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Liu procedure is applied to special relativistic fluids. It is shown that a reasonable relativistic theory is an extended one, where the basic state space contains the momentum density. This property follows from the structure of the energy-momentum balance and the Second Law of thermodynamics. Moreover, we derive that the entropy depends on both the energy density and the momentum density in a specific way, indicating that the local rest frame energy density cannot be interpreted as the internal energy, and that the local rest frame momentum density should be considered, too. The corresponding constitutive relations for the stress and the energy flux are derived.

1. Introduction

Nonrelativistic nonequilibrium thermodynamics separates the dissipative and nondissipative parts of the evolving physical quantities. This separation is based on the construction of the internal energy balance [Eckart 1940a; Groot and Mazur 1962; Gyarmati 1970]. According to the classical interpretation, the internal energy is the difference of the total energy and the special energies of known type. The entropy function depends directly on the internal energy. The internal energy is distributed equally among the molecular degrees of freedom. The process by which other energy types are converted to internal energy is called dissipation. This approach is common in all theories of nonequilibrium thermodynamics, including classical irreversible thermodynamics, where the hypothesis of local equilibrium applies. However, there is no internal energy in this sense in relativistic irreversible thermodynamics. In fact, there is practically no relativistic irreversible thermodynamics at all because the local equilibrium theory is plagued by serious inconsistencies. Only extended theories, theories beyond local equilibrium, are considered as viable. The reason for this is that the classical theory of Eckart [1940b] is for relativistic fluids simple and elegant, but produces generic instabilities [Hiscock and Lindblom 1985]. The more developed extended theories incorporate the theory of Eckart, but suppress the instabilities [Hiscock and Lindblom 1987; Geroch 1995; Lindblom 1996].

In this paper we investigate the possibility of local equilibrium in relativistic hydrodynamics by methods of continuum thermodynamics. In Section 2 the balances of energy-momentum and entropy are introduced. In Section 3, we calculate the dissipation inequality for first-order (local equilibrium) relativistic hydrodynamics using the Liu procedure. The need of second-order (extended, or weakly nonlocal) theories is indicated by the emergent structure. A new concept of relativistic internal energy follows. Based on these results, Section 4 shows the constitutive equations of the simplest extended theory by the heuristic arguments of irreversible thermodynamics.

Keywords: relativistic nonequilibrium thermodynamics, Liu procedure, relativistic internal energy.
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2. Basic balances of relativistic fluids

For the metric (Lorentz form) we use $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, employing the usual convention that the speed of light $c = 1$. Therefore, for a four-velocity u^α we have $u_\alpha u^\alpha = -1$. $\Delta^\alpha{}_\beta = g^\alpha{}_\beta + u^\alpha u_\beta$ denotes the u-orthogonal projection. With these conventions in mind, we proceed to form the basic balances of energy-momentum and entropy.

The energy-momentum density tensor is given with the help of the rest-frame quantities

$$T^{\alpha\beta} = eu^\alpha u^\beta + u^\alpha q^\beta + u^\beta q^\alpha + P^{\alpha\beta}, \tag{1}$$

where $e = u_\alpha u_\beta T^{\alpha\beta}$ is the *density of the energy*, $q^\beta = -u_\alpha \Delta^\beta{}_\gamma T^{\alpha\gamma}$ is the *energy flux* or *heat flow*, $q^\alpha = -u_\beta \Delta^\alpha{}_\gamma T^{\gamma\beta}$ is the *momentum density*, and $P^{\alpha\beta} = \Delta^\alpha_\gamma \Delta^\beta_\mu T^{\gamma\mu}$ is the *pressure tensor*. The momentum density, energy flux, and pressure are spacelike in the comoving frame, therefore $u_\alpha q^\alpha = 0$, $u_\beta q^\beta = 0$, and $u_\alpha P^{\alpha\beta} = u_\alpha P^{\beta\alpha} = 0^\beta$. The energy-momentum tensor is symmetric, because we assume that the internal spin of the material is zero. In this case, the energy flux and the momentum density are equal. Let us emphasize that the form (1) of the symmetric energy-momentum tensor is completely general for one-component fluids, but it is expressed by the local rest frame quantities.

Now the conservation of energy-momentum $\partial_\beta T^{\alpha\beta} = 0$ is expanded to

$$\partial_\beta T^{\alpha\beta} = \dot{e}u^\alpha + eu^\alpha \partial_\beta u^\beta + e\dot{u}^\alpha + u^\alpha \partial_\beta q^\beta + q^\beta \partial_\beta u^\alpha + \dot{q}^\alpha + q^\alpha \partial_\beta u^\beta + \partial_\beta P^{\alpha\beta}, \tag{2}$$

where $\dot{e} = \frac{d}{d\tau} e = u^\alpha \partial_\alpha e$ denotes the derivative of e by the proper time τ. Its timelike part in a local rest frame gives the balance of the energy

$$-u_\alpha \partial_\beta T^{\alpha\beta} = \dot{e} + e\partial_\alpha u^\alpha + \partial_\alpha q^\alpha + q^\alpha \dot{u}_\alpha + P^{\alpha\beta} \partial_\beta u_\alpha = 0. \tag{3}$$

The spacelike part in the local rest frame describes the balance of the momentum

$$\Delta^\alpha{}_\gamma \partial_\beta T^{\gamma\beta} = e\dot{u}^\alpha + q^\alpha \partial_\beta u^\beta + q^\beta \partial_\beta u^\alpha + \Delta^\alpha{}_\gamma \dot{q}^\gamma + \Delta^\alpha{}_\gamma \partial_\beta P^{\gamma\beta} = 0^\alpha. \tag{4}$$

The entropy density and flux can also be combined into a four-vector, using local rest frame quantities:

$$S^\alpha = su^\alpha + J^\alpha, \tag{5}$$

where $s = -u_\alpha S^\alpha$ is the *entropy density* and $J^\alpha = S^\alpha - u^\alpha s = \Delta^\alpha{}_\beta S^\beta$ is the *entropy flux*. The entropy flux is u-spacelike, therefore $u_\alpha J^\alpha = 0$. In this framework, the Second Law of thermodynamics is expressed by the following inequality

$$\partial_\alpha S^\alpha = \dot{s} + s\partial_\alpha u^\alpha + \partial_\alpha J^\alpha \geq 0. \tag{6}$$

3. Thermodynamics

The thermodynamical background in relativistic theories is usually based on analogies with nonrelativistic thermostatics. However, nonequilibrium thermodynamics has developed beyond the simple, 'let us substitute everything into the entropy balance and see what happens' theory since Eckart. It is important to check the dynamic consistency of the Second Law, considering the evolution equations as constraints for the entropy balance. This method of nonequilibrium thermodynamics is constructive, gives important information for new theories, and reveals some deeper interrelations. Here we exploit the Second Law by Liu's procedure [Liu 1972], introducing a first-order weakly nonlocal state space in all basic variables, and thus restricting ourselves to a local equilibrium theory. One can find a general treatment of nonrelativistic classical and extended irreversible thermodynamics from this point of view in [Ván 2003]. Our aim here is to investigate the relativistic fluids with similar methods, and to get the relativistic equivalent of the classical Fourier–Navier–Stokes system of equations for one component fluids.

Our most important assumption regarding relativistic thermodynamics is that the constitutive equations are local rest frame expressions. As material interactions are local, this is natural from a physical point of view.

The *basic state space* of the theory is spanned by the energy density e and by the velocity field u^α. The *constitutive state space* is spanned by the basic state variables and their first derivatives, is therefore first-order weakly nonlocal. Hence, the constitutive functions depend on the variable set $C = (e, u_\alpha, \partial_\alpha e, \partial_\alpha u_\beta)$. The *constitutive functions* are the energy flux/momentum density q^α, the pressure $P^{\alpha\beta}$, the entropy density s and the entropy flux J^α. The derivatives of the constitutive functions are denoted by the number of the corresponding variable in the constitutive space, for example, $\frac{\partial s}{\partial(\partial_\alpha e)} = \partial_3 s$. With this notation we can distinguish easily between the derivatives by the constitutive and spacetime variables. A nonequilibrium thermodynamic theory is considered to be solved if all other constitutive quantities are expressed by the entropy density and its derivatives.

According to the procedure of Liu, the balance of energy-momentum (2) is a constraint to the entropy balance (6) with the Lagrange–Farkas multiplier Λ_α,

$$\partial_\alpha S^\alpha - \Lambda_\alpha \partial_\beta T^{\alpha\beta} \geq 0. \tag{7}$$

Let us remember that here, the spacelike components of the four quantities and the entropy density are the constitutive quantities depending on the introduced constitutive variables C. Therefore, in the above inequality we can develop the derivatives of the composite functions. The coefficients of the derivatives that are not in the constitutive space must be zero. As a result, we get the following Liu-equations:

$$\begin{aligned} \partial_{\alpha\beta} e: &\quad (\partial_3 S^\alpha)^\beta - \Lambda_\mu (\partial_3 T^{\mu\alpha})^\beta = 0^{\alpha\beta}, \\ \partial_{\alpha\beta} u_\gamma: &\quad (\partial_4 S^\alpha)^{\beta\gamma} - \Lambda_\mu (\partial_4 T^{\mu\alpha})^{\beta\gamma} = 0^{\alpha\beta\gamma}. \end{aligned} \tag{8}$$

The simple structure of the Liu equations suggests the assumption that the Lagrange multiplier is a local function, and does not depend on the derivatives of the basic state variables

$$\Lambda_\gamma = \Lambda_\gamma(n, e). \tag{9}$$

A general solution of (8) is

$$S^\alpha - \Lambda_\gamma T^{\gamma\alpha} - A^\alpha = 0^\alpha, \tag{10}$$

where $A^\alpha = A^\alpha(n, e)$ is an arbitrary local function.

Let us introduce the splitting of the vector multiplier and the four-vector A^α into spacelike and timelike parts in the local rest frame as

$$\begin{aligned} \Lambda^\alpha &= -\Lambda u^\alpha + l^\alpha, \\ A^\alpha &= A u^\alpha + a^\alpha, \end{aligned}$$

where for the spacelike components $u_\alpha l^\alpha = u_\alpha a^\alpha = 0$. Now, Equation (10) gives

$$u^\alpha(s - \Lambda e - l_\gamma q^\gamma - A) + (J^\alpha - \Lambda q^\alpha - l_\gamma P^{\gamma\alpha} - a^\alpha) = 0^\alpha. \tag{11}$$

Here both the timelike and spacelike parts are zero, resulting in

$$s = \Lambda e + l_\gamma q^\gamma + A, \tag{12}$$

$$J^\alpha = \Lambda q^\alpha + l_\gamma P^{\gamma\alpha} + a^\alpha. \tag{13}$$

After the identification of the Liu equations, we expand the *dissipation inequality* as

$$\begin{aligned}\partial_\alpha e &\left[(\partial_1 s)u^\alpha + \partial_1 J^\alpha - \Lambda u^\alpha - \Lambda\partial_1 q^\alpha - l_\gamma \partial_1 P^{\gamma\alpha} - l_\gamma \partial_1 q^\gamma u^\alpha\right] \\ &+ \partial_\alpha u_\beta \left[\left(s - \Lambda e - l_\gamma q^\gamma\right)\Delta^{\alpha\beta} + (\partial_2 s)^\beta u^\alpha + (\partial_2 J^\alpha)^\beta \right.\\ &\left. - l^\beta e u^\alpha - l^\beta q^\alpha - \Lambda(\partial_2 q^\beta)^\alpha - \Lambda_\gamma(\partial_2 P^{\gamma\alpha})^\beta - \Lambda_\gamma u^\alpha(\partial_2 q^\gamma)^\beta\right] \geq 0.\end{aligned} \tag{14}$$

Here we exploited the fact that partial differentiation by e can be exchanged with a multiplication by the four velocity u^α.

In the dissipation inequality one should consider the solution of the Liu equations. Substituting (12) and (13) into (14) we get

$$\begin{aligned}\partial_\alpha e&\left[(\partial_1 s - \Lambda - l_\gamma \partial_1 q^\gamma)u^\alpha + q^\alpha \partial_1 \Lambda + P^{\gamma\alpha}\partial_1 l_\gamma + \partial_1 a^\alpha\right] \\ &+ \partial_\alpha u_\beta \left[A\Delta^{\alpha\beta} + q^\alpha(\partial_2\Lambda)^\beta + P^{\gamma\alpha}(\partial_2 l_\gamma)^\beta + (\partial_2 a^\alpha)^\beta \right.\\ &\left. + u^\alpha\left((\partial_2 s)^\beta - l_\gamma(\partial_2 q^\gamma)^\beta - l^\beta e - \Lambda q^\beta\right) - l^\beta q^\alpha - \Lambda P^{\alpha\beta}\right] \geq 0,\end{aligned} \tag{15}$$

where the following identities

$$u_\gamma \partial_{u_\beta} q^\gamma = \partial_{u_\beta}(u_\gamma q^\gamma) - q^\gamma \partial_{u_\beta} u_\gamma = -q^\gamma \Delta_\gamma^{\ \beta} = -q^\beta,$$

$$u_\gamma \partial_{u_\beta} P^{\gamma\alpha} = \partial_{u_\beta}(u_\gamma P^{\gamma\alpha}) - P^{\gamma\alpha}\partial_{u_\beta} u_\gamma = -P^{\gamma\alpha}\Delta_\gamma^{\ \beta} = -P^{\beta\alpha}.$$

were applied to simplify the last term ($\partial_2 = \partial_{u_\beta}$).

Observing the first term in the last form of the dissipation inequality, one can eliminate the direct velocity dependence of the entropy function, recognizing that the entropy may depend on the energy flux in the form

$$s(e, u^\alpha, \partial_\alpha e, \partial_\alpha u^\beta) = \hat{s}(e, q^\gamma(e, u^\alpha, \partial_\alpha e, \partial_\alpha u^\beta)). \tag{16}$$

Therefore, the entropy is local, and is independent of the derivatives of the basic state space variables and the velocity field. Entropy does, however, depend on the energy flux, which can depend on the derivatives because it is, according to our initial assumptions, a constitutive function. Taking this into account, the Lagrange–Farkas multipliers are determined by the entropy derivatives

$$\partial_e \hat{s} = \Lambda, \quad \partial_{q^\alpha}\hat{s} = l_\alpha. \tag{17}$$

We introduce a temperature T as

$$\partial_e \hat{s} = \Lambda = \frac{1}{T}. \tag{18}$$

We recognize that a full thermostatic compatibility requires that in (12), $A := \frac{p}{T}$, where p is the pressure. This consequence is completely analogous to the results of the nonrelativistic nonequilibrium thermodynamic theory, where thermostatics arises from the structure of the balance form evolution equations used as constraints to the Second Law.

Finally, we assume that entropy flux is classical, and the additional term a^α [Müller 1967] is zero

$$a^\alpha = 0^\alpha. \tag{19}$$

The dissipation inequality, then, reduces to the following simple form

$$q^\alpha \partial_\alpha \frac{1}{T} - \frac{1}{T}\big(P^{\alpha\beta} + Tl^\beta q^\alpha - p\Delta^{\alpha\beta}\big)\partial_\alpha u_\beta - P^{\alpha\gamma}\partial_\alpha l_\gamma - \Big(\frac{q^\alpha}{T} + el^\alpha\Big)\dot{u}_\alpha \geq 0. \tag{20}$$

As we do not want an acceleration-dependent entropy production, we require that the last term vanishes. According to (17) and (18)

$$e\partial_{q_\alpha}\hat{s} + q^\alpha \partial_e \hat{s} = 0. \tag{21}$$

The general solution of (21) can be given as

$$\hat{s} = \tilde{s}(e^2 - q^\alpha q_\alpha) + B, \tag{22}$$

where B=const. The entropy must depend on the energy density e and the momentum density q^α in a very particular but simple way. As a consequence of this functional form of the entropy function, the Gibbs relation can be given with the help of the entropy derivatives (17) as

$$de - \frac{q^\alpha}{e} dq_\alpha = T ds. \tag{23}$$

We may require first-order homogeneity of the entropy density (extensivity) in (22) without loss of generality. To do so, we introduce $E = \sqrt{|e^2 - q_\alpha q^\alpha|}$ as a variable of the entropy density. In this way, the entropy is a first-order homogeneous functions both of the energy density e and the momentum density q^α. With this property, it is unique.

The corresponding potential relation can be constructed according to the first-order homogeneity (extensivity) of the physical quantities as

$$e - \frac{q^\alpha q_\alpha}{e} = Ts - p. \tag{24}$$

The previous thermostatic relations require the interpretation of E as internal energy. On the other hand, let us recognize that E is the absolute value of the energy vector

$$E = \|E^\alpha\| = \|-u_\beta T^{\beta\alpha}\| = \|eu^\alpha + q^\alpha\| = \sqrt{|e^2 - q_\alpha q^\alpha|}. \tag{25}$$

One should note that the $1/T$ introduced in (18) is not the derivative of the entropy function according to E.

Finally, the entropy flux from (13) and (19) is

$$J^\alpha = \frac{1}{T} q^\alpha - \frac{q_\gamma}{eT} P^{\gamma\alpha}. \tag{26}$$

The final form of the dissipation inequality is

$$q^\alpha \partial_\alpha \frac{1}{T} - \frac{1}{T}\Big(P^{\alpha\beta} + \frac{q^\beta q^\alpha}{e} - p\Delta^{\alpha\beta}\Big)\partial_\alpha u_\beta - P^{\alpha\gamma}\partial_\alpha \frac{q_\gamma}{Te} \geq 0. \tag{27}$$

The last term in this expression with a derivative of one of the constitutive quantities indicates that we cannot give proper thermodynamic fluxes and forces as a solution of the inequality. Another problem appears with (21), because l_α, the spacelike part of the Lagrange multiplier in a local rest frame, was assumed independent of the derivatives of e and u^α. Thus, the Fourier heat conduction is excluded as a possible constitutive function. Both problems indicate that a complete theory may exist only either in an enlarged constitutive space or in an extended basic state space. One possible means of resolution is to introduce higher order derivatives of the basic state space into the constitutive state space, and construct a second-order weakly nonlocal theory. Another possibility is to enlarge the basic state space and construct an extended theory. In both cases, the key that may lead beyond the traditional Müller–Israel–Stewart theory is the new internal energy E.

4. Extended irreversible thermodynamics of relativistic fluids

Motivated by the results of the previous section we calculate the entropy production by a direct substitution of the balance of the energy into the entropy balance. We are to construct an extended theory, introducing q^α as an independent variable, but exploiting the fact that the entropy depends both on the energy and momentum densities in the specific way derived above.

The entropy flux is assumed to have the essentially classical form

$$J^\alpha = \frac{1}{T} q^\alpha. \tag{28}$$

Substituting the energy balance (3) into the entropy balance equation, we arrive at the following entropy production formula:

$$\begin{aligned}\partial_\alpha S^\alpha &= \dot{s}(e^2 + q^\alpha q_\alpha, s) + s\partial_\alpha u^\alpha + \partial_\alpha J^\alpha \\ &= -\frac{1}{T}(e\partial_\alpha u^\alpha + \partial_\alpha q^\alpha + q^\alpha \dot{u}_\alpha + P^{\alpha\beta}\partial_\beta u_\alpha) + \frac{q^\alpha}{Te}\dot{q}_\alpha + s\partial_\alpha u^\alpha + \partial_\alpha\Big(\frac{1}{T}q^\alpha\Big) \\ &= -\frac{1}{T}\Big(P^{\alpha\beta} - (-e + sT)\Delta^{\alpha\beta}\Big)\partial_\alpha u_\beta + q^\alpha\Big(\partial_\alpha \frac{1}{T} - \frac{\dot{u}^\alpha}{T} - \frac{\dot{q}^\alpha}{eT}\Big) \geq 0. \end{aligned}\tag{29}$$

In isotropic continua, the above entropy production results in constitutive functions assuming a linear relationship between the thermodynamic fluxes and forces. The thermodynamic fluxes are the *viscous stress* $\Pi^{\alpha\beta} = \big(P^{\alpha\beta} - (sT - e)\Delta^{\alpha\beta}\big)$, and the energy flux q^α. For these, we get

$$\Pi^{\alpha\beta} = P^{\alpha\beta} - \Delta^{\alpha\beta}\Big(p - \frac{q^\beta q_\beta}{e}\Big) = -2\eta(\Delta^{\alpha\gamma}\Delta^{\beta\mu}\partial_\gamma u_\mu)^{s0} - \eta_v \partial_\gamma u^\gamma \Delta^{\alpha\beta}, \tag{30}$$

$$q^\alpha = -\lambda \frac{1}{T^2}\Delta^{\alpha\gamma}\Big(\partial_\gamma T + T\dot{u}^\alpha + \frac{T\dot{q}^\alpha}{e}\Big), \tag{31}$$

where ^{s0} denotes the symmetric traceless part of the corresponding second order tensor, for example

$$(A^{ij})^{s0} = \tfrac{1}{2}(A^{ij} + A^{ji}) - \tfrac{1}{3}A^{ll}\delta^{ij},$$

and we have introduced the scalar thermostatic pressure according to (24), making $p \neq P^{\alpha}_{\alpha}/3$. Equations (30) and (31) are the relativistic generalizations of the Newtonian viscous stress function and the Fourier law of heat conduction. The shear and bulk viscosity coefficients, η and η_v, and the heat conduction coefficient, λ, are nonnegative according to the inequality of the entropy production (29).

Equations (3) and (4) are the evolution equations of a relativistic, heat conducting ideal fluid, together with the constitutive functions (30) and (31). As special cases we can get the relativistic Navier–Stokes equation by substituting (30) into (4) and assuming $q^{\alpha} = 0$, or the relativistic heat conduction equation by substituting (31) into (3) and assuming that $\Pi^{\alpha\beta} = 0$. The heat conduction part results in a special extended theory, where only the energy flux appears as an independent variable.

5. Summary and discussion

In the first part of the paper we investigated the local equilibrium theory of special relativistic fluids. We saw that there may be no such theory that could give a complete solution of the entropy inequality with the conditions that there be

(i) local Lagrange–Farkas multipliers;

(ii) local entropy (16);

(iii) no additional term in the entropy flux (19).

The first two assumptions were necessary to get a particular solution of the Liu equations and the dissipation inequality. On the other hand, they are natural in local equilibrium.

We conclude that either an extension of the basic state space or an enlargement of the constitutive state space may give a complete solution. Our investigations indicate a particular dependence of the entropy on the energy and momentum densities, leading to a distinction of internal and total energy densities of relativistic fluids.

The local rest frame energy density $e = u_{\alpha} T^{\alpha\beta} u_{\beta}$ is usually interpreted as internal energy in thermodynamic theories. However, the symmetry of the energy-momentum tensor can hide fact that while energy flux is related to dissipation, momentum density is not. This is a property of the relativistic theory, and not apparent in the nonrelativistic case because the nonrelativistic limit results in asymmetric energy-momentum. According to the previous investigations, the total energy density e (minus the time-timelike part of the energy-momentum tensor) is not a suitable internal energy, and the entropy density should be a function of the absolute value of the energy vector $E^{\alpha} = -u_{\beta} T^{\alpha\beta}$, the timelike part of the energy momentum.

To compare our proposal to the traditional Müller–Israel–Stewart theory [Israel 1976; Israel and Stewart 1980], it is instructive to expand the internal energy into the series, assuming that $e^2 > q^{\alpha} q_{\alpha}$:

$$E = \sqrt{|e^2 - q^{\alpha} q_{\alpha}|} \approx e - \frac{\boldsymbol{q}^2}{2e} + \dots . \tag{32}$$

The last, quadratic term in the above expression is what appears in the Müller–Israel–Stewart theory. However, in our case the corresponding relaxation time is fixed $\tau = 1/e$; the quadratic term is only the first approximation; and only the energy flux is introduced as an independent variable in our extended theory, with no need for viscous stress.

The series expansion is an instructive comparison to nonrelativistic hydrodynamics. Therein, the internal energy is the difference of the total energy and the relative kinetic energy. In (32) the quadratic expression is what one could consider as a kind of energy of the flow, considering only the local rest frame momentum density without any connection to an external observer. In a sense, our expression shows that by introducing E as internal energy, we declared that the momentum of the flow does not make a dissipative contribution.

The extension of the present calculations considering the balance of particle number is straightforward. Moreover, one can show that the above system of equations gives a stable homogeneous equilibrium in linear stability investigations, contrary to the theory of Eckart [Ván and Bíró 2008], and can therefore be considered as a minimal viable extension of the local equilibrium theory. The advantages of our approach over the Müller–Israel–Stewart one are that there are no additional material parameters compared to the Eckart theory and the stability of the homogeneous equilibrium does not require additional assumptions beyond the inequalities of thermodynamic stability.

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