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VIBRATIONS OF HIGHLY INHOMOGENEOUS SHELLS OF REVOLUTION UNDER STATIC LOADING

ELENA BESPALOVA AND GALINA URUSOVA

An approach to determining natural frequencies of middle-thickness inhomogeneous shells of revolution acted upon by static axisymmetrical loads is proposed. The approach is based on application of the nonclassic shell model that takes into account transverse shears and reduction across the wall thickness. In solving the problem posed, decomposition into two interconnected problems is used. The first problem is related to determination of the initial stress-strain state of shells under static axisymmetrical loads. The second problem is related to determination of natural frequencies of shells relative to this state that appears in the form of parametric terms. To solve the problems, the numerical-analytical technique developed by the authors is used. The efficiency of allowance for reduction across the wall thickness is illustrated on the example of an inhomogeneous middle-thickness cylindrical shell under local loads. Analysis of natural frequencies of the shell having the form of a pneumatic tire is presented depending on the value of internal pressure.

1. Introduction

Shell structures in real conditions operate, as a rule, in the fields of static actions, such as heat or radioactive emanation, aggressive environment, mechanical loading, and so on. These fields cause in shells some initial stress-strain state. Allowance for this state makes it possible to study correctly more complex processes in shells such as processes of stationary and nonstationary dynamics, stability, contact interaction, and so on.

The present paper addresses the problem on small vibrations of middle-thickness essentially inhomogeneous across the thickness anisotropic shells of revolution with a meridian of arbitrary form acted upon by axisymmetric mechanical and heat loads of a general kind.

A large body of publications in this field deals with the study of free vibrations of shells of revolution without taking into account the initial stresses. In the case of thin shells, it can be judged by references of different years, for example, by [Xi et al. 1996; Tan 1998; Wang and Redekop 2005; Grigorenko et al. 2006]. Detailed analysis of studies on vibrations of some classes of thick shells of revolution is presented in [Redekop 2006; Kang 2007].

A number of works, where vibrations of shells of revolution have been studied with allowance for preliminary stresses, are rather limited. Only publications [Rao et al. 1974; Karmishin et al. 1975; Grigorenko et al. 1986; Bespalova et al. 1991; Sivadas 1995; Lam and Hua 1997; 1998; 2000; Hua and Lam 2000; Wang et al. 1997; Yuan and Liu 2007] directly refer to the object of the given paper. These works mostly consider, based on the Love model, homogeneous across the thickness isotropic

Keywords: shells of revolution, inhomogeneity, preliminary stresses, natural frequencies, nonclassic model, numerical-analytical technique.

and orthotropic shells acted upon by initial pressure. In [Lam and Hua 1997; 1998; 2000; Hua and Lam 2000], initial pressure in rotating cylindrical and conical shells is caused by centrifugal forces and coriolis accelerations. The effect of boundary conditions, geometrical parameters, and rotating speeds on frequency characteristics of shells was studied. Inhomogeneous across the thickness orthotropic shells of revolution are addressed in [Karmishin et al. 1975] using the same Love model. Such classes of inhomogeneous shells of revolution and classes of cylindrical shells with arbitrary cross-section have been considered in [Grigorenko et al. 1986; Bespalova et al. 1991] based on the classic and Timoshenko–Reissner–Mindlin shear models. The purely shear first-order model has been used in [Sivadas 1995] to analyze vibrations of thick cones.

It should be noted that the distinguishing features of the class of shells being considered here are characterized by such factors as thick-walledness, essential inhomogeneity across the thickness (particularly, lamination), a complex pattern of distribution of the initial stresses caused by static loads of general form (particularly, localized ones). Owing to these complicating factors, vibrations of such shells should be studied based on the three-dimensional statement of the elasticity theory or on the shell models, taking into account all the kinds of transverse strain.

The present paper employs the nonclassic shell model with allowance for transverse shears and reduction. This makes it possible, in contrast to purely shear shell models, to take into account spatial effects of the chosen class of shells without increasing, in contrast to three-dimensional statement, the dimensionality of appropriate boundary-value problems. The last fact is rather important at the stage of realization of the chosen model of deformation, since the order of the resolving system of equations (algebraic or ordinary differential ones) rises by the power law with the problem dimensionality.

Note that the present-day tendency in analysis of compound deformable systems is connected in the majority of cases with employing finite-element and boundary-element methods. Allowing for the peculiarity of the class of shells under consideration, the authors propose the semianalytical approach, which is based on the accurate reduction of the dimensionality of an initial two-dimensional problem by analytical means and on the numerical solution of one-dimensional problems that makes it possible to obtain results with high accuracy.

2. The problem formulation and initial guidelines

Let us consider the class of inhomogeneous middle-thickness shells, which can be referred to the some surface of revolution. This surface in a general case is chosen by informal way and is known as a coordinate or reference surface (in particular, it is a median surface) [Ambartsumyan 1961]. Let this surface be generated by rotation of a some plane piecewise-smooth curve about the Oz-axis. It is convenient to describe the shell geometry in a spatial curvilinear orthogonal system of coordinates α, β, and γ, where the coordinate α changes along the generatrix (meridian), β is the angle in the cross-sectional plane $\alpha = \text{const}$, and γ varies across the shell thickness and is reckoned from the coordinate surface $\gamma = 0$ (see Figure 1a and b).

In the general case, the shell is inhomogeneous across the thickness and may be composed of an arbitrary number M of layers with constant or variable thickness along the α-axis. The interface of the adjacent mth and $(m+1)$th layers is specified by the equation $\gamma = \gamma_m(\alpha)$ $(m = \overline{1, M-1})$ (see Figure 1c and d). The material of each of the layers may be both isotropic or anisotropic with three planes of

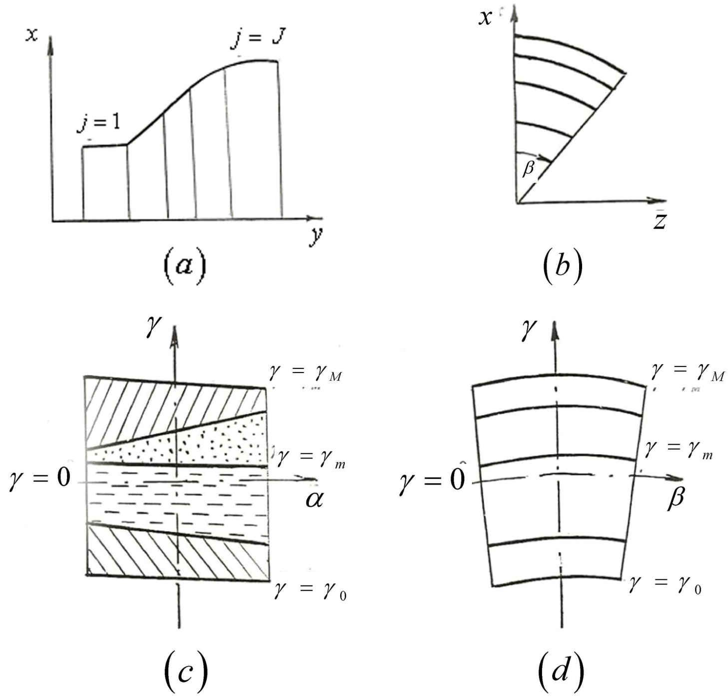

Figure 1. Schematic drawing of a laminated shell of revolution: (a) the overall view of the generatrix in the section $\beta = \text{const}$; (b) the view of the shell in the section $\alpha = \text{const}$; (c), (d) the structure of the shell across the thickness in the sections $\beta = \text{const}$ and $\alpha = \text{const}$, respectively.

elastic symmetry and with principal directions of elastic symmetry coinciding with the lines of principal curvatures $\alpha = \text{const}$ and $\beta = \text{const}$.

To describe the given class of shells, we divide their generatrix into J segments (see Figure 1a). Within the limits of each of the segments, geometrical parameters, values of the thickness, and physical-mechanical properties of the material are specified by smooth functions of the variable α. Besides, any kinds of boundary conditions, which take into account the character of loading and displacement of a contour, are admitted at the shell ends $\alpha = \alpha_0$ and $\alpha = \alpha_J$.

The static axisymmetrical fields imposed on the shell can be caused by the following actions: the normal $q_\gamma(\alpha)$ and meridional $q_\alpha(\alpha)$ loads applied at the points of the coordinate surface $\gamma = 0$; contour forces-moments, which act at the ends $\alpha = \alpha_0$ and $\alpha = \alpha_J$; concentrated forces-moments applied at the meridional sections $\alpha = \alpha_j = \text{const}$ ($j = \overline{1, J-1}$); and heating, which is described by the function $T = T(\alpha, \gamma)$.

To study vibrations of the prestressed shells, we will use the following assumptions:

(a) The problem on vibrations of statically loaded shells is formulated based on the geometrically nonlinear theory in the quadratic approximation [Mushtari and Galimov 1957; Grigorenko and Mukoed 1983].

(b) Vibrations of the shells are treated as small perturbations relative to the initial state caused by the static loading.

(c) Material of the layers is linearly elastic and obeys the generalized Hooke's law within the whole range of applied loads.

(d) Separation and sliding of the shell layers are absent.

(e) The nonclassic deformation model taking into account transverse shears and reduction in linear approximation of all components of a displacement vector across the thickness is adopted for the whole package of layers [Grigorenko et al. 1987] in the form

$$\begin{aligned} u_\alpha^m(\alpha,\beta,\gamma,t) &= u(\alpha,\beta,t)+\gamma\psi_\alpha(\alpha,\beta,t),\\ u_\beta^m(\alpha,\beta,\gamma,t) &= v(\alpha,\beta,t)+\gamma\psi_\beta(\alpha,\beta,t) \qquad (m=\overline{1,M}),\\ u_\gamma^m(\alpha,\beta,\gamma,t) &= w(\alpha,\beta,t)+\gamma\psi_\gamma(\alpha,\beta,t), \end{aligned}$$

where u_α^m, u_β^m, and u_γ^m are the displacements of points of the mth layer, u, v, and w are the displacements of points of the coordinate surface $\gamma=0$ in the directions α, β, and γ, respectively, ψ_α and ψ_β are the total rotation angles of a straight element, ψ_γ is the transverse normal strain, and t is the time variable.

(f) The inertial forces related to the shell translation, rotation of the straight element and its reduction are taken into account.

(g) Temperature actions are allowed for based on the Duhamel–Neumann hypothesis.

According to the assumptions adopted, the following basic relations are used:

- Motion equations

$$\begin{aligned}
&\frac{\partial(BN_\alpha)}{\partial\alpha}-\frac{\partial B}{\partial\alpha}N_\beta+\frac{\partial(AN_{\beta\alpha})}{\partial\beta}+ABk_1Q_\alpha+AB\Big(q_\alpha-I_0\frac{\partial^2u}{\partial t^2}-I_1\frac{\partial^2\psi_\alpha}{\partial t^2}\Big)=0,\\
&\frac{\partial(BN_{\alpha\beta})}{\partial\alpha}+\frac{\partial(AN_\beta)}{\partial\beta}+\frac{\partial B}{\partial\alpha}N_{\beta\alpha}+ABk_2Q_\beta+AB\Big(q_\beta-I_0\frac{\partial^2v}{\partial t^2}-I_1\frac{\partial^2\psi_\beta}{\partial t^2}\Big)=0,\\
&\frac{\partial(BQ_\alpha)}{\partial\alpha}+\frac{\partial(AQ_\beta)}{\partial\beta}-ABk_1N_\alpha-ABk_2N_\beta+AB\Big(q_\gamma-I_0\frac{\partial^2w}{\partial t^2}-I_1\frac{\partial^2\psi_\gamma}{\partial t^2}\Big)=0,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial(BM_\alpha)}{\partial\alpha}-\frac{\partial B}{\partial\alpha}M_\beta+\frac{\partial(AM_{\beta\alpha})}{\partial\beta}-ABQ_\alpha-AB\big[(N_\alpha+k_1M_\alpha)\vartheta_\alpha+N_{\alpha\beta}\vartheta_\beta\big]\\
&\qquad\qquad -AB\Big(-m_\alpha+I_1\frac{\partial^2u}{\partial t^2}+I_2\frac{\partial^2\psi_\alpha}{\partial t^2}\Big)=0,\\
&\frac{\partial(BM_{\alpha\beta})}{\partial\alpha}+\frac{\partial(AM_\beta)}{\partial\beta}+\frac{\partial B}{\partial\alpha}M_{\beta\alpha}-ABQ_\beta-AB\big[(N_\beta+k_2M_\beta)\vartheta_\beta+N_{\beta\alpha}\vartheta_\alpha\big]\\
&\qquad\qquad -AB\Big(-m_\beta+I_1\frac{\partial^2v}{\partial t^2}+I_2\frac{\partial^2\psi_\beta}{\partial t^2}\Big)=0,\\
&\frac{\partial(BP_\alpha)}{\partial\alpha}+\frac{\partial(AP_\beta)}{\partial\beta}-ABk_1M_\alpha-ABk_2M_\beta-AB(C_{13}\varepsilon_\alpha+C_{23}\varepsilon_\beta+C_{33}\psi_\gamma+K_{13}\kappa_\alpha+K_{23}\kappa_\beta-N_{\gamma T})\\
&\qquad\qquad -AB\Big(-\gamma_Mq_\gamma^+-\gamma_0q_\gamma^-+I_1\frac{\partial^2w}{\partial t^2}+I_2\frac{\partial^2\psi_\gamma}{\partial t^2}\Big)=0. \qquad (1)
\end{aligned}$$

- Expressions for strains in terms of displacements

$$\varepsilon_\alpha = \frac{1}{A}\frac{\partial u}{\partial \alpha} + k_1 w + \frac{1}{2}\vartheta_\alpha^2, \qquad \varepsilon_\beta = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u + k_2 w + \frac{1}{2}\vartheta_\beta^2,$$

$$\varepsilon_{\alpha\beta} = \frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}v + \frac{1}{A}\frac{\partial v}{\partial \alpha} + \vartheta_\alpha\vartheta_\beta, \qquad \kappa_\alpha = \frac{1}{A}\frac{\partial \psi_\alpha}{\partial \alpha} + k_1(\psi_\gamma - \varepsilon_\alpha),$$

$$\kappa_\beta = \frac{1}{B}\frac{\partial \psi_\beta}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}\psi_\alpha + k_2(\psi_\gamma - \varepsilon_\beta),$$

$$2\kappa_{\alpha\beta} = \frac{1}{A}\frac{\partial \psi_\beta}{\partial \alpha} + \frac{1}{B}\frac{\partial \psi_\alpha}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}\psi_\beta - k_1\frac{1}{A}\frac{\partial v}{\partial \alpha} - k_2\left(\frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}v\right),$$

$$\varphi_\alpha = \psi_\alpha - \vartheta_\alpha, \qquad \varphi_\beta = \psi_\beta - \vartheta_\beta,$$

$$\vartheta_\alpha = -\frac{1}{A}\frac{\partial w}{\partial \alpha} + k_1 u, \qquad \vartheta_\beta = -\frac{1}{B}\frac{\partial w}{\partial \beta} + k_2 v. \tag{2}$$

- Elasticity relations

$$N_\alpha = C_{11}\varepsilon_\alpha + C_{12}\varepsilon_\beta + C_{13}\psi_\gamma + K_{11}\kappa_\alpha + K_{12}\kappa_\beta - N_{\alpha T},$$

$$N_\beta = C_{12}\varepsilon_\alpha + C_{22}\varepsilon_\beta + C_{23}\psi_\gamma + K_{11}\kappa_\alpha + K_{12}\kappa_\beta - N_{\beta T},$$

$$N_{\alpha\beta} = C_{66}\varepsilon_{\alpha\beta} + K_{66}2\kappa_{\alpha\beta} + k_2(K_{66}\varepsilon_{\alpha\beta} + D_{66}2\kappa_{\alpha\beta}) - N_{\alpha\beta T},$$

$$N_{\beta\alpha} = C_{66}\varepsilon_{\alpha\beta} + K_{66}2\kappa_{\alpha\beta} + k_1(K_{66}\varepsilon_{\alpha\beta} + D_{66}2\kappa_{\alpha\beta}) - N_{\beta\alpha T},$$

$$M_\alpha = K_{11}\varepsilon_\alpha + K_{12}\varepsilon_\beta + K_{13}\psi_\gamma + D_{11}\kappa_\alpha + D_{12}\kappa_\beta - M_{\alpha T},$$

$$M_\beta = K_{12}\varepsilon_\alpha + K_{22}\varepsilon_\beta + K_{23}\psi_\gamma + D_{12}\kappa_\alpha + D_{22}\kappa_\beta - M_{\beta T},$$

$$M_{\alpha\beta} = M_{\beta\alpha} = K_{66}\varepsilon_{\alpha\beta} + D_{66}2\kappa_{\alpha\beta} - M_{\alpha\beta T},$$

$$N_\gamma = C_{13}\varepsilon_\alpha + C_{23}\varepsilon_\beta + C_{33}\psi_\gamma + K_{13}\kappa_\alpha + K_{23}\kappa_\beta - N_{\gamma T},$$

$$Q_\alpha = K_1\varphi_\alpha + D_1\frac{1}{A}\frac{\partial \psi_\gamma}{\partial \alpha} - Q_{\alpha T}, \qquad Q_\beta = K_2\varphi_\beta + D_2\frac{1}{B}\frac{\partial \psi_\gamma}{\partial \beta} - Q_{\beta T},$$

$$P_\alpha = D_1\varphi_\alpha + C_1\frac{1}{A}\frac{\partial \psi_\gamma}{\partial \alpha} - P_{\alpha T}, \qquad P_\beta = D_2\varphi_\beta + C_2\frac{1}{B}\frac{\partial \psi_\gamma}{\partial \beta} - P_{\beta T}. \tag{3}$$

Here A and B and k_1 and k_2 are the coefficients of the first quadratic form and the principal curvatures of the chosen coordinate surface, N_α, $N_{\alpha\beta}$, and Q_α are the normal, shearing, and transverse forces in the section $\alpha = \text{const}$, M_α and $M_{\alpha\beta}$ are the bending and twisting moments, P_α is the first-order moment caused by the tangential stress of transverse shear in the same section, N_β, $N_{\beta\alpha}$, Q_β, M_β, $M_{\beta\alpha}$, and P_β are the same factors in the section $\beta = \text{const}$, q_α, q_β, q_γ, m_α, and m_β are the components of the intensity of the distributed load and bending moments statically equivalent to the body forces and forces, which are applied to the bounding surfaces $\gamma = \gamma_0$ and $\gamma = \gamma_M$, q_γ^+ and q_γ^- are the intensities of the normal force on outer and inner shell surfaces, ε_α, ε_β, and $\varepsilon_{\alpha\beta}$ are the tensile and shear strains of the coordinate surface, κ_α, κ_β, and $2\kappa_{\alpha\beta}$ are the bending and twisting strains, ϑ_α and ϑ_β are the rotation angles of the normal in the planes $\alpha = \text{const}$ and $\beta = \text{const}$, φ_α and φ_β are the rotation angles attributed to the transverse shears, $N_{\alpha T}, N_{\beta T}, \ldots, P_{\beta T}$ are the integral characteristics of the temperature field $T = T(\alpha, \beta)$, I_k $(k = 0, 1, 2)$

are the kth-order moments of the material density, and $C_{11}, \dots, K_{11}, \dots, D_{66}, K_1, \dots, D_1, \dots, C_2$ are the integral characteristics of the shell stiffness determined in terms of the thickness of layers and their elastic properties, for example

$$C_{11} = \sum_{m=1}^{M} \int_{\gamma_{m-1}}^{\gamma_m} \frac{E_\alpha}{1-\nu_\alpha \nu_\beta} d\gamma, \ \dots, \ K_{11} = \sum_{m=1}^{M} \int_{\gamma_{m-1}}^{\gamma_m} \frac{E_\alpha}{1-\nu_\alpha \nu_\beta} \gamma \, d\gamma, \ \dots, \ D_{66} = \sum_{m=1}^{M} \int_{\gamma_{m-1}}^{\gamma_m} G_{\alpha\beta} \gamma^2 d\gamma,$$

$$K_1 = \sum_{m-1}^{M} \int_{\gamma_{m-1}}^{\gamma_m} G_{\beta\gamma}(1-k_1\gamma) d\gamma, \ \dots, \ D_1 = \sum_{m-1}^{M} \int_{\gamma_{m-1}}^{\gamma_m} G_{\beta\gamma} \gamma \, d\gamma, \ \dots, \ C_2 = \sum_{m-1}^{M} \int_{\gamma_{m-1}}^{\gamma_m} G_{\alpha\gamma} \gamma^2 d\gamma.$$

In these expressions, E_α, E_β and E_γ are the elastic moduli in the directions α, β and γ, respectively, $G_{\beta\gamma}$, $G_{\alpha\gamma}$, and $G_{\alpha\beta}$ are the shear moduli for the planes parallel to the coordinate surfaces $\alpha = \text{const}$, $\beta = \text{const}$, and $\gamma = \text{const}$, and ν_α and ν_β are Poisson's ratios.

Relations (1)–(3) make it possible to describe the shell state under consideration in the form of a two-dimensional nonlinear boundary-value problem. In the given paper, components of a vector-function $\boldsymbol{N} = \{N_n(\alpha, \beta, t), n = \overline{1, 12}\} = \{\boldsymbol{Q}, \boldsymbol{U}\}$ are chosen as basic unknowns. They include static,

$$\boldsymbol{Q} = \{N_\alpha, N_{\alpha\beta}, Q_\alpha, M_\alpha, M_{\alpha\beta}, P_\alpha\}^T, \tag{4}$$

and kinematic,

$$\boldsymbol{U} = \{u, v, w, \psi_\alpha, \psi_\beta, \psi_\gamma\}^T, \tag{5}$$

characteristics of the stress state [Grigorenko et al. 1987]. Then, expressing the functions N_β, M_β, ε_α, κ_α, $\varepsilon_{\alpha\beta}$, and $2\kappa_{\alpha\beta}$ with the help of Equations (2) and (3) through the functions N_α, M_α, $M_{\alpha\beta}$, ε_β, κ_β, and ψ_γ and the functions ψ_α, $\partial\psi_\gamma/\partial\alpha$, Q_β, and P_β through the functions Q_α, P_α, ψ_β, and $\partial\psi_\gamma/\partial\beta$, after linear but sufficiently cumbersome transformations, we arrive at the formulation of a two-dimensional problem in the form

$$\begin{aligned}
&\frac{\partial \boldsymbol{N}}{\partial \alpha} = \mathbf{L}\boldsymbol{N} + \boldsymbol{G} + \boldsymbol{q}^{\mathbf{0}} + \mathbf{C}\frac{\partial^2 \boldsymbol{N}}{\partial t^2}, && \alpha \in (\alpha_{j-1}, \alpha_j) \quad (j = \overline{1, J}), \qquad \beta \in [0, 2\pi],\\
&\mathbf{S}_j \boldsymbol{N} = \mathbf{S}_{j+1}\boldsymbol{N} + \boldsymbol{F}_j^{\mathbf{0}}, && \alpha = \alpha_j \qquad (j = \overline{1, J-1}),\\
&\mathbf{B}_j \boldsymbol{N} = \boldsymbol{b}_j^{\mathbf{0}}, && \alpha = \alpha_j \qquad (j = 0; J),\\
&\boldsymbol{N}(\alpha, \beta + 2\pi, t) = \boldsymbol{N}(\alpha, \beta, t).
\end{aligned} \tag{6}$$

Here

$$\mathbf{L} = \left\{ l_{m,n} = \frac{1}{A}\left(A^0_{m,n} + A^1_{m,n}\frac{\partial}{\partial\beta} + A^2_{m,n}\frac{\partial^2}{\partial\beta^2} \right) \right\} \qquad (m, n = \overline{1, 12}),$$

is the matrix differential second-order operator with respect to the variable β, constructed with the help of relations (1)–(3) of the deformation model adopted. Components of the vector $\boldsymbol{G} = \{g_m\}$ are the quadratic functions of the components of the stress-strain state according to the geometrically nonlinear shell theory in the quadratic approximation; $\mathbf{C}$ is the matrix whose elements characterize the inertial properties of the shell; $\mathbf{S}_j$ and $\mathbf{B}_j$ are the matrices allowing for the discontinuities in representation of physical-mechanical and geometrical parameters of the shell in the section $\alpha = \alpha_j$ $(j = \overline{1, J-1})$, as well as the type of boundary conditions at the contour $\alpha = \alpha_j$ $(j = 0; J)$; and $\boldsymbol{q}^0$, $\boldsymbol{F}^0_j$, and $\boldsymbol{b}^0_j$ are the vectors characterizing the axisymmetrical distributed loads and temperature fields, concentrated forces-moments in the section $\alpha = \alpha_j$ $(j = \overline{1, J-1})$ and contour actions at $\alpha = \alpha_j$ $(j = 0;\, J)$.

Note that choice of unknowns in Equations (4) and (5) makes it possible to formulate arbitrary boundary conditions at edges $\alpha = \alpha_0$ and $\alpha = \alpha_J$ in the simplest form.

According to assumption (b), the state of the shell being analyzed can be presented as

$$\boldsymbol{N} = \boldsymbol{N}^{\mathrm{st}} + \boldsymbol{N}^{\mathrm{d}} \qquad \big(|\boldsymbol{N}^{\mathrm{d}}| \ll |\boldsymbol{N}^{\mathrm{st}}|\big),$$

where the vector-function $\boldsymbol{N}^{\mathrm{st}}$ characterizes the stress-strain state of the shell under static actions (initial state), and $\boldsymbol{N}^{\mathrm{d}}$ are the small undamping vibrations about this state. Correspondingly, initial nonlinear problem (6) is reduced to the following two coupled problems:

(i) The problem on deformation of a shell under specified axisymmetrical loads; this is formulated relative to the vector $\boldsymbol{N}^{\mathrm{st}}$ as

$$\begin{aligned} &\frac{d\boldsymbol{N}^{\mathrm{st}}}{d\alpha} = \mathbf{L}^0 \boldsymbol{N}^{\mathrm{st}} + \boldsymbol{G}(\alpha, \boldsymbol{N}^{\mathrm{st}}, \dots) + \boldsymbol{q}^0, && \alpha \in (\alpha_{j-1}, \alpha_j) && (j = \overline{1, J}),\\ &\mathbf{S}_j \boldsymbol{N}^{\mathrm{st}} = \mathbf{S}_{j+1} \boldsymbol{N}^{\mathrm{st}} + \boldsymbol{F}_{\boldsymbol{j}}^{\boldsymbol{0}}, && \alpha = \alpha_j && (j = \overline{1, J-1}),\\ &\mathbf{B}_j \boldsymbol{N}^{\mathrm{st}} = \boldsymbol{b}_{\boldsymbol{j}}^{\boldsymbol{0}}, && \alpha = \alpha_j && (j = 0;\, J), \end{aligned} \tag{7}$$

(here $\mathbf{L}^0$ is the matrix zeroth-order differential operator).

(ii) The problem on vibrations of a prestressed shell; this is formulated relative to the vector $\boldsymbol{N}^{\mathrm{d}}$ as

$$\begin{aligned} &\frac{\partial \boldsymbol{N}^{d}}{\partial \alpha} = \widetilde{\mathbf{L}} \boldsymbol{N}^{\mathrm{d}} + \mathbf{C} \frac{\partial^2 \boldsymbol{N}^{\mathrm{d}}}{\partial t^2}, && \alpha \in (\alpha_{j-1}, \alpha_j) && (j = \overline{1, J}), && \beta \in [0, 2\pi],\\ &\mathbf{S}_j \boldsymbol{N}^{\mathrm{d}} = \mathbf{S}_{j+1} \boldsymbol{N}^{\mathrm{d}}, && \alpha = \alpha_j && (j = \overline{1, J-1}),\\ &\mathbf{B}_j \boldsymbol{N}^{\mathrm{d}} = 0, && \alpha = \alpha_j && (j = 0;\, J),\\ &\boldsymbol{N}^{\mathrm{d}}(\alpha, \beta + 2\pi, t) = \boldsymbol{N}^{\mathrm{d}}(\alpha, \beta, t). \end{aligned} \tag{8}$$

Here $\widetilde{\mathbf{L}}$ is the matrix differential operator corresponding to the operator $\mathbf{L}$ in (6). It was obtained as the result of linearization of the function $\boldsymbol{G}$ relative to the vector $\boldsymbol{N}^{\mathrm{d}}$ and includes components of the vector $\boldsymbol{N}^{\mathrm{st}}$ as parametric terms.

3. Problem-solving technique

Problems (7) and (8) methodically may be considered as two successive stages of solving the initial problem. At the first stage, we determine the stress-strain state of a shell under specified axisymmetrical loads. In the case of finite strains, the one-dimensional problem is nonlinear, and for its solving we can employ methods of quasilinearization or simple iteration [Bellman and Kalaba 1965]. One-dimensional linearized problems are solved numerically. In the present work, the study is limited by loads which cause small strains. For this reason problem (7) can be considered in the linear formulation. To solve it, we will use the numerical orthogonal-sweep method [Godunov 1961]. This method proved itself to be effective in solving the stationary problems of the shell theory by using different deformation models

[Grigorenko et al. 1986; 1987]. Detailed description of this method is presented in [Grigorenko and Rozhok 2003].

For the second stage, to solve the problem on small vibrations of the shell with allowance for the initial stress state $\boldsymbol{N}_d$, we use a numerical-analytical approach. This approach includes the method of separation of variables, the method of reverse iterations with constructing the Rayleigh ratio, and numerical solving of one-dimensional boundary-value problems by the orthogonal-sweep method.

The first step in solving problem (8) is separation of the time multiplier $e^{i\omega t}$ for components of the vector-function $\boldsymbol{N}^{\mathrm{d}}$ and their representation in the form of the single trigonometric Fourier series along the circumferential coordinate β by

$$\boldsymbol{N}^{\mathrm{d}} = \left\{ N_n^d(\alpha, \beta, t) = \sum_{k=0}^{\infty} N_{nk}^d(\alpha) \begin{bmatrix} \sin k\beta \\ \cos k\beta \end{bmatrix} e^{i\omega t}, \quad n = \overline{1, 12} \right\}. \tag{9}$$

Here ω is the natural frequency of shell vibrations, k is the parameter characterizing the shape of a wave along the circumference (this parameter is equal to the number of waves, which go fully in this direction). The expression in square brackets indicates that one part of the components of the vector-function $\boldsymbol{N}^{\mathrm{d}}$, namely, $(N_\alpha^d, Q_\alpha^d, M_\alpha^d, P_\alpha^d, u^d, w^d, \psi_\alpha^d, \psi_\gamma^d)$ is represented in $\cos k\beta$, whereas the other part $(N_{\alpha\beta}^d, M_{\alpha\beta}^d, v^d, \psi_\beta^d)$ is represented in $\sin k\beta$. As a result, the two-dimensional problem (8) is reduced exactly to the following sequence of uncoupled single-parametric one-dimensional problems with respect to the functional coefficients $\boldsymbol{N}_k^d = \{N_{nk}^d(\alpha), n = \overline{1, 12}\}$ in (9):

$$\begin{aligned}
&\frac{d\boldsymbol{N}_k^d}{d\alpha} = (\mathbf{A}_k - \lambda \mathbf{C})\boldsymbol{N}_k^d, && \alpha \in (\alpha_{j-1}, \alpha_j) && (j = \overline{1, J}), \\
&\mathbf{S}_j \boldsymbol{N}_k^d = \mathbf{S}_{j+1} \boldsymbol{N}_k^d, && \alpha = \alpha_j && (j = \overline{1, J-1}), \\
&\mathbf{B}_j \boldsymbol{N}_k^d = 0, && \alpha = \alpha_j && (j = 0; J) \qquad (k = 0, 1, 2, \dots),
\end{aligned} \tag{10}$$

where $\mathbf{A}_k$ is the squared 12th-order matrix defined by the operator $\widetilde{\mathbf{L}}$ in (8) in accordance with approximation (9), and $\lambda = \omega^2$ is the unknown numerical parameter. Expressions for elements of matrices $\mathbf{A}_k, \mathbf{S}_j, \mathbf{S}_{j+1}$ $(j = \overline{1, J-1})$, and $\mathbf{B}_j$ $(j = 0; J)$ are presented in [Grigorenko et al. 1987]. Nonzero elements of the matrix $\mathbf{C}$ are

$$\begin{aligned}
c_{1,7} = c_{2,8} = c_{3,9} = I_0 &= \sum_{m=1}^{M} \int_{\gamma_{m-1}}^{\gamma_m} \rho_m(\alpha, \gamma)\, d\gamma, \\
c_{1,10} = c_{2,11} = c_{3,12} = c_{4,7} = c_{5,8} = c_{6,9} = I_1 &= \sum_{m=1}^{M} \int_{\gamma_{m-1}}^{\gamma_m} \rho_m(\alpha, \gamma)\gamma\, d\gamma, \\
c_{4,10} = c_{5,11} = c_{6,12} = I_2 &= \sum_{m=1}^{M} \int_{\gamma_{m-1}}^{\gamma_m} \rho_m(\alpha, \gamma)\gamma^2 d\gamma,
\end{aligned} \tag{11}$$

where $\rho_m(\alpha, \gamma)$ is the density of the mth layer.

To solve the eigenvalue problem (10), we employ the method of reverse iterations with the shift of the spectrum of eigenvalues [Kollatz 1963]. Application of this method to the problems of the given class for the classic and Timoshenko–Mindlin type models is outlined in [Grigorenko et al. 1986; Bespalova et al. 1991].

According to the method of reverse iterations, the minimum eigenvalue of the shifted spectrum is determined as the limit of the numeric sequence λ_n constructed by the Rayleigh ratio

$$\lambda_n = \frac{\left(\boldsymbol{U}^{d,(n)}, \mathbf{C}^* \boldsymbol{U}^{d,(n-1)}\right)}{\left(\boldsymbol{U}^{d,(n)}, \mathbf{C}^* \boldsymbol{U}^{d,n}\right)}, \qquad n = 1, 2, \dots, \tag{12}$$

where $\boldsymbol{U}^{d,(n)}$ is the vector of kinematic characteristics of the deformed state (5), $\mathbf{C}^*$ is the nonsingular matrix with elements (11), n is the iteration number, and $(\dots, \dots)$ is the scalar product. The vector-function $\boldsymbol{U}^{d,(n)}$ being the component of the vector-function $\boldsymbol{N}^{d,(n)}$ at each nth step of the iteration ($n = 1, 2, \dots$) is determined from the solution of the nonuniform problem

$$\begin{aligned}
&\frac{d\boldsymbol{N}_k^{d,(n)}}{d\alpha} = (\mathbf{A}_k - \tau\mathbf{C})\boldsymbol{N}_k^{d,(n)} - \mathbf{C}\boldsymbol{N}_k^{d,(n-1)}, && \alpha \in (\alpha_{j-1}, \alpha_j) \quad (j = \overline{1, J}),\\
&\mathbf{S}_j \boldsymbol{N}_k^{d,(n)} = \mathbf{S}_{j+1}\boldsymbol{N}_k^{d,(n)}, && \alpha = \alpha_j \quad (j = \overline{1, J-1}),\\
&\mathbf{B}_j \boldsymbol{N}_k^{d,(n)} = 0, && \alpha = \alpha_j \quad (j = 0; J) \quad (k = 0, 1, 2, \dots).
\end{aligned} \tag{13}$$

According to [Kollatz 1963], the sequence of such problems ($n = 1, 2, \dots$) is obtained from (10) by adding the term which characterizes the shift of a frequency spectrum of value τ, and by changing the term with the coefficient λ to $\boldsymbol{N}_k^{d,(n-1)}$. As in the first stage, problems (13) are solved numerically by the orthogonal-sweep method. If the sequence $\boldsymbol{N}_k^{d,(n)}$ ($n = 1, 2, \dots$) converges, the vector-functions $\boldsymbol{U}_k^{d,(n)}$ will tend to the eigenvector-functions and the numerical sequence λ_n in (12) will tend to the sought for eigenvalue λ. Fulfillment of the condition $|1 - \lambda^{n+1}/\lambda^{(n)}| < \varepsilon$ is the natural criterion for the finish of the iterative process. The issue of the selection of the initial approximation and convergence of the process in the problems of the shell theory has been considered in [Grigorenko et al. 1986].

4. Vibration analysis of prestressed inhomogeneous shells

As an example of employment of the approach developed, we will solve two problems. The first one is a model problem and demonstrates the efficiency of the variant, in which reduction and transverse shear are taken into account, in contrast to the Kirchhoff–Love and Timoshenko–Mindlin theories. The second problem concerns analysis of natural frequencies of the shell similar to the real structure having the form of a pneumatic tire.

***Problem* 1.** Here, using as an example a shell with a simple geometric shape which is homogeneous along the generatrix, we will estimate the validity of the technique depending on two factors: inhomogeneity of physical-mechanical properties across the thickness and localization of the acting load.

Let us consider a sandwich cylindrical shell of length $2l$ ($s \in [-l, l]$) with the radius of a median surface R_0 and general thickness h (s is the arc length of the generatrix). Layers of the shell are isotropic with different elastic properties and are placed symmetrically relative to the median surface. The elastic modulus and density of outer layers are $E = E_0$ and $\rho = \rho_0$, respectively, and Poisson's ratio is ν. These characteristics for the inner layer are $E = E_0/d$ and $\rho = \rho_0/d$ with the same Poisson's ratio. The volume of the inner layer is equal to that of two outer ones. The inhomogeneity of such laminated structures is characterized by the parameters d or $\eta = \lg d$. The case $\eta = 0$ corresponds to a homogeneous isotropic shell made of a material of outer layers. Note that with η being varied, the velocity of propagation of elastic waves for the cylinder as a whole remains the same. The shell is under the action of the normal axisymmetrical load with intensity q. The load is applied on the circular ring of the length $2l^*$ with a center in the section $s = 0$ ($s \in [-l^*, l^*]$). To characterize the level of its localization, we will use the parameter $\delta = l^*/l$. As the limiting case of the load localization, we will consider the circular radial force Q_s (for $\delta = 0$) concentrated in the section $s = 0$. It is assumed that both cylinder ends $s = \pm l$ are hinged. In this case, the prestressed state is defined mainly by the circumferential force N_β.

Natural frequencies ω of the shell are analyzed depending on two parameters: the degree of the inhomogeneity across the thickness (parameter $\eta \in [0; 3]$) and the degree of the load localization (parameter $\delta \in [0; 0.2]$). We will compare the results obtained by different shell models using the load value q^*, for which the minimum frequency becomes equal to zero ($\omega_{min}(q) \simeq 0$). Note that in accordance with the dynamic criterion of the shell stability, the value q^* can be adopted as the upper quantity of a critical load.

To calculate the magnitudes of the critical load, we will employ the following shell models: the classic model (transverse strains are neglected), the Timoshenko–Mindlin type model (transverse shears are regarded), and nonclassic model (transverse shears and reduction across the wall thickness are regarded). Comparison of corresponding solutions makes it possible to evaluate the contribution of each type of the transverse strain into correction of the results obtained by the classic theory. These results are presented in Figure 2 for the relative value of the critical load $\chi = q^*(\eta)/q_c^*(0)$, where $q_c^*(0)$ is the q^* which is obtained for an homogeneous shell by the classic model.

The dependencies $\chi = \chi(\eta)$ are presented for the above-mentioned shell models and for different kinds of load localization $\delta = 0.2, 0.1$, and 0. Data accepted for calculations are $R_0 = 100 l_0$, $2l/R_0 = 2$, and $h/R_0 = 1/5$, where l_0 is the typical linear dimension of the cylinder. It should be noted, that, as

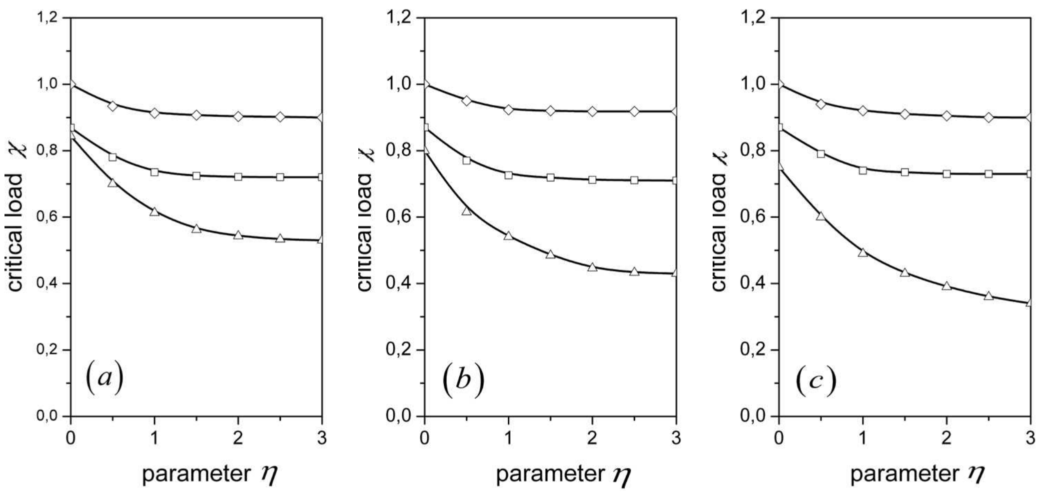

Figure 2. Dependency of the critical load $\chi = \chi(\eta)$ on the parameter of inhomogeneity η for different shell models: classic (◇), shear (□), and with allowance for reduction (△), and different values of the load localization: (a) $\delta = 0.2$; (b) $\delta = 0.1$; and (c) $\delta = 0$.

applied to these data, the calculation error, which takes place in determining the natural frequencies of unloaded shell by the nonclassic theory with reduction being taken into account, is not more then 2% in comparison with the results obtained by the three-dimensional theory [Bespalova and Urusova 2007].

Let us analyze in detail the dependencies $\chi = \chi(\eta)$.

In the case of a homogeneous shell ($\eta = 0$) and weak load localization (for example, at $\delta = 0.2$, see Figure 2a), the allowance for transverse shears makes it possible to refine the results obtained by the classic theory by 15%. The complementary allowance for the reduction does not result in any appreciable changes. For this reason, in the case of a homogeneous shell with small localization, we can limit ourselves to transverse shears only. The correct use of classic and purely shear models as applied to the inhomogeneous shells ($\eta > 0$) is possible only if $\eta < 1$ and ($d < 10$), that is, when the difference in properties of the material of layers is within the limit of one order.

When the inhomogeneity is considerable ($\eta > 1, d > 10$), both models do not depict variation in shell properties. The allowance for transverse shears yields only insignificant refinement (approximately for 20%) of results in comparison with a homogeneous shell. At the same time, the allowance for reduction results in considerable refinement for essentially inhomogeneous shells even if the localization of a load is insignificant. So, for $\eta = 3$, the magnitude of a critical load is refined by 40% compared with the classic theory and by 25% compared with the shear theory. With the localization increasing ($\delta = 0.1$, see Figure 2b) and especially in the limiting case of the point action ($\delta = 0$), the influence of allowance for reduction increases considerably. In the case of a concentrated load ($\delta = 0$, see Figure 2c) and homogeneous shell ($\eta = 0$), the effect of the allowance for transverse shears and reduction refines the magnitude of the critical load by 25% and reduction proper by 15%. For an inhomogeneous shell ($\eta = 3$) such refinement is more than 60% and 50%, respectively.

Thus, in determining the critical loads for essentially inhomogeneous shells ($\eta > 1$) and considerable localization of actions ($\delta < 0.2$), the allowance only for transverse shears may be inadequate. In these cases, for analysis to be correct, the shell models, which take into attention all kinds of the transverse strain including reduction, should be employed.

***Problem* 2.** As an example, let us determine natural frequencies of such standard shell construction as a pneumatic tire, which, as a preliminary, was loaded with internal pressure. This example in full measure represents the class of problems being considered. The shell has laminated structure and complicated geometric shape, variable thickness and inhomogeneous physical-mechanical properties both along the generatrix and across the thickness. These factors provoke the complex initial stress state even under the uniform action.

As the reference surface we will chose the inside surface of the shell. This surface has the shape of a torus with an elliptical cross-section, its half-axes are a and b, and distance to the axis of revolution is R_0. The shell can be divided conditionally into two segments with different thicknesses and different physical-mechanical properties: $\alpha \in [0, \alpha_1]$ and $\alpha \in [\alpha_1, \alpha_2]$, where the angle α characterizes the current position on the generatrix. On the first segment, the shell is composed of three layers. The inner layer with the thickness h_1 is made of orthotropic material with the characteristics $E_\alpha = 5.8E_0$, $E_\beta = E_\gamma = 0.12E_0$, $G_{\alpha\beta} = G_{\alpha\gamma} = 0.043E_0$, $G_{\beta\gamma} = 0.031E_0$, $\nu_\alpha = 0.42$, and $\rho = \rho_0$. The outer layer with the thickness h_3 is isotropic and has $E = 0.05E_0$, $\nu = 0.49$. The middle layer is composed of two orthotropic sublayers with the same thickness h_2, whose orthotropy axes in the sublayers are oriented relative to the shell generatrix at angles of $\pm 70^0$. Its properties are $E_1 = 18.4E_0$, $E_2 = E_3 = 0.074E_0$, $G_{12} = G_{13} = 0.021E_0$, $G_{23} = 0.019E_0$, $\nu_{12} = 0.47$, and $\rho = 1.54\rho_0$, where indices 1 and 2 denote the principal elasticity directions of the orthotropic material. This layer as a whole is considered as structurally orthotropic and its characteristics are determined by the known formulas which are related to the turn of coordinate axes [Lechnitskii 1977]. In this case the rigidities connecting the tension-compression strains with twisting strains can be neglected due to the symmetric location of the orthotropic axes of the material in sublayers. The general thickness of the shell on this segment is $h = h_1 + 2h_2 + h_3$. On the second segment $\alpha \in [\alpha_1, \alpha_2]$, the shell is composed of one orthotropic layer with the thickness h_1 and characteristics $E_\alpha = 5.8E_0$, $E_\beta = E_\gamma = 0.12E_0$, $G_{\alpha\beta} = G_{\alpha\gamma} = 0.043E_0$, $G_{\beta\gamma} = 0.031E_0$, $\nu_\alpha = 0.42$, and $\rho = \rho_0$.

An initial stress-strain state of the shell is caused by the normal axisymmetric pressure of the intensity q.

Let us analyze the natural frequencies for the shell in the case of $R_0 = 217l_0$, $h_1 = 0.75l_0$, $h_2 = 1.4l_0$, $h_3 = 6.45l_0$, $\alpha_1 = 0.95$ rad, $\alpha_2 = 2.168$ rad ($l_0 = 10^{-3}$ m, $E_0 = 10^2$ MPa, and $\rho_0 = 10^3$ kg·m^{-3}) as applied to the two following variants of geometric parameters: variant I, where $a = 84.6l_0$, and $b = 43.0l_0$ (elliptic-section torus), and variant II, where $a = b = 60.3l_0$ (circular-section torus).

The internal pressure causes in the shell the complex stress state. So, Figure 3a shows the distribution of dimensionless circumferential $\tilde{N}_\beta = N_\beta(\alpha)/g$ and meridional $\tilde{N}_\alpha = N_\alpha(\alpha)/g$ forces along the shell generatrix for two configurations (I, the elliptic cross-section, and II, the circular cross-section, $g = 10^3 q l_0$). In the case of an elliptic cross-section, the circumferential forces $\tilde{N}_\beta$ are dominating, moreover on the first segment they are two order higher then on the second one. At the point of the segment conjugation ($\alpha/\alpha_2 = 0.49$), we observe the jump attributed to the stepwise variation in the shell structure across

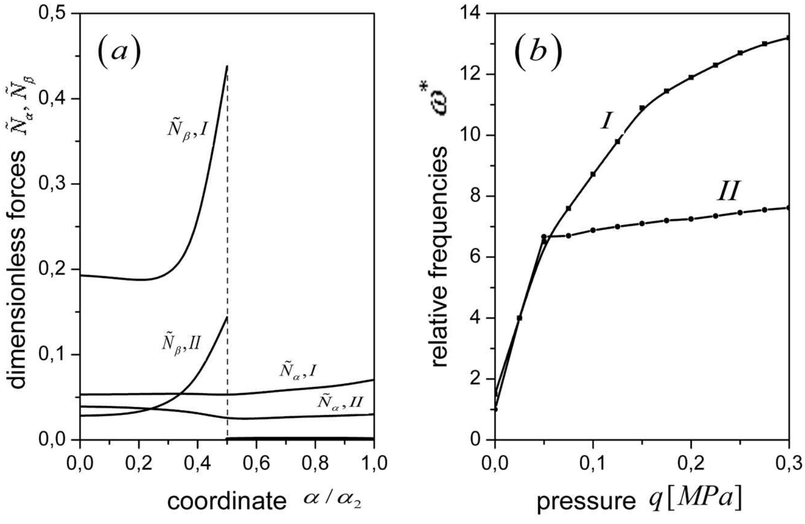

Figure 3. Effect of the initial stress state on the natural frequencies of the toroidal shell. (a) The forces $\tilde{N}_\alpha$, I and $\tilde{N}_\beta$, I for an elliptic torus and the forces $\tilde{N}_\alpha$, II and $\tilde{N}_\beta$, II for a circular torus. (b) The dependency of minimum frequencies ω^* for elliptic ($i =$ I) and circular ($i =$ II) toruses on the internal pressure q.

the thickness. The meridional forces $\tilde{N}_\alpha$ are distributed along the generatrix nearly uniformly. The qualitative pattern of the force distribution for the circular torus is kept the same.

The natural frequencies of such a prestressed shell are determined depending on the value of the internal pressure q, which varies through the range [0; 0.3] MPa. The values of minimum frequencies for elliptic $\omega_{\mathrm{I}}(q)$ and circular $\omega_{\mathrm{II}}(q)$ toruses referred to the minimum frequency of the unloaded circular torus $\omega_{\mathrm{II}}(0)$ ($\omega^* = \omega_i(q)/\omega_{\mathrm{II}}(0)$, $i = \mathrm{I}, \mathrm{II}$) are presented in Figure 3b.

For both variants of the geometry, the dependencies $\omega^* = \omega^*(q)$ are the monotonous piecewise smooth curves. Here inflection points correspond to the mode change at minimum frequency. If the internal pressure is low, difference in natural frequencies of both shells is negligibly small. With the pressure increasing ($q > 0.1$ MPa), difference in frequencies also increases and for $q = 0.3$ MPa frequencies of the elliptic torus exceed those for the circular one approximately by 70%.

5. Conclusions

The paper presents a numerical-analytical approach to the analysis of natural frequencies for middle-thickness inhomogeneous shells of revolution under axisymmetric loads. The approach includes the following points:

(i) Formulation of the problem based on the nonclassic two-dimensional model of shells with allowance for transverse shears and reduction within the frame of the geometrically nonlinear theory.

(ii) Decomposition of the problem into two interconnected problems: the problem on preliminary stresses in a shell under static loading and the problem on small shell vibrations with allowance for the preliminary stress-strain state.

(iii) Numerical-analytical technique for solving both problems using the following procedures:
   (a) Trigonometric Fourier series expansion along circumferential coordinate.
   (b) Inverse iterations method for solving eigenvalue problems.
   (c) Numerical orthogonal-sweep method for solving one-dimensional boundary-value problems.

The efficiency of the approach proposed is illustrated by the example of the inhomogeneous across the thickness nonthin cylindrical shell under localized loading in comparison with classical and shear models. It is shown that in analyzing the natural frequencies of a shell with the essential difference in physical-mechanical properties across the shell thickness (more than one order) and appreciable localization of static actions, it would be desirable to take into account not only transverse shears but also reduction.

We have analyzed the natural frequencies of a complicated pneumatic-tire shell system for two variants of cross-sectional configuration depending on the value of internal pressure. Calculation results are physically justified [Buchin 1988].

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