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THE OPTIMAL SHAPE PARAMETER OF MULTIQUADRIC COLLOCATION METHOD FOR SOLUTION OF NONLINEAR STEADY-STATE HEAT CONDUCTION IN MULTILAYERED PLATE

JAN ADAM KOŁODZIEJ AND MAGDALENA MIERZWICZAK

This paper deals with the numerical solution of the nonlinear heat transfer problem in a multilayered plate. Kansa's meshless method is used for the solution of this problem. In this approach, the unknown temperatures in layers are approximated by the linear combination of radial basis functions, while the governing equation and the boundary conditions are imposed directly at the collocation points. The multiquadrics [MQ] are used as the radial basis functions. In the presented method the radial basis functions contains a free parameter C, called the shape parameter. Usually, in the application of radial basis functions, this parameter is chosen arbitrarily depending on the author's experience. In the presented paper, special attention is paid to the optimal choice of the shape parameter for the radial basis functions. This optimal value of the shape parameter is obtained using a formula given by other authors for solution of the linear case.

1. Introduction

In the last two decades, meshless methods were introduced to computational mechanics. The essential feature of these methods is that they only require a set of unconnected nodes to construct the approximation functions. Among all the meshless methods, Kansa's method [1990a; 1990b] has become quite popular due to its simplicity. In this approach, the solution is approximated by a linear combination of the radial basis functions, while the governing equation and the boundary conditions are imposed directly at the collocation points. The most popular radial basis functions are multiquadrics [Hardy 1971]. In the presented method, the radial basis functions contain a free parameter C, called the shape parameter. Usually in the application to radial basis functions this parameter is chosen arbitrarily, depending on the author's experience. However, the shape parameter affects both the accuracy of the approximation and the conditioning of the system of equation, and there are papers in which this parameter is chosen optimally in proposed algorithms of solution, for example, [Golberg et al. 1996; Rippa 1999; Wertz et al. 2006; Huang et al. 2007].

The purpose of this paper is to determine the optimal choice of the shape parameter for the radial basis functions when a nonlinear heat conduction problem in multilayered solid structures is considered. Walls of heat treatment furnaces usually consist of several layers of different materials, with a different temperature inside and outside of the furnace. There are some electronic devices in which heat flow exists in the multi-layered device materials. Accurate thermal analysis of the high-temperature

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devices must take into account the dependence of the thermal conductivity on the temperature. Usually, for problems with temperature-dependent thermal conductivity, the Kirchhoff transformation is used to convert a nonlinear heat equation into a linear one with nonlinear boundary conditions. However, in the multilayered walls, the nonlinear boundary conditions appear between layers, which makes this transformation generally problematic. In the paper [Bonani and Ghione 1995], the heat flow in only two layers has been considered. Moreover, the authors assumed that the thermal conductivity in layers is linearly dependent, which permits them to use the Kirchhoff transformations. Similarly, in the paper [Pesare et al. 2001], the authors linearized the boundary conditions between layers and used the Fourier transformation.

In this paper we apply Kansa's method for the numerical solution of nonlinear heat transfer in multi-layered solid structures. Special attention is paid to the optimal choice of the shape parameter for radial basis functions. This optimal value of the shape parameter is obtained using the formula given in the paper [Huang et al. 2007], which was used in the two-dimensional linear case. Here, this formula is examined for the nonlinear one-dimensional case.

2. Formulation of the problem

Let's consider three cases of multi-layer walls as shown in Figure 1:

(1) constant thermal conductivity of layers $\lambda_{(i)} = AW(i)$,

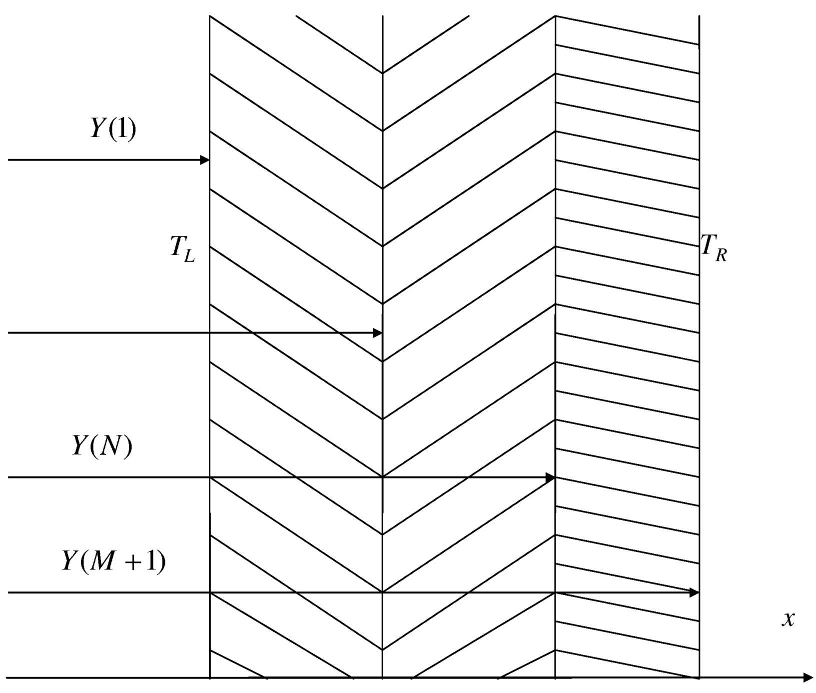

Figure 1. Plane multi-layer wall.

(2) linear temperature-dependent thermal conductivity $\lambda^{(i)} = AW(i) + BW(i)T^{(i)}$,

(3) temperature-dependent thermal conductivity of layers

$$\lambda^{(i)} = AW(i) + BW(i)T^{(i)} + CW(i)(T^{(i)})^2, \qquad i = 1, 2, \ldots, M, \tag{2-1}$$

where $T^{(i)}(x)$ is the temperature field in ith layer, and $AW(i)$, $BW(i)$, and $CW(i)$ are known constants for each layer. On the left and the right hand of the walls the temperatures are T_L and T_R respectively.

The one-dimensional governing equation for steady state heat transfer in multi-layered walls with thermal conductivity dependent on temperature is given as

$$\frac{d}{dx}\left[\lambda\left(T^{(i)}\right)\frac{dT^{(i)}}{dx}\right] = 0, \qquad \text{for } x \in [Y(i), Y(i+1)], \quad i = 1, \ldots, M. \tag{2-2}$$

Equation (2-2) can be expressed as

$$\frac{d^2T}{dx^2} = -\frac{1}{\lambda\left(T^{(i)}\right)}\frac{d\lambda}{dT^{(i)}}\left(\frac{dT^{(i)}}{dx}\right). \tag{2-3}$$

Substituting Equation (2-1) into Equation (2-3) we have

$$\frac{d^2T}{dx^2} = -\frac{BW(i) + 2CW(i)T^{(i)}}{AW(i) + BW(i)T^{(i)} + CW(i)(T^{(i)})^2}\left(\frac{dT^{(i)}}{dx}\right)^2. \tag{2-4}$$

Equation (2-4) should be solved with the following boundary conditions:

(1) on the left boundary:

$$T^{(1)} = T_L \quad \text{for } x = Y(1), \tag{2-5}$$

(2) on the right boundary:

$$T^{(M)} = T_R \quad \text{for } x = Y(M+1), \tag{2-6}$$

(3) continuity of temperature and heat flux between layers:

$$\begin{aligned} T^{(i)} &= T^{(i+1)}, \\ \lambda\left(T^{(i)}\right)\frac{dT^{(i)}}{dx} &= \lambda\left(T^{(i+1)}\right)\frac{dT^{(i+1)}}{dx}, \\ \text{for } x = Y(i+1), &\qquad i = 1, 2, \ldots, M. \end{aligned} \tag{2-7}$$

We introduce the nondimensional variables in the form $\breve{T}^{(i)} = T^{(i)}/T_L$, $\breve{x}^{(i)} = x^{(i)}/D$, where $D = Y(M+1) - Y(1)$ is the width of a wall. Now, the nondimensional thermal conductivity has the form

$$\breve{\lambda}^{(i)} = 1 + \frac{BW(i)}{AW(i)}T_L\breve{T}^{(i)} + \frac{CW(i)}{AW(i)}(T_L\breve{T}^{(i)})^2 \tag{2-8}$$

$$= 1 + \breve{B}W(i)\breve{T}^(i) + \breve{C}W(i)(\breve{T}^{(i)})^2, \tag{2-9}$$

where $\breve{B}W(i) = \dfrac{BW(i)}{AW(i)}T_L$, $\breve{C}W(i) = \dfrac{CW(i)}{AW(i)}(T_L)^2$.

The governing Equation (2-4), in the nondimensional thermal conductivity, is now the following:

$$\frac{d^2\breve{T}^{(i)}}{d\breve{x}^2} = -\frac{\breve{B}W(i)+2\breve{C}W(i)\breve{T}^{(i)}}{1+\breve{B}W(i)T^{(i)}+\breve{C}W(i)(T^{(i)})^2}\left(\frac{d\breve{T}^{(i)}}{d\breve{x}}\right)^2, \tag{2-10}$$

and is solved with the following boundary conditions in dimensionless form:

(1) on the left boundary:

$$\breve{T}^{(1)} = 1, \qquad \text{for } \breve{x} = \frac{Y(1)}{D}, \tag{2-11}$$

(2) on the right boundary:

$$\breve{T}^{(M)} = \frac{T_R}{T_L}, \qquad \text{for } \breve{x} = \frac{Y(M+1)}{D}, \tag{2-12}$$

(3) continuity between layers:

$$\begin{gathered}\breve{T}^{(i)} = \breve{T}^{(i+1)},\\ \breve{\lambda}\left(\breve{T}^{(i)}\right)\frac{d\breve{T}^{(i)}}{d\breve{x}} = \beta^{(i+1)}\cdot\breve{\lambda}\left(\breve{T}^{(i+1)}\right)\frac{d\breve{T}^{(i+1)}}{d\breve{x}},\\ \breve{x} = \frac{Y\,(i+1)}{D}, \quad i = 1, 2, \ldots, M,\end{gathered} \tag{2-13}$$

where $\beta^{(i+1)} = \dfrac{AW(I+1)}{AW(I)}$.

3. Method of solution

According to Kansa's method the approximate solution is assumed in the form

$$\breve{T}^{(i)} = \sum_{j=1}^{N} D(i,j)\phi_j(\breve{x}, \breve{x}w(i,j), C), \qquad i = 1, 2, \ldots, M, \tag{3-1}$$

where $\phi_j(\breve{x}, \breve{x}w(i,j), C) = \sqrt{(\breve{x}-\breve{x}w(i,j))^2+C^2}$ are multiquadrics as radial basis functions, $D(i,j)$ are coefficients to be determined, i is related to ith layer, j is related to interpolation nodes, N is the number of interpolation points in each layer, $\breve{x}w(i,j)$ are interpolation points which are determined by the formula

$$\breve{x}w(i,j) = \frac{\left(\frac{Y(i+1)}{D}-\frac{Y(i)}{D}\right)\cdot(j-i)}{N-1}+\frac{Y(i)}{D},$$

and C is the shape factor for which the optimal value will be determined by using error estimation.

Then we can write the solution, Equation (3-1), as

$$\breve{T}^{(i)} = \sum_{j=1}^{N} D(i,j)\sqrt{(\breve{x}-\breve{x}w(i,j))^2+C^2}, \qquad i = 1, 2, \ldots, M, \tag{3-2}$$

The problem can be solved if the coefficients $D(i,j),\ i = 1, 2, \ldots, M,\ j = 1, 2, \ldots, N$, are known. These $NG = M\cdot N$ unknown coefficients will be determined with the following equations:

(1) from determination of boundary condition (2-11) at the left side of wall:

$$\breve{T}^{(1)}(\breve{x}w(1,1)) = 1, \tag{3-3}$$

(2) determination of boundary condition (2-12) at the right side of wall:

$$\breve{T}^{(M)}(\breve{x}w(M,N)) = \frac{T_R}{T_L}, \tag{3-4}$$

(3) from the continuity conditions (2-13) between layers, which lead to $2(M-1)$ equations in the form

$$\breve{T}^{(i)}(\breve{x}w(i,N)) = \breve{T}^{(i+1)}(\breve{x}w(i+1,1)), \tag{3-5}$$

$$\breve{\lambda}(\breve{T}^{(i)})\frac{d\breve{T}}{d\breve{x}}\Big|_{\breve{x}w(i,M))} = \beta^{(i+1)}\breve{\lambda}(\breve{T}^{(i+1)})\frac{d\breve{T}^{(i+1)}}{d\breve{x}}\Big|_{\breve{x}w(i+1,j)}. \tag{3-6}$$

(4) from pointwise satisfaction of Equation (2-10) in the inner nodes on each layer, which leads to $M(N-2)$ equations of the form

$$\frac{d^2\breve{T}^{(i)}}{d\breve{x}^2}\Big|_{\breve{x}w(i,j))} = \left\{-\frac{\breve{B}W(i)+2\breve{C}W(i)\breve{T}^{(i)}}{1+\breve{B}W(i)T^{(i)}+\breve{C}W(i)(T^{(i)})^2}\left(\frac{d\breve{T}^{(i)}}{d\breve{x}}\right)^2\right\}\Big|_{\breve{x}w(i,j))}, \tag{3-7}$$

where $i = 1, 2, \ldots, M$ and $j = 2, \ldots, N-1$.

Together we have $1+1+2(M-1)+M(N-2) = MN$ nonlinear equations, the same as the number of unknowns $D(i, j)$.

In the presented method, the radial basis functions contain a free parameter C called the shape parameter. Usually in the application of radial basis functions this parameter is chosen arbitrarily, depending on author's experience. However, the shape parameter affects both the accuracy of the approximation and the conditioning of the system of equations, and there are papers in which this parameter is chosen optimally in a proposed algorithm of solution, for example, [Rippa 1999]. For MQ collocation, the shape factor in the basis functions should be increased to its limit. When we push $C \to \infty$, the theoretical accuracy can be achieved but condition number of solutions matrix becomes huge which leads to the loss of accuracy. We establish an error estimate of

$$\epsilon \approx O\big(\exp(aC^{3/2} + (\ln\lambda)C^{1/2}h^{-1})\big), \qquad \text{with } 0 < \lambda < 1 \text{and } a > 0,$$

given in [Huang et al. 2007]. A finite C value for which the error is minimized exists. This optimal value is found to be $C_{\text{opt}} = C_{\max} = -\ln\lambda/(3ah)$. To determine C_{opt}, knowledge of the constants λ and a in the error estimate is needed. In a real world problem, the error is not known because the true solution is not given. Without data, λ and a cannot be determined. This difficulty is overcome by utilizing the residual error, which is a good measure of the error trend, but not the error magnitude. Using residual errors corresponding to a number of C and h values, these two constants λ and a can be estimated by least square data fitting.

For numerical experiments we solve three cases: constant, linear, and nonlinear temperature-dependent thermal conductivity of layers. In order to verify the exactness of the proposed method, as a first example, one layer with known temperature at the left and right walls and linearly temperature-dependent thermal conductivity $\lambda(T) = AW + BW \cdot T$ was considered. The analytical solution is

$$T(x) = \frac{-AW + \sqrt{AW^2 - 2BW(\kappa_1 \cdot x + \kappa_2)}}{BW},$$

where

$$\kappa_1 = \frac{AW(T_R - T_L) + \frac{1}{2}BW(T_K^2 - T_L^2)}{G} \quad \text{and} \quad \kappa_2 = AWT_L + \tfrac{1}{2}BWT_L^2,$$

and we can make a comparison of the result from the MQ collocation method with the optimal shape factor and check the accuracy of the method using the maximum error

$$\epsilon_{\max} = \max_{q=1,\dots,N-1} \frac{|T(xr_q) - \breve{T}(xr_q)|}{T_{\max}},$$

and square error

$$\epsilon_{\text{sqr}} = \frac{\sqrt{\dfrac{1}{N-1}\displaystyle\sum_{q=1}^{N-1}[T(xr_q) - \breve{T}(xr_q)]^2}}{T_{\max}}.$$

4. Residual error

In a real life problem, we have no knowledge about the exact solution; hence we do not have error data to use at all. We need to find an alternative to the above procedure and estimate the residual error. If we check the residual error at a node xr_q not belonging to the collocation set,

$$\epsilon_R(xr_q) = \frac{d}{dx}\left(\lambda \breve{T}(xr_q) \frac{d\breve{T}(xr_q)}{dx}\right),$$

the error is generally not zero. The residual error can be used as a good indication of error trend, but it does not give the error magnitude. We can write the estimate of residual error as follows: $\epsilon_r = A\epsilon$, where A is a constant of an unknown order of magnitude. For a given grid h, we can perform two computations using two different C values, C_k and C_{k+1}, to obtain the residual errors $\epsilon_r(C_k)$ and $\epsilon_R(C_{k+1})$. With two such data points, their ratio gives the following linear equation in the two unknowns a and $\ln\lambda$:

$$\ln \frac{\epsilon_R(C_k)}{\epsilon_R(C_{k+1})} = (C_k^{3/2} - C_{k+1}^{3/2})a - \frac{C_k^{1/2} - C_{k+1}^{1/2}}{h} \ln\lambda,$$

The three computations with different C's can form two equations for the determination of a and $\ln\lambda$. In practice, it is better to obtain a larger number of data points to perform the least squares fitting. Then the obtained constants can be used to determine the C_{opt} value for a finer grid.

5. Numerical results

Linear temperature-dependent thermal conductivity of one layer. In these cases, the exact solutions are unknown so we can estimate the method error magnitude (maximum and square error). In these numerical calculations, 11 and 21 collocation points in one layer were chosen and 10^{-4} for maximal

C	$\ln \epsilon_R$
1.6	0.00121982413
1.8	0.000886213166
2.0	0.000691965229
2.2	0.000571502605
2.4	0.00048876342
2.6	0.000562506018
2.8	1.6346723
3.0	2.36810104
$C_{\mathbf{opt}}$	$\ln \epsilon_R$
2.45619734	0.00046022506

Table 1. Residual error as a function of the shape parameter C for linear temperature-dependent thermal conductivity.

C	$\ln \epsilon_R$
0.005	0.792708032
0.155	0.629296421
0.305	0.0521441337
0.455	0.00555486248
0.605	0.000754166689
0.755	0.000130104558
$C_{\mathbf{opt}}$	$\ln \epsilon_R$
0.786010956	0.000104694988

Table 2. Residual error as a function of the shape parameter C for linear temperature-dependent thermal conductivity.

error in Newton's method was accepted. The first approximation was the solution to the temperature distribution for a constant thermal conductivity coefficient (independent of the temperature).

For 11 collocation points the optimal shape factor is $C_{\text{opt}} = 2.45619734$ for which the residual error $\epsilon_R = 0.00046$ and maximum and square error between approximated and analytic solution is, $\epsilon_{\max} = 3.81 \times 10^{-005}$, $\epsilon_{\text{sqr}} = 2.27 \times 10^{-005}$ respectively (Table 1).

For 21 collocation points $C_{\text{opt}} = 0.786010956$, $\epsilon_R(C_{\text{opt}}) = 0.0001047$, $\epsilon_{\max} = 3.46 \times 10^{-005}$, $\epsilon_{\text{sqr}} = 2.84 \times 10^{-005}$ (Table 2). The values of maximum and square errors show that the accuracy of the method is high and the approximate solution agrees with the theoretical solution. The above figures show that the numerical and theoretical solutions are similar and the MQ collocation method with optimized shape procedure is an effective tool to solve heat transfer problems.

Constant thermal conductivity of multilayer plane. We considered three cases:

C	$\ln \epsilon_R$
0.05	3.0710643
0.2	0.447483879
0.35	0.150444561
0.5	0.086741511
0.65	0.065856967
0.8	0.057433939
0.95	0.053670263
1.1	0.051987961
1.25	0.095730805
1.4	0.074601817
1.55	0.064468031
$C_{\mathbf{opt}}$	$\ln \epsilon_R$
0.594536825	0.000310218

C	$\ln \epsilon_R$
0.1	2.64531052
0.2	0.53068379
0.3	0.174990457
0.4	0.09438661
0.5	0.068759911
0.6	0.058639665
0.7	0.054148311
0.8	0.052033416
0.9	0.05100849
1.0	0.050664394
1.1	0.08348719
$C_{\mathbf{opt}}$	$\ln \epsilon_R$
0.386502402	0.015555844

Table 3. Residual error as a function of the shape parameter C for constant thermal conductivity. Left: one-layer wall with 21 collocation points. Right: two-layer wall with 11 collocation points.

The first case experiment is performed at one layer with 21 collocation points. The calculated shape factor is $C_{\text{opt}} = 0.594536825$ for which the residual error is $\epsilon_R(C_{\text{opt}}) = 0.00031$ (Table 3, left).

In the second case, a two layer wall is taken with 11 collocation points. The calculated shape factor is $C_{\text{opt}} = 0.386502402$ for which the residual error is $\epsilon_R(C_{\text{opt}}) = 0.01556$ (Table 3, right).

In the third case, a three layer wall is considered with 11 collocation points. The optimal shape factor is $C_{\text{opt}} = 0.687803178$ and the residual error is $\epsilon_R(C_{\text{opt}}) = 0.0328$ (Table 4).

C	$\ln \epsilon_R$	C	$\ln \epsilon_R$	C	$\ln \epsilon_R$
0.1	1.97888189	0.35	0.042408028	0.6	0.033871399
0.15	0.532822138	0.4	0.037617988	0.65	0.056527116
0.2	0.177169659	0.45	0.035414293	0.7	0.074507336
0.25	0.083325356	0.5	0.034366375	0.75	0.19534613
0.3	0.053618929	0.55	0.033893021	0.8	0.886799178
		$C_{\mathbf{opt}}$	$\ln \epsilon_R$		
		0.687803178	0.032803874		

Table 4. Residual error as a function of the shape parameter C for constant thermal conductivity, for a three-layer wall with 11 collocation points.

C	$\ln \epsilon_R$
0.01	1.53630798
0.11	1.41237822
0.21	0.246053512
0.31	0.048526885
0.41	0.010647406
0.51	0.00258324
0.61	0.000698235
0.71	0.000209848
0.81	0.00048872
$C_{\mathbf{opt}}$	$\ln \epsilon_R$
0.791210985	8.34×10^{-05}

C	$\ln \epsilon_R$
0.005	2.11501089
0.13	1.66547618
0.255	0.168228737
0.38	0.023131902
0.505	0.004399075
0.63	0.001125429
0.755	0.000375851
0.88	0.000161322
1.005	0.066491517
$C_{\mathbf{opt}}$	$\ln \epsilon_R$
0.812596263	0.000245721

Table 5. Residual error as a function of the shape parameter C. Left: one-layer wall with 21 collocation points. Right: two-layer wall with 11 collocation points.

Temperature-dependent thermal conductivity of layers. Next we considered the nonlinear temperature-dependent thermal conductivity, so we solved the nonlinear system using Newton's method.

In the first case, the experiment is performed at one layer with 21 collocation points. The calculated shape factor is $C_{\text{opt}} = 0.791210985$ for which the residual error is $\epsilon_R(C_{\text{opt}}) = 8.34 \times 10^{-005}$ (Table 5, left).

In the second case, a two layer wall is taken into account with 11 collocation points. The calculated shape factor is $C_{\text{opt}} = 0.812596263$ for which the residual error is $\epsilon_R(C_{\text{opt}}) = 2.46 \times 10^{-04}$ (Table 5, right).

In the third case, a three layer wall is taken into account with 11 collocation points. The optimal shape factor is $C_{\text{opt}} = 0.618039263$ and the residual error is $\epsilon_R(C_{\text{opt}}) = 1.11 \times 10^{-04}$ (Table 6).

C	$\ln \epsilon_R$
0.005	4.89989774
0.13	0.910910183
0.255	0.0331283856
0.38	0.00237394996
0.505	0.000305350558
0.63	0.00172510558
$C_{\mathbf{opt}}$	$\ln \epsilon_R$
0.618039263	0.000111371465

Table 6. Residual error as a function of the shape parameter C, for a three-layer wall with 11 collocation points.

6. Conclusion

In this paper, the meshless method has been successfully used to solve the nonlinear heat transfer problem in multilayer wall insulation with a temperature-dependent thermal conductivity. Special attention was paid to the optimal choice of the shape parameter for the radial basis functions. For a calculated optimal value $C_{\text{opt}} = -(\ln\lambda)/(3ah)$, we can minimize the solution error. We find a constant $\ln\lambda$ using the residual error and least square method. The proposed method can be very easily implemented. The proposed algorithm of calculation is based on the Kansas's method, which numerically leads to a relatively simple nonlinear system of algebraic equations. The use of the calculated optimal shape factor guarantees a quick convergence with Newton's method for nonlinear system equations.

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