu_1}}\frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}\left[\rho\dot{u}_\alpha\dot{u}_\alpha+\nu_1\left(u_{3,\alpha}u_{3,\alpha}+u_{\alpha,3}u_{\alpha,3}\right)\right]\mathrm{d}a \\
& + \frac{\max\left(\sqrt{c_{11}c_{33}},c_{55}\right)}{\sqrt{2\rho\nu_2}}\frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}\left[\rho\dot{u}_3\dot{u}_3+\nu_2\left(u_{\alpha,\alpha}u_{\rho,\rho}+u_{3,3}^2\right)\right]\mathrm{d}a. \qquad (3.23)
\end{aligned}
$$

Thus, if we set

$$
\gamma = \max\left(\frac{c_{55}}{\sqrt{2\rho\nu_1}},\frac{\max\left(\sqrt{c_{11}c_{33}},c_{55}\right)}{\sqrt{2\rho\nu_2}}\right), \qquad (3.24)
$$

by means of the relations (3.20) and (3.23) we obtain the first-order partial differential inequality

$$
\left|\frac{\partial J_\varkappa}{\partial t}(x_3,t)\right| \leq -\gamma\frac{\partial J_\varkappa}{\partial x_3}(x_3,t), \qquad (x_3,t)\in[0,\infty)\times[0,\infty). \qquad (3.25)
$$

Using an estimating procedure like the one above we obtain the first-order differential inequality

$$
\frac{\sigma}{\gamma}\left|J_\varkappa(x_3,t)\right| \leq -\frac{\partial J_\varkappa}{\partial x_3}(x_3,t), \qquad (x_3,t)\in[0,\infty)\times[0,\infty). \qquad (3.26)
$$

We now proceed to find the spatial behavior of solutions as described by the differential inequalities above. We start with the study of differential inequality (3.25) and note that it is equivalent to

$$\frac{1}{\gamma}\frac{\partial J_\varkappa}{\partial t}(x_3,t)+\frac{\partial J_\varkappa}{\partial x_3}(x_3,t)\leq 0,\qquad (x_3,t)\in[0,\infty)\times[0,\infty) \tag{3.27}$$

and

$$-\frac{1}{\gamma}\frac{\partial J_\varkappa}{\partial t}(x_3,t)+\frac{\partial J_\varkappa}{\partial x_3}(x_3,t)\leq 0,\qquad (x_3,t)\in[0,\infty)\times[0,\infty). \tag{3.28}$$

Let us fix $t_0>0$ and assume that $x_3^0>\gamma t_0$. If we set $x_3=x_3^0+\gamma\,(t-t_0)$ in relation (3.27), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left[J_\varkappa\left(x_3,\,t_0+\frac{x_3-x_3^0}{\gamma}\right)\right]\leq 0, \tag{3.29}$$

so that, recalling that $0\leq x_3=x_3^0-ct_0\leq x_3^0$, we have

$$J_\varkappa\left(x_3^0,\,t_0\right)\leq J_\varkappa\left(x_3^0-ct_0,\,0\right)=0. \tag{3.30}$$

Moreover, by setting $x_3=x_3^0-\gamma\,(t-t_0)$ in (3.28), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left[J_\varkappa\left(x_3,\,t_0-\frac{x_3-x_3^0}{\gamma}\right)\right]\leq 0, \tag{3.31}$$

and hence we have

$$0=J_\varkappa\left(x_3^0+\gamma t_0,\,0\right)\leq J_\varkappa\left(x_3^0,\,t_0\right). \tag{3.32}$$

Thus, by making $x_3^0\to\infty$ in relations (3.30) and (3.32), we get

$$J_\varkappa\left(\infty,\,t_0\right)\equiv\lim_{x_3\to\infty}J_\varkappa\left(x_3,\,t_0\right)=0. \tag{3.33}$$

Consequently, by using the relations (3.14) and (3.32), we obtain

$$\begin{aligned}
&J_\varkappa(x_3,t)\\
&\;=\frac{1}{2}\int_{B(x_3,t)}e^{-\sigma t}\rho\dot{u}_i\dot{u}_i\,\mathrm{d}v\\
&\;\quad+\frac{1}{2}\int_{B(x_3,t)}e^{-\sigma t}[c_{11}(u_{1,1}+u_{2,2})^2+c_{66}(u_{1,2}-u_{2,1})^2+c_{33}u_{3,3}^2+2(c_{13}+\varkappa)(u_{1,1}+u_{2,2})u_{3,3}]\,\mathrm{d}v\\
&\;\quad+\frac{1}{2}\int_{B(x_3,t)}e^{-\sigma t}[c_{55}(u_{3,1}^2+u_{3,2}^2+u_{1,3}^2+u_{2,3}^2)+2(c_{55}-\varkappa)(u_{1,3}u_{3,1}+u_{2,3}u_{3,2})]\,\mathrm{d}v\\
&\;\quad+\frac{\sigma}{2}\int_0^t\int_{B(x_3,t)}e^{-\sigma s}\rho\dot{u}_i\dot{u}_i\,\mathrm{d}v\,\mathrm{d}s\\
&\;\quad+\frac{\sigma}{2}\int_0^t\int_{B(x_3,s)}e^{-\sigma s}[c_{11}(u_{1,1}+u_{2,2})^2+c_{66}(u_{1,2}-u_{2,1})^2+c_{33}u_{3,3}^2+2(c_{13}+\varkappa)(u_{1,1}+u_{2,2})u_{3,3}]\,\mathrm{d}v\,\mathrm{d}s\\
&\;\quad+\frac{\sigma}{2}\int_0^t\int_{D(x_3,s)}e^{-\sigma s}[c_{55}(u_{3,1}^2+u_{3,2}^2+u_{1,3}^2+u_{2,3}^2)+2(c_{55}-\varkappa)(u_{1,3}u_{3,1}+u_{2,3}u_{3,2})]\,\mathrm{d}v\,\mathrm{d}s\\
&\;\geq 0.
\end{aligned} \tag{3.34}$$

This proves that $J_\varkappa(x_3,t)$ represents an acceptable measure for the solution in concern.

Further, we choose $t>0$ and $x_3\geq\gamma t$. By setting $x_3=\gamma t$ in (3.27) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[J_\varkappa(\gamma t,t)\right]\leq 0 \tag{3.35}$$

and hence we have

$$J_\varkappa(\gamma t,t)\leq J_\varkappa(0,0)=0. \tag{3.36}$$

On the other hand, in view of relation (3.20) and recalling that $x_3\geq\gamma t$ we have

$$J_\varkappa(x_3,t)\leq J_\varkappa(\gamma t,t)\leq 0. \tag{3.37}$$

Consequently, the relations (3.34) and (3.37) imply relation (3.12). Moreover, the integration of the differential inequality (3.26) leads to the spatial estimate (3.13) and the proof of theorem is complete. □

4. Rhombic systems

Suppose the cylinder is filled by a rhombic elastic material with the group $\mathscr{C}_3$ generated by $\mathbf{R}^{\pi}_{e_3}$, $\mathbf{R}^{\pi}_{e_2}$ (here $\mathbf{R}^{\theta}_{e}$ is the orthogonal tensor corresponding to a right-handed rotation through the angle $\theta\in(0,2\pi)$, about an axis in the direction of the unit vector $\mathbf{e}$). According to Gurtin [1972], such class of materials is characterized by

$$\begin{gathered}
C_{1123}=C_{1131}=C_{1112}=C_{2223}=C_{2231}=C_{2212}=0,\\
C_{3323}=C_{3331}=C_{3312}=C_{2331}=C_{2312}=C_{3112}=0,\\
c_{11}=C_{1111},\qquad c_{22}=C_{2222},\qquad c_{33}=C_{3333},\\
c_{12}=C_{1122},\qquad c_{23}=C_{2233},\qquad c_{31}=C_{3311},\\
c_{44}=C_{2323},\qquad c_{55}=C_{1313},\qquad c_{66}=C_{1212}.
\end{gathered}$$

The strong ellipticity condition (2.4) becomes

$$\begin{aligned}
c_{11}n_1^2m_1^2+c_{22}n_2^2m_2^2+c_{33}n_3^2m_3^2+c_{66}(n_1m_2+n_2m_1)^2+c_{44}(n_3m_2+n_2m_3)^2\\
+c_{55}(n_1m_3+n_3m_1)^2+2c_{12}n_1m_1n_2m_2+2c_{23}n_2m_2n_3m_3+2c_{31}n_3m_3n_1m_1>0,
\end{aligned} \tag{4.1}$$

for all nonzero vectors (m_1,m_2,m_3) and (n_1,n_2,n_3). It is equivalent to the conditions (see [Chiriţă and Ciarletta 1999])

$$\begin{gathered}
c_{11}>0,\quad c_{22}>0,\quad c_{33}>0,\quad c_{44}>0,\quad c_{55}>0,\quad c_{66}>0,\\
-2c_{66}+\varkappa_3^i\sqrt{c_{11}c_{22}}<c_{12}<\varkappa_3^s\sqrt{c_{11}c_{22}},\\
-2c_{44}+\varkappa_1^i\sqrt{c_{22}c_{33}}<c_{23}<\varkappa_1^s\sqrt{c_{22}c_{33}},\\
-2c_{55}+\varkappa_2^i\sqrt{c_{11}c_{33}}<c_{13}<\varkappa_2^s\sqrt{c_{11}c_{33}},
\end{gathered} \tag{4.2}$$

where $(\varkappa_1^i,\varkappa_1^s)$, $(\varkappa_2^i,\varkappa_2^s)$ and $(\varkappa_3^i,\varkappa_3^s)$ are solutions with respect to x, y and z of the equation

$$x^2+y^2+z^2-2xyz-1=0, \tag{4.3}$$

with

$$|x|<1,\qquad |y|<1,\qquad |z|<1,$$

$$x \in \left\{ \frac{c_{23}}{\sqrt{c_{22}c_{33}}}, \frac{c_{23}+2c_{44}}{\sqrt{c_{22}c_{33}}} \right\}, \quad y \in \left\{ \frac{c_{13}}{\sqrt{c_{11}c_{33}}}, \frac{c_{13}+2c_{55}}{\sqrt{c_{11}c_{33}}} \right\}, \quad z \in \left\{ \frac{c_{12}}{\sqrt{c_{11}c_{22}}}, \frac{c_{12}+2c_{66}}{\sqrt{c_{11}c_{22}}} \right\}. \tag{4.4}$$

The statement above is equivalent with relation (4.2), and all points $P\,(x, y, z)$ bounded by (4.4) and (4.4) lie inside the surface $S(x, y, z)$, defined by relation (4.3) above.

The basic equations (2.5) become

$$\begin{aligned} &c_{11}u_{1,11} + c_{66}u_{1,22} + c_{55}u_{1,33} + (c_{12}+c_{66})\,u_{2,21} + (c_{13}+c_{55})\,u_{3,31} = \rho\ddot{u}_1, \\ &(c_{12}+c_{66})\,u_{1,12} + c_{66}u_{2,11} + c_{22}u_{2,22} + c_{44}u_{2,33} + (c_{23}+c_{44})\,u_{3,32} = \rho\ddot{u}_2, \\ &(c_{13}+c_{55})\,u_{1,13} + (c_{23}+c_{44})\,u_{2,23} + c_{55}u_{3,11} + c_{44}u_{3,22} + c_{33}u_{3,33} = \rho\ddot{u}_3, \end{aligned} \tag{4.5}$$

and consequently we have the identity

$$\begin{aligned} &\frac{\partial}{\partial t}\Big\{\tfrac{1}{2}\Big[\rho\dot{u}_i\dot{u}_i + c_{11}u_{1,1}^2 + c_{22}u_{2,2}^2 + c_{33}u_{3,3}^2 + c_{66}(u_{1,2}^2+u_{2,1}^2) + c_{55}(u_{3,1}^2+u_{1,3}^2) + c_{44}(u_{3,2}^2+u_{2,3}^2) \\ &\qquad + 2(c_{12}+\varkappa_3)u_{1,1}u_{2,2} + 2(c_{66}-\varkappa_3)u_{1,2}u_{2,1} + 2(c_{13}+\varkappa_2)u_{1,1}u_{3,3} \\ &\qquad + 2(c_{55}-\varkappa_2)u_{1,3}u_{3,1} + 2(c_{23}+\varkappa_1)u_{2,2}u_{3,3} + 2(c_{44}-\varkappa_1)u_{2,3}u_{3,2}\Big]\Big\} \\ &= \{\dot{u}_1[c_{55}u_{1,3}+(c_{55}-\varkappa_2)u_{3,1}] + \dot{u}_2[c_{44}u_{2,3}+(c_{44}-\varkappa_1)u_{3,2}] + \dot{u}_3[(c_{13}+\varkappa_2)u_{1,1}+(c_{23}+\varkappa_1)u_{2,2}+c_{33}u_{3,3}]\}_{,3} \\ &+ \{\dot{u}_1[c_{11}u_{1,1}+(c_{12}+\varkappa_3)u_{2,2}+(c_{13}+\varkappa_2)u_{3,3}] + \dot{u}_2[c_{66}u_{2,1}+(c_{66}-\varkappa_3)u_{1,2}] + \dot{u}_3[c_{55}u_{3,1}+(c_{55}-\varkappa_2)u_{1,3}]\}_{,1} \\ &+ \{\dot{u}_1[c_{66}u_{1,2}+(c_{66}-\varkappa_3)u_{2,1}] + \dot{u}_2[(c_{12}+\varkappa_3)u_{1,1}+c_{22}u_{2,2}+(c_{23}+\varkappa_1)u_{3,3}] + \dot{u}_3[c_{44}u_{3,2}+(c_{44}-\varkappa_1)u_{2,3}]\}_{,2}, \end{aligned} \tag{4.6}$$

where $\varkappa_1$, $\varkappa_2$ and $\varkappa_3$ are positive parameters at our disposal.

So we have to introduce the function

$$\begin{aligned} K_\varkappa(x_3, t) = -\int_0^t \int_{D(x_3,s)} e^{-\sigma s}\{&\dot{u}_1[(c_{55}-\varkappa_2)u_{3,1} + c_{55}u_{1,3}] + \dot{u}_2[(c_{44}-\varkappa_1)u_{3,2} + c_{44}u_{2,3}] \\ &+ \dot{u}_3[(c_{13}+\varkappa_2)u_{1,1} + (c_{23}+\varkappa_1)u_{2,2} + c_{33}u_{3,3}]\}\,\mathrm{d}a\,\mathrm{d}s \end{aligned} \tag{4.7}$$

and note that identity (4.6) implies that

$$\begin{aligned} -\frac{\partial K_\varkappa}{\partial x_3}(x_3, t) = \frac{1}{2}\int_{D(x_3,t)} &e^{-\sigma t}\rho\dot{u}_i\dot{u}_i\,\mathrm{d}a \\ &+ \frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}[c_{11}u_{1,1}^2 + c_{22}u_{2,2}^2 + c_{33}u_{3,3}^2 \\ &\qquad + 2(c_{12}+\varkappa_3)u_{1,1}u_{2,2} + 2(c_{13}+\varkappa_2)u_{1,1}u_{3,3} + 2(c_{23}+\varkappa_1)u_{2,2}u_{3,3}]\,\mathrm{d}a \\ &+ \frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}[c_{66}(u_{1,2}^2+u_{2,1}^2) + 2(c_{66}-\varkappa_3)u_{1,2}u_{2,1}]\,\mathrm{d}a \\ &+ \frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}[c_{44}(u_{2,3}^2+u_{3,2}^2) + 2(c_{44}-\varkappa_1)u_{2,3}u_{3,2}]\,\mathrm{d}a \\ &+ \frac{1}{2}\int_{D(x_3,t)} e^{-\sigma t}[c_{55}(u_{3,1}^2+u_{1,3}^2) + 2(c_{55}-\varkappa_2)u_{1,3}u_{3,1}]\,\mathrm{d}a \end{aligned}$$

$$
\begin{aligned}
&+\frac{\sigma}{2}\int_0^t\int_{D(x_3,s)} e^{-\sigma s}\rho\dot{u}_i\dot{u}_i\,\mathrm{d}a\,\mathrm{d}s\\
&+\frac{\sigma}{2}\int_0^t\int_{D(x_3,s)} e^{-\sigma s}[c_{11}u_{1,1}^2+c_{22}u_{2,2}^2+c_{33}u_{3,3}^2\\
&\qquad +2(c_{12}+\varkappa_3)u_{1,1}u_{2,2}+2(c_{13}+\varkappa_2)u_{1,1}u_{3,3}+2(c_{23}+\varkappa_1)u_{2,2}u_{3,3}]\,\mathrm{d}a\,\mathrm{d}s\\
&+\frac{\sigma}{2}\int_0^t\int_{D(x_3,s)} e^{-\sigma s}[c_{66}(u_{1,2}^2+u_{2,1}^2)+2(c_{66}-\varkappa_3)u_{1,2}u_{2,1}]\,\mathrm{d}a\,\mathrm{d}s\\
&+\frac{\sigma}{2}\int_0^t\int_{D(x_3,s)} e^{-\sigma s}[c_{44}(u_{2,3}^2+u_{3,2}^2)+2(c_{44}-\varkappa_1)u_{2,3}u_{3,2}]\,\mathrm{d}a\,\mathrm{d}s\\
&+\frac{\sigma}{2}\int_0^t\int_{D(x_3,s)} e^{-\sigma s}[c_{55}(u_{3,1}^2+u_{1,3}^2)+2(c_{55}-\varkappa_2)u_{1,3}u_{3,1}]\,\mathrm{d}a\,\mathrm{d}s. \qquad (4.8)
\end{aligned}
$$

In view of the assumptions (4.2) we can choose $\varkappa_1 \in [0, 2c_{44}]$, $\varkappa_2 \in [0, 2c_{55}]$, $\varkappa_3 \in [0, 2c_{66}]$ so that $P(x, y, z)$, with coordinates

$$
x=\frac{c_{23}+\varkappa_1}{\sqrt{c_{22}c_{33}}},\qquad y=\frac{c_{13}+\varkappa_2}{\sqrt{c_{11}c_{33}}},\qquad z=\frac{c_{12}+\varkappa_3}{\sqrt{c_{11}c_{22}}},
$$

lies inside the region limited by the surface $S(x, y, z)$. Then the Sylvester criterion provides a straightforward way toward conditions of positivity of the quadratic form

$$
\begin{aligned}
\mathscr{F}\left(u_{i,j}\right)=&\left[c_{11}u_{1,1}^2+c_{22}u_{2,2}^2+c_{33}u_{3,3}^2+2\left(c_{12}+\varkappa_3\right)u_{1,1}u_{2,2}+2\left(c_{13}+\varkappa_2\right)u_{1,1}u_{3,3}+2\left(c_{23}+\varkappa_1\right)u_{2,2}u_{3,3}\right]\\
&+\left[c_{66}\left(u_{1,2}^2+u_{2,1}^2\right)+2\left(c_{66}-\varkappa_3\right)u_{1,2}u_{2,1}\right]+\left[c_{44}\left(u_{2,3}^2+u_{3,2}^2\right)+2\left(c_{44}-\varkappa_1\right)u_{2,3}u_{3,2}\right]\\
&+\left[c_{55}\left(u_{3,1}^2+u_{1,3}^2\right)+2\left(c_{55}-\varkappa_2\right)u_{1,3}u_{3,1}\right] \qquad (4.9)
\end{aligned}
$$

in terms of $u_{i,j}$. Therefore, we have

$$
\mu_m u_{i,j}u_{i,j}\le\mathscr{F}\left(u_{r,s}\right)\le\mu_M u_{i,j}u_{i,j}, \qquad (4.10)
$$

where μ_m and μ_M are the positive minimum and maximum eigenvalues of the positive quadratic form $\mathscr{F}\left(u_{i,j}\right)$. Thus, the relations (4.8) to (4.10) qualify $K_\varkappa(x_3, t)$ as a valuable measure of the solution. Further the analysis for evolution of the measure $K_\varkappa(x_3, t)$ follows the procedure developed in the above section, thus leading to a spatial behavior like that described by the Theorem 1. Consequently, we can state the following result.

Theorem 2. *Let $u_i(x, t)$ be a solution of the initial boundary value problem defined by (4.5), (2.6), (2.7) and (2.8). Then $K_\varkappa(x_3, t)$ as defined by (4.7) represents a measure of the solution and, for each $t > 0$, it evolves as follows. There is a positive constant ν, depending on the elastic constants, such that*

(a) *for $x_3 \ge \nu t$ we have $K_\varkappa(x_3, t) = 0$ and hence*

$$
u_i(x_3, t) = 0 \quad \textit{for all} \quad x_3 \ge \nu t; \qquad (4.11)
$$

(b) *for $x_3 \le \nu t$, the spatial behavior is described by*

$$
0 \le K_\varkappa(x_3, t) \le K_\varkappa(0, t)\exp\left(-\frac{\sigma}{\nu}x_3\right). \qquad (4.12)
$$

5. Conclusions

The spatial behavior of solutions to the initial boundary value problems in linear elastodynamics has been established based on the assumption of strong ellipticity of the elasticity tensor. A domain of influence theorem has been obtained proving that the whole activity is zero in that part of the cylinder whose axial distance from the loaded end is greater than a critical value. Moreover, inside of the influence domain, a spatial estimate of Saint Venant type has been established, which describes the exponential decay of solutions with respect to the distance from the loaded end. However, these results have been obtained for transversely isotropic and rhombic elastic materials only, since the necessary and sufficient conditions characterizing the strong ellipticity have been discovered recently. Extension to the whole class of anisotropic materials seems to be impossible as long as the characterization of their strong ellipticity remains an open problem.

We have to point out that the influence domain result (relations (3.12) and (4.11)) is independent of the parameter σ entering in the definition of the functions $J_\varkappa$ and $K_\varkappa$, while it appears in the exponential estimates (3.13) and (4.12) as a scaling parameter. Our analysis works when $\sigma = 0$ to prove the influence domain result, but the exponential estimates (3.13) and (4.11) fail to give information on the spatial evolution of solutions. In such case we have to apply the method developed by Chiriţă [1997] for the model of linear viscoelasticity. It is worthwhile to point out that we cannot make $\sigma \to \infty$, because in this case the functions $J_\varkappa$ and $K_\varkappa$ cannot be considered as measures of solutions.

References

[Chiriţă 1997] S. Chiriţă, "On Saint-Venant's principle in dynamic linear viscoelasticity", *Quart. Appl. Math.* **55**:1 (1997), 139–149. MR 98d:73037

[Chiriţă 2006] S. Chiriţă, "On the strong ellipticity condition for transversely isotropic linearly elastic solids", *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **52**:2 (2006), 245–250. MR 2008f:74010

[Chiriţă and Ciarletta 1999] S. Chiriţă and M. Ciarletta, "Time-weighted surface power function method for the study of spatial behaviour in dynamics of continua", *Eur. J. Mech. A Solids* **18**:5 (1999), 915–933. MR 2000i:74044

[Gurtin 1972] M. E. Gurtin, *The linear theory of elasticity*, vol. VIa/2, edited by C. Truesdell, Springer, Berlin, 1972.

[Merodio and Ogden 2003] J. Merodio and R. W. Ogden, "A note on strong ellipticity for transversely isotropic linearly elastic solids", *Quart. J. Mech. Appl. Math.* **56**:4 (2003), 589–591. MR 2026873

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VINCENZO TIBULLO: vtibullo@unisa.it
Università degli Studi di Salerno, via Ponte Don Melillo, 84084 Fisciano (SA), Italy

MASSIMO VACCARO: massimo_vaccaro@libero.it
Università degli Studi di Salerno, via Ponte Don Melillo, 84084 Fisciano (SA), Italy